

Confinement limit of Dirac particles in scalar one-dimensional potentials

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We present a general proof that Dirac particles cannot be localized below their Compton length by symmetric but otherwise arbitrary scalar potentials. This proof does not invoke the Heisenberg uncertainty relation and thus does not rely on the nonrelativistic linear momentum relation. Further it is argued that the result is also applicable for more general potentials, as, e.g., generated by nonlinear interactions. Finally a possible realization of such a system is proposed.

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I. INTRODUCTION

It is generally accepted that relativistic particles with rest mass m_0 described by the Dirac equation cannot be localized better than half its corresponding Compton wavelength $\lambda_C \equiv \hbar/m_0c^2$ [1]. The argument to support this fact is based on the Heisenberg uncertainty relation $\Delta z \Delta p \geq \frac{\hbar}{2}$. Using the relation $\Delta p = m_0 \Delta v$ and observing that $\Delta v < c$, one arrives at the statement $\Delta z \geq \hbar/2m_0c^2 = \lambda_C/2$. However this argument is not stringent since it is based on the purely nonrelativistic relation $p = m_0v$ and cannot therefore be used for highly relativistic particles.

Recently there is a growing interest in various realizations of systems exposing Dirac-type features [2–4]. In [5] it was shown that stationary light [6] at small spatial length scales exhibits Dirac-type behavior, as, e.g., *Zitterbewegung* and Klein tunneling. It is in this context that the question about the lower confinement limit of Dirac particles arises.

For freely moving Dirac particles a stringent derivation of the lower localization bound can be found in [7].

In this Brief Report we want to provide a general proof of the statement that the coordinate variance $\langle \Delta z \rangle$ for any Dirac particle with rest mass m_0 in a one-dimensional (1D) symmetric scalar potential is strictly larger than half its corresponding Compton length. In doing so we use neither the Heisenberg uncertainty relation nor the nonrelativistic linear momentum relation. We also argue that the result can be generalized to potentials created by certain nonlinear interactions. A well-known example for this kind of interaction is the massive Thirring model [8].

II. PROOF

The one-dimensional Dirac equation for a particle with rest mass m_0 in external scalar and vector potentials $U(z)$ and $A(z)$ is given by

$$i\hbar \partial_t \Psi(t, z) = \{c\alpha[-i\hbar \partial_z - A(z)] + \beta m_0 c^2 + U(z)\} \Psi(t, z). \quad (1)$$

It can be shown [9] that the case of a vanishing scalar potential, i.e., $U(z) = 0$, Eq. (1) is equivalent to an effective Schrödinger equation. Thus confinement to arbitrarily small length scales is possible.

In the following we assume that $A(z) = 0$ and only a finite scalar potential $U(z)$ is present. As a particular representation of the Dirac algebra we choose

$$\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

This is a permissible choice, since it fulfills the restrictions $\alpha^2 = \beta^2 = \mathbb{1}_{2 \times 2}$, $\alpha\beta + \beta\alpha = 0$ and leaves Eq. (1) invariant under Lorentz transformations [10]. Hence, in one dimension it is sufficient to use a single spinor with two scalar components $\Psi(t, z) = (\phi(t, z), \chi(t, z))^T$. Making the ansatz $\Psi(t, z) = \Psi(z) \exp[-iEt/\hbar]$ and introducing dimensionless quantities

$$\gamma = \frac{E}{m_0 c^2}, \quad z \rightarrow \frac{z}{\lambda_C}, \quad \lambda_C = \frac{\hbar}{m_0 c}, \quad (3)$$

where λ_C is the Compton length, we arrive at the following coupled differential equations:

$$\gamma \phi(z) = -\frac{\partial}{\partial z} \chi(z) + [1 + f(z)] \phi(z), \quad (4)$$

$$\gamma \chi(z) = \frac{\partial}{\partial z} \phi(z) - [1 - f(z)] \chi(z). \quad (5)$$

Here $f(z) = U(z)/m_0c^2$ is the normalized external potential.

In the following we want to derive an estimation for the coordinate variance $\langle \Delta z \rangle = \sqrt{\langle z^2 \rangle - \langle z \rangle^2}$. Expectation values of any operator A are calculated by

$$\langle A \rangle = \int_{-\infty}^{\infty} dz \Psi^\dagger(z) A \Psi(z). \quad (6)$$

First we calculate the expectation value of the coordinate $\langle z \rangle$. Assuming a symmetric potential, i.e., $f(z) = f(-z)$, we find that $\phi^2(z) + \chi^2(z)$ is an even function. Further we may assume that for bound states $\phi(z)$ and $\chi(z)$ are real functions. Thus we observe for the expectation value

$$\langle z \rangle = \int_{-\infty}^{\infty} dz z [\phi^2(z) + \chi^2(z)] = 0. \quad (7)$$

Assuming that $\langle z^2 \rangle$ exists and that the normalization condition

$$\int_{-\infty}^{\infty} dz[\phi^2(z) + \chi^2(z)] = 1 < \infty \quad (8)$$

holds, integration by parts then yields

$$\int_{-\infty}^{\infty} dz z \phi(z) \frac{d\phi(z)}{dz} = -\frac{1}{2} \int_{-\infty}^{\infty} dz \phi^2(z). \quad (9)$$

The same holds for $\chi(z)$. Using these relations and Dirac equation (5) we find

$$\left| \int_{-\infty}^{\infty} dz z \phi(z) \chi(z) \right| = \frac{1}{4}. \quad (10)$$

Applying the arithmetic mean inequality

$$|\phi(z)\chi(z)| \leq \frac{1}{2}[\phi^2(z) + \chi^2(z)] \quad (11)$$

to Eq. (10) we find

$$\langle |z| \rangle = \int_{-\infty}^{\infty} dz |z| [\phi^2(z) + \chi^2(z)] \geq \frac{1}{2}. \quad (12)$$

Next we calculate the expectation value

$$\langle z^2 \rangle = \int_{-\infty}^{\infty} dz z^2 [\phi^2(z) + \chi^2(z)]. \quad (13)$$

Using the Cauchy-Schwarz or Jensen [11] inequality and Eq. (12) we obtain

$$\langle z^2 \rangle = \langle |z|^2 \rangle \geq \langle |z| \rangle^2 = \frac{1}{4}. \quad (14)$$

From Eq. (11) one concludes that equality only holds if $\chi(z) = \phi(z)$. However this linear dependence of the eigen-solutions of the Dirac equation never occurs and thus we have a strict inequality.

Reintroducing dimensional variables according to Eq. (3) we finally observe

$$\langle \Delta z \rangle = \sqrt{\langle z^2 \rangle - \langle z \rangle^2} > \frac{\lambda_C}{2}. \quad (15)$$

This result coincides with the one obtained from the Heisenberg uncertainty relation. Physically this inequality states any

Dirac particle is smeared out over a range larger than half its Compton length, irrespective of the confining energy. The reason for this is the increasing contribution of the negative energy solutions to the wave packet. In a multiparticle picture it corresponds to particle-antiparticle production which cannot be localized better than half the corresponding Compton length.

It is worth noticing that this result not only applies to external scalar potentials but also to potentials created by nonlinear interactions, as, e.g., given in the nonlinear massive Thirring model [8]. This can be seen by observing that any additional term entering Eq. (5) with the same sign cancels when calculating Eq. (10). Thus any solitonic solution of this model cannot become smaller than half its Compton wavelength.

This might have applications when combining the relativistic Dirac-type light [5] with an intrinsic Kerr-type nonlinearity [12]. This system then provides a natural realization of a massive Thirring model, where the relativistic particles are massive photons with an externally controllable mass and the nonlinearity is provided by interactions with an atomic ensemble.

III. CONCLUSION

In this Brief Report we derived a strict inequality for the lower bound on the coordinate variance $\langle \Delta z \rangle$ of a Dirac particle in a 1D scalar and symmetric external potential. We found that the result coincides with the estimate obtained using the Heisenberg uncertainty relation in conjunction with the nonrelativistic expression for momentum $\langle \Delta z \rangle > \lambda_C/2$. That is, any localization of a relativistic Dirac particle has to be larger than half the corresponding Compton length. Finally we noted that this result is also applicable to more general potentials generated, e.g., by nonlinear interactions, as in the Thirring model [8].

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- [1] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
 [2] K. S. Novoselov *et al.*, Nature (London) **438**, 197 (2005); Y. Zhang *et al.*, *ibid.* **438**, 201 (2005); M. I. Katsnelson *et al.*, Nat. Phys. **2**, 620 (2006).
 [3] J. Y. Vaishnav and C. W. Clark, Phys. Rev. Lett. **100**, 153002 (2008).
 [4] M. Merkl *et al.*, EPL **83**, 54002 (2008).
 [5] J. Otterbach, R. G. Unanyan, and M. Fleischhauer, Phys. Rev. Lett. **102**, 063602 (2009).
 [6] F. E. Zimmer *et al.*, Opt. Commun. **264**, 441 (2006).

- [7] V. V. Dodonov and S. S. Mizrahi, Phys. Lett. A **177**, 394 (1993).
 [8] W. E. Thirring, Ann. Phys. (N.Y.) **3**, 91 (1958).
 [9] A. De Martino, L. Dell'Anna, and R. Egger, Phys. Rev. Lett. **98**, 066802 (2007).
 [10] F. Scheck, *Quantisierte Felder* (Springer, Berlin, 2001).
 [11] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1971), Vol. II.
 [12] M. Fleischhauer, J. Otterbach, and R. G. Unanyan, Phys. Rev. Lett. **101**, 163601 (2008).