PHYSICAL REVIEW A **79**, 051605(R) (2009)

Cloaking of matter waves under the global Aharonov-Bohm effect

De-Hone Lin[1] and Pi-Gang Luan[2]
[1]*Department of Physics, National Sun Yat-sen University, Kaohsiung 804, Taiwan*
[2]*Department of Optics and Photonics, National Central University, Jhongli 320, Taiwan*

(Received 5 March 2009; published 27 May 2009)

We discuss the Aharonov-Bohm effect of a magnetic flux for its influence on a two-dimensional quantum cloak. It is shown that the matter wave of a charged particle under the global influence of the Aharonov-Bohm effect can still be perfectly cloaked and guided by the quantum cloak. Since the presence of the global influence of a magnetic flux on charged particles is universal, the perfect cloaking and guiding nature not only provides an ideal setup to cloak an object from matter waves but also provides an ideal setup to test the global physics of charged matter waves in the presence of a bare magnetic flux.

DOI: 10.1103/PhysRevA.79.051605 PACS number(s): 03.75.−b, 34.50.−s, 61.05.fd, 03.65.Vf

Based on Pendry's proposal [1], invisibility cloaks for classical waves that can hide an object from the detection of the probe waves and is thus externally invisible have been intensively studied both in the theoretical and experimental literatures [2–13]. Recently, Zhang *et al.* [14] explored the possibility of cloaking matter waves. They showed that by way of controlling the effective mass of a quantum particle and the potential acting on it, the matter wave of the particle can indeed be cloaked. A quantum cloak with appropriate medium parameters can thus be designed. A beam of matter wave incident on the cloak will be guided along the cloaking shell without any distortion and loss when it returns to the original propagation direction. Following Zhang's extended idea, Greenleaf *et al.* [15] proposed to use the aggregation of infinite many ring-shaped potential profiles to mimic the cloaking condition which may offer another feasible approach to realize the quantum cloaking system.

In this Rapid Communication, we generalize Zhang's result to include the Aharonov-Bohm (AB) effect of a flux hidden inside the cloaked region of a two-dimensional (2D) quantum cloak. It is shown that the cloaking ability of the 2D quantum cloak does not diminish or degrade even when the global AB effect is present. Since the quantum cloak can guide the matter wave detouring the cloaked region, it provides us an ideal setup to manifest the global AB effect, prohibiting the matter wave from penetrating into the region of nonzero magnetic field. Accordingly, the quantum cloak not only can cloak an object from matter waves but also is an ideal instrument for testing the global influence of a magnetic flux. The ideal setup discussed here may be generalized, which can be utilized to detect other kinds of nonlocal quantum effects, such as the scalar AB effect, the AB-EPR effect, and the Aharonov-Casher effect [16].

This Rapid Communication is organized as follows: we first analyze the cloaking of matter waves under the AB effect both theoretically and numerically in detail. Following it, the discrepancy between the classical and quantum particles in the presence of the cloak and AB effect is explained. Our conclusions are summarized in the final section.

The Schrödinger equation for a mass M_0 particle with energy E in a general curvilinear coordinate system is given by

$$\sqrt{g}(E-V)\Psi = \frac{1}{2}\{-\hbar^2\partial_i(M^{ij}\partial_j\Psi) + iq\hbar(\partial_i M^{ij}A_j)\Psi + 2iq\hbar M^{ij}A_j\partial_i\Psi + q^2M^{ij}A_jA_i\Psi\}, \qquad (1)$$

where $g=|g_{ij}|$ is the determinant of the generic metric tensor g_{ij}, V is the scalar potential, $M^{ij}=g^{ij}\sqrt{g}/M_0$ is the effective mass tensor with g^{ij} being the inverse of g_{ij}, q is the charge carried by the quantum particle, and A_j is the components of the vector potential $\mathbf{A}$ that describes the magnetic field. The property of a quantum cloak is completely determined by the effective mass tensor M^{ij} in the sense of material interpretation [1]. Such a property can be achieved through the design of media as proposed by Zhang with Bose-Einstein condensation and Greenleaf with potentials approach. In view of more feasible consideration, we shall study a 2D cloak and assume the potential $V=0$. For the AB effect under consideration, the vector potential of a magnetic flux along the z direction hidden in the cloaked region can be expressed as $\mathbf{A}=\Phi(-ye_x+xe_y)/[2\pi(x^2+y^2)]$, where e_x and e_y stand for the unit vectors along the x and y axes, respectively. Introducing the azimuthal angle $\varphi(\mathbf{x})=\tan^{-1}(y/x)$ around the flux tube, the components of the vector potential can be expressed as $A_i=(\Phi/2\pi)\partial_i\varphi(\mathbf{x})$. The associated magnetic field is confined to an infinitely thin tube, $B_3=(\Phi/2\pi)\epsilon_{3ij}\partial_i\partial_j\varphi(\mathbf{x})=\Phi\delta(\mathbf{x}_\perp)$, where $\mathbf{x}_\perp$ stands for the transverse coordinates $\mathbf{x}_\perp=(x,y)$. The magnetic flux along the z axis is then found to be $\int\int dxdyB_3=\Phi$. Here we see the vector potential of the flux spreads out through the entire x-y plane such that the charged particles would be influenced by the potential through the minimal coupling $(\mathbf{p}-q\mathbf{A})$ in the Hamiltonian even though they do not enter the region of nonzero magnetic field $\mathbf{B}$.

Outside the cloak, the matter wave of a charged quantum particle Ψ is mastered by the Schrödinger equation $(-i\hbar\nabla-q\mathbf{A})^2\Psi/2M_0=E\Psi$. Due to the rotational symmetry of $\mathbf{A}$, the method of "separation of variables" is applicable. Trying $\Psi=R(\rho)\exp(im\varphi)$, with $m=0,\pm1,\pm2,\cdots$, we get

$$ER=\left(-\frac{\hbar^2}{2M_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{[m\hbar-(q/2\pi)\Phi]^2}{2M_0\rho^2}\right)R. \qquad (2)$$

The general solution for the Schrödinger equation is then given by

$$\Psi = \sum_{m=-\infty}^{\infty} [a_{|m-\mu|} J_{|m-\mu|}(k_0\rho) + b_{|m-\mu|} H_{|m-\mu|}(k_0\rho)] e^{im\varphi}, \quad (3)$$

where $k_0 = \sqrt{2M_0E}/\hbar$, $\mu = \Phi/\Phi_0$ with $\Phi_0 = h/q$ the fundamental flux quantum, and J_m (H_m) is the Bessel (Hankel) function as usual. Obviously, the first term of the solution represents the incident wave of the charged particle Ψ^i, and the second term represents its scattered wave Ψ^s. For a 2D cylindrical quantum cloak with inner radius $\rho = a$ and outer radius $\rho = b$ that hides an object from the matter waves in the region $\rho < a$, the cloaking transformation can be taken as $\bar{\rho} = f(\rho)$, $\bar{\varphi} = \varphi$, where $f(\rho)$ is radial scaling function with $f(a) = 0$ and $f(b) = b$. The cloaking transformation requires that the metric g_{ij} and the effective mass tensor M_{ij} in $a \leq \rho \leq b$ have to satisfy the relations

$$\sqrt{g} = f'(\rho)f(\rho)/\rho, \quad M_{\rho\rho} = M_0\rho f'(\rho)/f(\rho),$$

$$M_{\varphi\varphi} = M_0 f(\rho)/[\rho f'(\rho)]. \quad (4)$$

Substituting the transformations for Eq. (1), the matter wave within the cloak is found to be

$$\Psi^c = \sum_{m=-\infty}^{\infty} c_{|m-\mu|} J_{|m-\mu|}[k_0 f(\rho)] e^{im\varphi}. \quad (5)$$

With the help of the boundary conditions of wave functions, we have $\Psi(b) = \Psi^c(b)$ and $M_0^{-1}\partial_\rho \Psi(b) = M_{\rho\rho}^{-1}\partial_\rho \Psi^c(b)$, which lead to

$$a_{|m-\mu|} J_{|m-\mu|}(k_0 b) + b_{|m-\mu|} H_{|m-\mu|}(k_0 b) = c_{|m-\mu|} J_{|m-\mu|}(k_0 b),$$

$$a_{|m-\mu|} J'_{|m-\mu|}(k_0 b) + b_{|m-\mu|} H'_{|m-\mu|}(k_0 b) = (M_0/M_{\rho\rho}) f'(\rho) c_{|m-\mu|} J'_{|m-\mu|}(k_0 b) = c_{|m-\mu|} J'_{|m-\mu|}(k_0 b). \quad (6)$$

Equation (6) shows $b_{|m-\mu|} = 0$, which means the incident matter wave passes through the cloaking shell without suffering any scattering and distortion. Actually, $\Psi^c = \sum_{m=-\infty}^{\infty} a_{|m-\mu|} J_{|m-\mu|}[k_0 f(\rho)] e^{im\varphi} = \Psi^i[f(\rho), \varphi]$ exhibits that the probability flow of the matter wave is smoothly connected on both sides of the cloak boundary and returns to the original incident direction. This fact reveals that the cloak is invisible at the quantum level. The numerical results for the wave-function patterns (snapshots) are shown in Fig. 1. In all cases the matter waves are incident from the left and leave toward the right. For comparison we have also plotted the patterns of matter waves scattered by an ideal bare magnetic flux of infinitely thin at the origin, in the absence of the quantum cloak. We see that the patterns are greatly influenced by the AB effect [17]. The matter-wave patterns in the presence of the bare magnetic flux are shown in Figs. 1(a)–1(d), whereas the wave patterns in the presence of both the magnetic flux and quantum cloak are shown in Figs. 1(e)–1(h). In these simulations, the relation $b = 2a$ is assumed. They exhibit that the quantum cloak indeed cloaks and guides the matter waves through the cloaking shell perfectly, keeping the wave patterns outside the cloak system unaltered. The result is interesting since a charged particle scattered by a bare magnetic flux is an important starting point of studying the global effect of quantum physics. Nevertheless, it is hard to realize such a bare flux experiment under the assumption that the matter wave does not touch the region of nonzero magnetic field. Here the quantum cloak provides a possible feasible tool to check the physics of scattering by a bare magnetic flux.

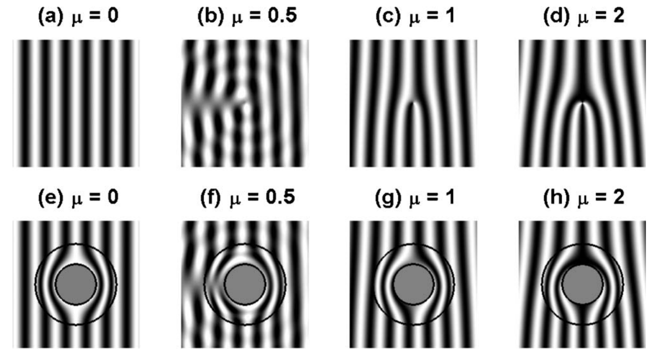

FIG. 1. The cloaking of charged matter waves under the Aharonov-Bohm effect of an infinitely thin magnetic flux along the z axis. In all cases, the matter waves are incident from the left and leave toward the right. (a)–(d) show the scattering patterns of wave functions under the magnetic flux in the absence of quantum cloak, where $\mu = \Phi/\Phi_0$ is in unit of the fundamental flux quantum $\Phi_0 = h/q$. (e)–(h) exhibit that the matter waves are perfectly cloaked and guided by the quantum cloak in the presence of the AB effect. Outside the cloak, the outgoing waves completely coincide with the patterns shown in the first row.

The perfectness of the cloaking system is sensitive to the incident energy of the charged particle. If the energy of the particle is different from the designed cloaking energy, the matter wave will be scattered, degrading the perfectness of the cloaking system. The sensitivity can be analyzed and examined by the methods of perturbation. Although for the case of slight energy deviation the perturbation approach can be employed to analyze the imperfectness of the cloaking system, it does not provide useful information when the energy deviation is arbitrarily large. The exact solution of Schrödinger equation is thus desirable; however, such a solution is not only difficult to obtain but also is too involved to analyze. For these reasons, we analyze this problem relying on the classical limit approach. An appropriate model in the classical limit can be exactly solved so that all trajectory families of arbitrary incident energies are available. Since the classical exact solution in general gives the profile of a quantum system, the classical limit affords us some useful insight on how critical the cloaking condition needs to be satisfied. Besides, in contrast with a study through the classical limit, the nonlocal quantum nature of the AB interference effect can be revealed.

Inside the cloaking system, the classical Hamiltonian is given by

$$H = \frac{p_\rho^2}{2M_{\rho\rho}} + \frac{(p_\varphi - q\Phi/2\pi c)^2}{2M_{\varphi\varphi}\rho^2} + \tilde{V}, \quad (7)$$

where the effective potential $\tilde{V} = E + \sqrt{g}(V - E)$ [14]. The rotational symmetry of the system, i.e., H is independent of φ, implies that the canonical momentum $p_\varphi = M_{\varphi\varphi}\rho^2\dot{\varphi} + q\Phi/2\pi c$ is a constant of motion. In addition, since in this Rapid Communication we consider only a static flux, the angular momentum $M_{\varphi\varphi}\rho^2\dot{\varphi} = (p_\varphi - q\Phi/2\pi c)$ is also a constant of motion given by $L = w\sqrt{2M_0 E}$, with w being the impact parameter. Now we consider the case that the incident particle has energy E, whereas the cloaking energy is designed to be E_0, corresponding to the Hamiltonian

$$H = \frac{p_\rho^2}{2M_{\rho\rho}} + \frac{L^2}{2M_{\varphi\varphi}\rho^2} + \widetilde{V}(\rho, E_0). \tag{8}$$

Although the particle trajectories are now influenced by the energy deviations $\Delta E = E - E_0$, the particle still does not enter the region of $\mathbf{B} \neq 0$, and thus the dynamics of the classical particle is not affected by the presence of the flux. Therefore, hiding a magnetic flux in the cloaked region does not cause any observable physical effect in the classical level. As a specific explanation, the scaling transformation is taken as $f(\rho) = (\rho - a)/\lambda$, with $\lambda = (b-a)/b$ [14]. The differential equation of the trajectory inside the cloak for a particle of incident energy E can be derived from the Hamiltonian of Eq. (8) and is given by

$$\frac{d\varphi}{d\rho} = -\frac{1}{(\rho - a)\sqrt{c_1(\rho - a)^2 + c_2(\rho - a) - 1}}, \tag{9}$$

where $c_1 = (E - E_0 + E_0/\lambda^2)/(w^2 E)$ and $c_2 = (E - E_0)a/(w^2 E)$. Equation (9) is integrable which gives an analytic representation of the trajectory family as

$$\rho = a + \frac{2}{c_2 + \sqrt{c_2^2 + 4c_1}\sin(\varphi - c_3)}, \tag{10}$$

where c_3 is a constant determined by the initial condition. Numerical results are shown in the Fig. 2. Figure 2(a) exhibits the perfect conformal paths in the cloaking shell when the particle is injected with the designed cloaking energy $E = E_0$. Figure 2(b) shows the path deflection effect when the energy deviation is 10% of the designed cloaking energy, i.e., $\Delta E = 0.1E_0$.

The comparison of the numerical results in Figs. 1 and 2(a) shows the significant discrepancy between the classical and quantum effects when magnetic flux is present and the incident particle has exactly the cloaking energy E_0. In classical level the charged particle is guided through and then leaves the cloaking shell without any modification to its energy and momentum. In contrast, in quantum level the matter wave of the charged particle is interfered by the global AB effect of the magnetic flux even when it is perfectly cloaked and guided to pass though the cloaking shell. Note that outside the cloak the wave pattern is the same as that of the bare flux case. Because of the perfect nature of guiding matter waves, the quantum cloak provides an ideal instrument to test the global effects of a magnetic flux such as the scattering cross section $\sigma_{\mathrm{AB}}(\varphi) = \sin^2(\pi\mu)/[2\pi k_0 \cos^2(\varphi/2)]$ caused by a bare flux predicted by Aharonov and Bohm [18], avoiding the possibility of wave penetration into the region of $\mathbf{B} \neq 0$.

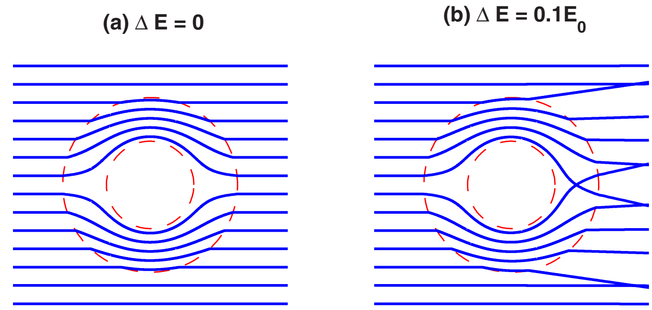

FIG. 2. (Color online) The sensitivity of the quantum cloak to the incident energy. (a) shows the perfect conformal trajectories of the classical particles when they are injected with the designed cloaking energy, where $\Delta E = E - E_0$. (b) exhibits the departures of paths from the perfect trajectories when the deviation of the incident energy is $\Delta E = 0.1E_0$.

To summarize, we have investigated the AB effect of a magnetic flux for its influence on a 2D quantum cloak. It turns out that the efficiency of the quantum cloak is unaltered even when there is a magnetic flux inside the cloaked region. Owing to the perfect nature of guiding matter waves, the cloak not only cloaks an object hidden in the cloaked region from the probe matter waves but also provides an ideal instrument to test the physical effects of a single bare flux. Other models that are possible to apply the quantum cloak to test their global effects are such as the scalar AB effect, the Aharonov-Casher effect and so forth. These effects all associate with the guiding matter waves propagating along two different paths and are then recombined to return to the original incident direction. A perfect experiment for them requires that the partial matter waves guided along different paths without suffering any leakage out of the paths. Thus, quantum cloak is naturally the ideal setup for these kinds of experiments.

[1] J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
[2] U. Leonhardt, Science **312**, 1777 (2006).
[3] A. Alù, and N. Engheta, Phys. Rev. E **72**, 016623 (2005).
[4] D. A. B. Miller, Opt. Express **14**, 12457 (2006).
[5] U. Leonhardt, New J. Phys. **8**, 118 (2006).
[6] S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, and J. B. Pendry, Phys. Rev. E **74**, 036621 (2006).
[7] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, Science **314**, 977 (2006).
[8] G. W. Milton, M. Briane, and J. R. Willis, New J. Phys. **8**, 248 (2006).
[9] F. Zolla, S. Guenneau, A. Nicolet, and J. B. Pendry, Opt. Lett. **32**, 1069 (2007).
[10] W. Cai, U. K. Chettiar, A. V. Kildishev, and V. M. Shalaev, Nat. Photonics **1**, 224 (2007).

[11] H. Chen and C. T. Chan, Appl. Phys. Lett. **90**, 241105 (2007).
[12] Z. Ruan, M. Yan, C. W. Neff, and M. Qiu, Phys. Rev. Lett. **99**, 113903 (2007).
[13] S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, J. Pendry, M. Rahm, and A. Starr, Phys. Rev. Lett. **100**, 024301 (2008).
[14] S. Zhang, D. A. Genov, C. Sun, and X. Zhang, Phys. Rev. Lett. **100**, 123002 (2008).
[15] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, Phys. Rev. Lett. **101**, 220404 (2008).
[16] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).
[17] M. V. Berry, R. G. Chambers, M. D. Large, C. Upstill, and J. C. Walmsley, Eur. J. Phys. **1**, 154 (1980).
[18] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).