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Semiclassical quantization of the one- and two-kink dark solitons

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Drawing from an analogy with a linear particle-chain model, we develop a picture of the dark optical soliton as a composite of a background and notch each enjoying a certain degree of independence. The semiclassical quantization procedure of Dashen *et al.* [Phys. Rev. D **11**, 3424 (1975)] is modified to allow for the separate quantization of these two entities. We apply our results to the one-kink and two-kink dark solitons. For both we find that the fluctuations about the notch can be understood as a bound state of bosons held by an attractive delta potential while the background fluctuations are seen as an ensemble of oscillators with a slightly repulsive interaction. We show that the collision effects are small for the two-kink soliton. The quantum numbers emerging from this analysis are interpreted in terms of the number of particles. Moreover, the topological character of the dark soliton also appears naturally in our description. We speculate on the possibility of fractional quantum numbers and derive the repulsive interaction of two dark solitons from the framework developed for the two-kink soliton.

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I. INTRODUCTION

For over four decades now the ability of solitons to preserve their shapes and identities during collision and evolution and their remarkable robustness have been the subject of continuing interest and investigation. A large body of phenomena describing solitons is well encapsulated by the nonlinear Schrödinger equation (NLSE) which has two distinct types of localized solutions: the bright and dark solitons. While much progress with the former has been made, for various reasons, dark solitons have been observed only relatively recently, for instance, in trapped Bose-Einstein condensates (BECs) [1], in optical fibers [2], and in thin magnetic films [3]. Here we focus on dark optical spatial solitons in one dimension, which are low-intensity dips on a constant wave (cw) background field and which do not diffract during propagation of the beam [4]. Such solitons have been regarded as the reflectionless modes of the optical waveguide they induce (contrasting with bright spatial solitons which are the bound modes) [4,5].

The development of our understanding of optical dark solitons might be said to parallel somewhat the odyssey of modern optics: for a long time the semiclassical quantization of the electromagnetic field was quite adequate for the elucidation of the optical phenomena than known and only later, with the advent of the laser, did the full quantization of the electromagnetic field become urgently necessary [6]. Thus one might argue that solitons in BEC are necessarily quantum objects because these condensates are purely quantum phenomena to start with, whereas optical solitons on the other hand may be treated as classical objects since the cw background seems classical. To be sure, a full quantum theory of solitons, optical, or otherwise is not available. Nevertheless if we recall that many nonlinear field theories describing elementary particles have soliton solutions, then the quantization of solitons should appear naturally because the soliton remains a particle even in the classical limit, $\hbar \to 0$. An obvious way to go about this is thus by carrying out an expansion in the Lagrangian in powers of $\hbar$ or some other parameter occurring with $\hbar$, wherein the classical soliton emerges as the leading order term and the higher order terms describe quantum effects. In fact, such a procedure, the semiclassical quantization in field theory, is known in particle physics [7] and has found application in other problems [8]. Although a small coupling approximation, the semiclassical quantization method is not a perturbation theory because it contains all orders of perturbation theory (see [7] and also remarks after Eq. (18)). In the context of solitons this method had been applied before by Dashen, Hasslacher, and Neveu (DHN) [9] in their seminal work on the quantization of the doublet solution of the sine-Gordon equation [Eq. (9)]. But a direct link to dark optical solitons was not made. Hence the goal of this paper is to carry out the semiclassical quantization of the two-kink dark soliton. However, our approach will depart somewhat from DHN, both in method and concept of quantization, and we believe that it is much simpler and also more physically direct. We should also note that DHN were interested in manifestly relativistic field theories whereas optical solitons will be treated here as nonrelativistic systems.

The case for the quantization of nonlinear field theories, of nonlinear effects in optical systems, and of optical solitons in particular has been articulated by several workers in the past few years. Recently, for instance, there has been much interest in creating a strongly interacting atom-photon system [10]. Although such a system does not involve the soliton, one must contend with both photonic and atomic degrees of freedom. (At this stage it will suffice to define quantization as the calculation of the energy spectrum of a nonlinear system about a suitable classical stable configuration. We will make this more precise in Sec. III.) Chiao and his co-workers [11] pioneered the study of a "photon fluid" confined in a Fabry-Pérot resonator in which the photons appear as weakly interacting massive bosons and where the Bose-Einstein con-

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densation for photons might be observed. Although these workers do not explicitly discuss solitons, the underlying physics is governed by the nonlinear Schrödinger equation. If the photon-photon interactions of a photon fluid are replaced by exciton-polariton scattering instead we are led to current studies on cavity polaritons which share much in common with BEC. A large body of work on this exotic system and on the prospects of a polariton laser is already in existence [12]. In another development novel quantum effects including the possibility of a phase transition have also been reported in optical pulses propagating in a Kerr-effect ring cavity [13]. Thus there is a growing awareness that the nonclassical aspects of solitons cannot be glossed over for long.

To be specific we will be considering in this paper the quantization of a one-dimensional optical system generically described by the NLSE,

$$i\varphi_t + \varphi_{xx} - \tfrac{1}{2}|\varphi|^2\varphi = 0 \tag{1}$$

with a view to arriving at a particle interpretation of it. Here φ is the complex electric field. As pointed out above the semiclassical quantization method is appropriate for treating this case because of its superiority over a perturbative approach. The last term is the interaction term and as it stands this equation describes a system with a repulsive interaction [4,14]. One can verify that a solution is the one-kink soliton $\varphi(x,t)=\sqrt{2}\eta e^{-i\eta^2 t}\tanh(\eta x/\sqrt{2})$, η=const, which is also called the dark soliton. If $|\varphi|^2$ is proportional to intensity then this describes a dark spot, or notch, at $x=0$ in an otherwise uniform background field. We will, however, define the term notch differently in Sec. II. In a recent paper we showed that the semiclassical quantization of this soliton complemented a parallel Bogoliubov–de Gennes (BG) analysis [15]. In this paper we explain the dark soliton as a composite quantum structure involving a background whose fluctuations are quantized harmonic oscillators interacting with a slightly repulsive anharmonicity and a localized notch supporting massive bosons with attractive interaction. In *distinguishing* these features of the dark soliton, we must be careful not to *separate* them for in doing so we would risk loosing the nonlinear interaction itself and thus speaking of two objects quite unrelated to the single unified reality being studied.

Going back to Eq. (1), if we have the positive sign for the third term then the corresponding one-soliton solution is $\varphi(x,t)=2\eta e^{i\eta^2 t}\,\mathrm{sech}(\eta x)$, which is termed a bright soliton and, unlike the dark soliton, it clearly represents a localized pulse. (In our work, this will correspond to the notch just mentioned above.) This soliton has been shown to sustain a bound system of bosons with attractive point delta interaction [16,17]. The field-theoretical basis for characterizing this soliton and its properties in the limit of large quantum numbers had been carried out by Nohl [16] and more recently studies of a possible phase transition indicate that it is an object with a clear distinguishable identity [13]. For a review of the quantum theory of point-particle bosons see [18]. An important difference between the dark soliton and the bright one is that the former is topological, i.e., its behavior at spatial infinity is nontrivial [7,19]. Clearly an adequate description of the dark soliton must address the issue.

This paper is organized as follows. Section II discusses the excitations of a linear chain model of particles which shares some features in common with the dark optical soliton. We develop an analogy between this physically appealing model and the less intuitively accessible dark optical soliton and explore some of its consequences. We emphasize that the linear chain, while useful in our development, has properties quite unrelated to it. In Sec. III we introduce the one-kink dark soliton and carry out the semiclassical quantization of this system. On account of a significant feature of the dark soliton, namely, the fact that the Lagrangian can be split into two distinct terms, we also explain how our procedure differs from DHN. This section has several parts: (a) first we introduce the one-kink soliton, (b) then summarize the DHN method as it pertains to our case, (c) apply a modified DHN method to the kink, taking its topological character into account, and (d) discuss the results. The two-kink dark soliton is studied in Sec. IV and we quantize it in Sec. V where we also relate it with the linear chain model. Here we touch on fractional soliton number and the conservation of topological charge. Then in Sec. VI we derive an expression for the repulsive interaction between two well separated dark solitons, an issue of relevance to the control over dark solitons. We are unaware of a direct derivation of our result and believe it would be of interest and utility. Finally in Sec. VII we give our conclusions.

II. ANALOGY WITH A LINEAR CHAIN OF PARTICLES

In this section we study the dynamics of a linear monatomic chain of particles interacting via nearest neighbors through harmonic and quartic potentials. By focusing on the antisymmetric kink solution of this system we hope to arrive at an analogy which will provide us with a useful picture of the dark optical soliton. There has been much work on this model, which is only tangentially related to solitons, and our aim is simply to gather insight useful for our immediate goal, the quantization of the dark soliton. For our purposes it will suffice to note that much of the effort on the chain model is on the search for stationary localized modes brought about by the quartic potential [20]. Our interest here then is in identifying qualitative similarities, not necessarily quantitatively exact correspondences.

We denote the particle mass by m, the longitudinal displacement of the nth particle by u_n, and the harmonic and anharmonic force constants by K_2 and K_4, respectively. The equations of motion are (see Fig. 1)

$$m\ddot{u}_n + K_2(2u_n - u_{n+1} - u_{n-1}) + K_4 u_n^3 = 0. \tag{2}$$

We are studying therefore the case of on-site (negative) anharmonicity. (A study of the corresponding translationally invariant case yields qualitatively similar results. Other scenarios, such as boundary conditions, can be introduced and studied, but these will take us too far afield.) To solve for stationary solutions of Eq. (2) we follow Bortolani *et al.* [21] by first assuming that

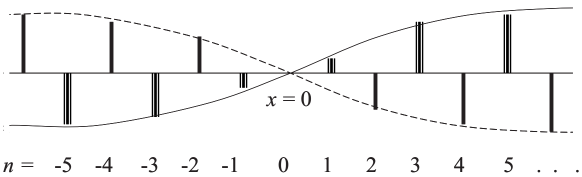

FIG. 1. Displacement pattern of the antisymmetric kink. The dashed and solid curves show the fitting curves with tanh behavior which connect displacements corresponding to a kink. Notice that the patterns to the right and left of $x=0$ display their regular reversal in sign from site to site. However on passing $x=0$ this regularity suffers a phase change in π (e.g., $n=1$ should be "down" in relation to the particles on the left half). If the pattern for positive x is reflected about the horizontal axis, the phase change still remains (since now the solid fitting curve also must be reversed to connect "like" particles).

$$u_n = \xi_n \cos \omega t, \quad \omega = \omega_1 + \Omega \tag{3}$$

and use the rotating wave approximation, familiar from optics [22], to simplify Eq. (2),

$$m\omega^2 \xi_n = K_2[(\xi_n - \xi_{n-1}) + (\xi_n - \xi_{n+1})] + \tfrac{3}{4}K_4\xi_n^3. \tag{4}$$

An approximate expression for the energy, which will be useful later, can also be given by following similar steps:

$$E = \frac{1}{2}m\omega^2 \sum_n \xi_n^2 \sin^2 \omega t + \frac{1}{2}K_2 \sum_n (\xi_n - \xi_{n+1})^2 \cos^2 \omega t + \frac{3}{4}K_4 \sum_n \xi_n^2 \cos^2 \omega t. \tag{5}$$

Next we build into the equations of motion the reversal in sign from site to site through (see Fig. 1)

$$\xi_n = (-1)^n \psi_n, \quad n \geq 0 \tag{6}$$

(ψ_n is independent of n) and introduce the continuous variable $x=na$, a=lattice constant. Equation (6) is also used in other problems, for instance, in polyacetylene [23]. The displacements can be expanded to second order as $\psi_n=\psi(x)$, $\psi_{n\pm1}=\psi(x)\pm a\psi_x+\frac{1}{2}a^2\psi_{xx}$. The subscript x indicates differentiation with respect to x. We expect the contribution from the harmonic part of the excitation, ω_1, to be much larger than Ω, which is the contribution due to the nonlinearity. Thus we can ignore Ω^2 and choose the maximum frequency $\omega_1^2=4K_2/m$. We obtain from Eq. (4)

$$-2m\omega_1\Omega\psi(x) + a^2K_2\frac{d^2\psi}{dx^2} + \frac{3}{4}K_4\psi^3(x) = 0. \tag{7}$$

To keep track of signs, it will be convenient to write $K_4 = |K_4|\mathrm{sgn}(K_4)$, $\Omega=|\Omega|\mathrm{sgn}(\Omega)$. We also define the positive parameters,

$$\alpha = \frac{2m\omega_1|\Omega|}{a^2K_2}, \quad \beta = \frac{\frac{3}{4}|K_4|}{a^2K_2}, \tag{8}$$

and set $x=y/\sqrt{\alpha}$ so we can now recast Eq. (7) as

$$\frac{d^2\psi}{dy^2} - \mathrm{sgn}(\Omega)\psi + \mathrm{sgn}(K_4)\frac{\beta}{\alpha}\psi^3 = 0. \tag{9}$$

Further simplification is made by letting $\chi=\sqrt{\beta/\alpha}\psi$ and $\mu = d\chi/dy$. We integrate Eq. (9) directly,

$$\tfrac{1}{2}\mu^2 - \tfrac{1}{2}\mathrm{sgn}(\Omega)\chi^2 + \tfrac{1}{4}\mathrm{sgn}(K_4)\chi^4 = C, \tag{10}$$

where C is an arbitrary constant. Anticipating a kink, i.e., a tanh solution for χ, we choose the boundary conditions $\mu = 0$, $\chi=\pm1$ as $y\to\pm\infty$. For the case we are interested in, namely, $\mathrm{sgn}(K_4)=-$ and $\mathrm{sgn}(\Omega)=-$, we must select $C=\frac{1}{4}$. Solving Eq. (10) for χ we finally arrive at the displacements

$$\psi(x) = \pm\left(\frac{4m\omega_1|\Omega|}{3|K_4|}\right)^{1/2} \tanh\left[\left(\frac{m\omega_1|\Omega|}{a^2K_2}\right)^{1/2}(x-x_0)\right]. \tag{11}$$

This then gives the envelope for the particle displacements of the antisymmetric kink [see Figs. 1 and 2(a)]. An expression for the anharmonic contribution Ω can be obtained by taking a large value for n, say $N \gg 1$. That is, we put $\xi_{N+1}=\xi_{N-1}=-\xi_N$ and substitute back into Eq. (4) to find

$$m\omega^2\xi_N = 4K_2\xi_N + \tfrac{3}{4}K_4\xi_N^3. \tag{12}$$

Once again, invoking the definition of ω_1 and ignoring Ω^2 we have

$$\Omega \cong \frac{3K_4}{8m\omega_1}\xi_N^2, \tag{13}$$

which is negative for $K_4=-$ and quadratic in the mode amplitudes ξ_n.

The net output of this continuum-limit calculation is the antisymmetric kink solution (11) and the result that the shift in frequency due to the nonlinearity is below the harmonic band (that is, the energy spectrum due only to K_2) because $\Omega<0$. An important feature is that, Ω, the anharmonic contribution to the frequency, is quadratic with respect to the amplitude of the mode. Much of the literature on the chain model focuses on numerical studies of the equations of motion and we now briefly touch on a comparison of the above results with numerical studies. For small displacements $|\psi_{\pm1}|$ of the particles adjacent to the particle at rest ($n=0$ which corresponds to $x=0$ in Fig. 1), the displacement patterns for the kink fit very well with envelope (11). When the particle displacements $|\psi_{\pm1}|$ are large, however, two localized excitations of sech form appear which are superimposed on the kink excitation at the regions adjacent to $n=0$ [21] [see Fig. 2(b)]. This is obviously more complicated than the first case and there is no analytical treatment, similar to the above, available for this. Nevertheless, we observe that when $|\psi_{\pm1}|$ is not small (a) the expansion to second order and (b) the argument leading to Eq. (13) cannot be valid for the particles near $n=0$ so we expect to encounter difficulties there. For our picture of the dark soliton we consider the occurrence of the localized excitations as evidence for two phenomena associated with different scales (from K_2 and K_4). When the nonlinearity (i.e., K_4) is small only the first phenomenon connected with K_2 is important. Indeed examining Eq. (7) for large x (so $|\psi|^2\approx1$, say) we have a linear equation.

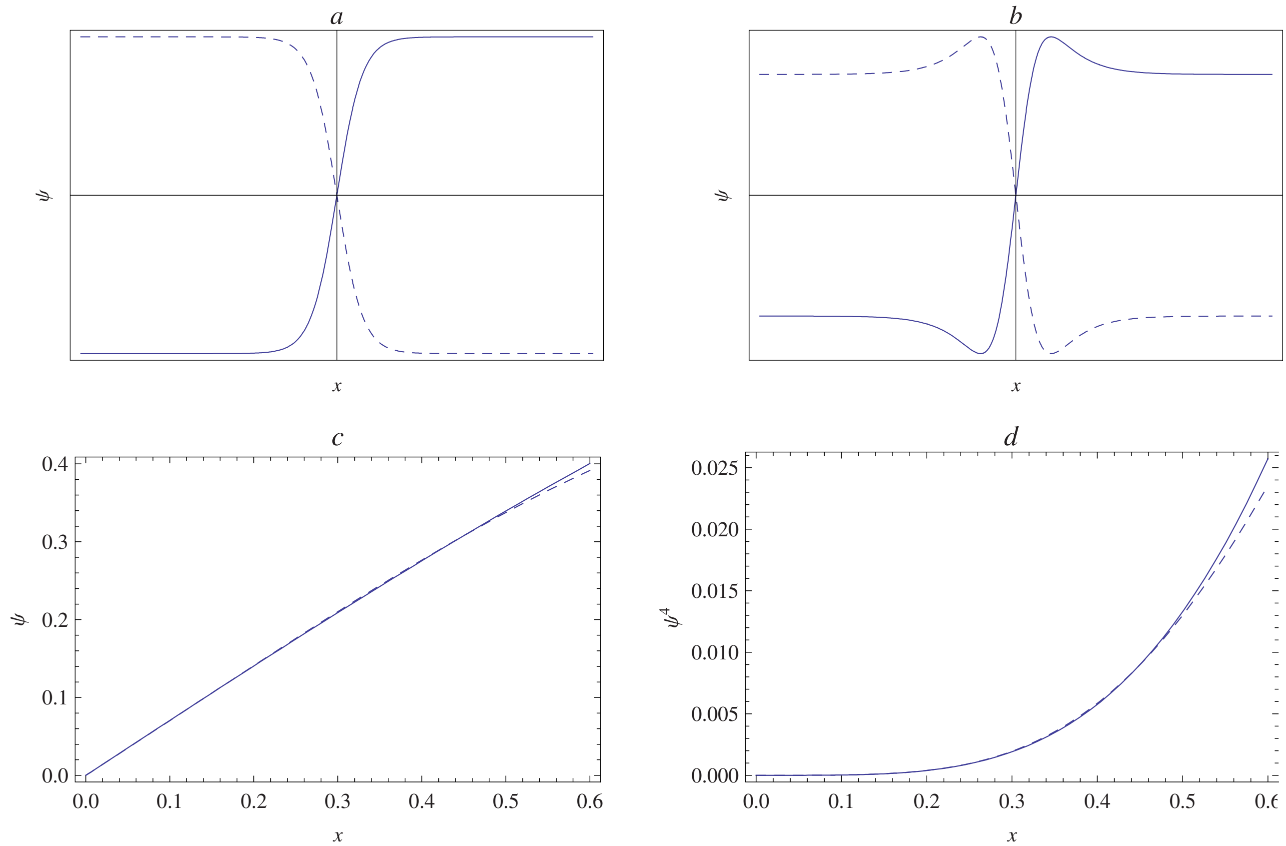

FIG. 2. (Color online) (a) Envelope solutions [Eq. (11)] of the form $\tanh(x/\sqrt{2})$ displaying the kink at the origin. These fit well with numerical results for small displacements around the kink. (b) When the displacements around the kink are large, sech type excitations appear near the origin. An analytical theory for this is not available. Comparison of the (c) field envelopes $\tanh x/\sqrt{2}$ (solid) and $\sqrt{2}[\text{sech}(x-1)-\text{sech}(1)]$ (dashed) and (d) their fourth power close to the origin and for positive x. These functions were discussed in Sec. I for the dark and bright solitons, respectively. The term sech(1) is included so the envelope vanishes at the origin. These figures suggest that the features of the dark soliton have a "bright soliton" component for the kink. This connection is explored in Sec. III.

Let us now draw up an analogy that will be useful for interpreting the optical dark soliton. Based on the results above for small displacements around the dip, the large set of particles excluding those few around $n=0$ can be viewed as a collection of oscillators with a natural frequency (i.e., ω_1) dictated by the harmonic constant K_2. We refer to this large ensemble as the *off-center particles*. The effect of the nonlinearity K_4 is to reduce this frequency by an amount proportional to the square of the mode amplitude [cf. Eq. (13)]. We will refer to the relatively few particles around and including the $n=0$ particle as the *center particles* (see Fig. 1). Now if we remove the center particles and simply replace them with a fresh set of off-center particles instead, we are effectively overestimating the energy of the system. This is because the amplitudes of the center particles are smaller than those of the off-center particles and, according to Eq. (5), the energy at $t=\pi/2\omega$ is $E=\frac{1}{2}m\omega^2\Sigma\,\xi_n^2$, that is, a sum of squares of amplitudes. If we imagine retaining the off-center particles in place of the center ones, while keeping constant the energy of the original configuration, we must subtract some energy as compensation for the overcount.

We carry over this analogy into our picture of the optical dark soliton and we now explicitly state this picture of the dark soliton as follows: the optical dark soliton can be considered as a composite object of (a) a large set of off-center particles (or *background* as it is currently called in the literature) extending *throughout* the entire range of the system and (b) a smaller set of particles, that is, a negative energy *notch* positioned over the center particles (negative so as to compensate for the overcount as noted in the previous paragraph). We emphasize once more that the notch particles are *not* the same as the center particles because, by definition, at the center, the background-particle energy minus the notch-particle energy gives the center-particle energy. (Of course if the background particles are bosons, then the notch particles are bosons as well. They simply differ in frequency, energy, position, and nearest-neighbor interaction.) Note that currently the term notch describes the kink at $x=0$, which is not the way we have defined that term here. (Since x and y are proportional we use them interchangeably.)

There is still another consequence of the above picture. Because the amplitudes of the notch particles are smaller than those of the background particles then in virtue of Eq. (13) the frequency of the notch particles must be larger than that of the background in order to effect a cancellation at $n=0$ (where the particle energy is exactly zero). That is, we must have $\Omega>0$ for the notch particles and this in turn implies that $K_4>0$. This latter result can be verified directly from Eq. (9) when the signs above for Ω and K_4 are used: the solution is now sech y instead of $\tanh(y/\sqrt{2})$, which had led to Eq. (11). The sech solution [cf. discussion around Eq. (1)], a localized solution centered at $y=0$, reminds us of the bright soliton mentioned in Sec. I, which can be shown to correspond to a system with an *attractive* point-particle interaction (instead of a repulsive interaction as in the case $K_4<0$) [13]. We will show later that the bright soliton has negative energy because it is a bound state [see Eq. (29) and discussion after it]. Hence the set of background particles around $y=0$ and the notch *together* serve to describe the kink at $y=0$.

We make two additional comments of the above picture. First, the particle-chain model we have been discussing corresponds to the *black* soliton, one whose notch has a point of zero intensity. However we can equally well apply this discussion to the case of the *gray* soliton for which the intensity does not drop all the way to zero (see Fig. 4). Second, our picture of the dark soliton distinguishes the background from the notch. To correctly represent the kink, the notch *and* the background have to be superimposed. The notch and background can be viewed as disparate objects. In the following sections, we will apply this picture as well as amplify it further for optical dark solitons (see Secs. III D, III E, and V).

Let us summarize the two important observations gathered in this section. First, two phenomena connected with two different length scales—from Eq. (11) these are $a|K_2/m\omega_1\Omega|^{1/2}$ and L, total length of the system—manifest themselves distinctly in the particle chain. They are not scattered or diffused but clear and distinguishable. Second, whereas the main feature of the system away from the kink is described by $\tanh x \cong \pm 1$, the envelope at the kink is approximately given by a sech function as shown in Fig. 2(c). A similar result can be written for $x<0$. [The fourth power is included in Fig. 2(d) because of the nonlinear interaction; see Eq. (14).] These functions are precisely the dark and bright soliton solutions discussed in Sec. I. They suggest that the two features are related to these envelope functions. In Sec. III B, we take advantage of these *qualitative* observations to develop a *quantitative* picture of the dark soliton. Thus the graphs are intended to motivate the study to be undertaken in Sec. III.

III. ONE-KINK DARK SOLITON

A. One-kink dark soliton

In this section we study the one-kink dark soliton formed by the internal field envelope of laser light propagating in a cavity filled with a nonlinear polarizable medium. In dimensionless units, the system is governed by the Lagrangian,

$$L(x,t) = \frac{i}{2}(\varphi^*\varphi_t - \varphi_t^*\varphi) - |\varphi_x|^2 + m|\varphi|^2 - \frac{1}{4}|\varphi|^4, \quad (14)$$

where subscripts denote derivatives with respect to time and space. In the context of the propagation of light in a cavity fitted with suitable highly reflective mirrors and with a bulk optical medium possessed with intensity-dependent refractive index, the last (nonlinear) term arises from a self-defocusing Kerr nonlinearity and the field φ denotes the complex scalar electric field. The same Lagrangian may also describe the propagation of nonlinear pulses in optical fibers with positive group-velocity dispersion. The quantity m is the detuning, that is, the difference in angular frequencies between the field and the natural cavity frequency. More detail about this system can be found elsewhere [12–15] (see also Ref. [4]). Here we will emphasize on the application of the semiclassical quantization method to this system as well as the two-soliton system in Sec. V. We assume that m is positive (we verify this later). The equation of motion takes the form

$$i\varphi_t + \varphi_{xx} + m\varphi - \tfrac{1}{2}|\varphi|^2\varphi = 0, \quad (15)$$

which reminds us of Eq. (9). The one-kink solution of Eq. (15) is

$$\varphi(x,t) = \sqrt{2m}\,\eta e^{im(1-\eta^2)t}\tanh\left(\sqrt{\frac{m}{2}}\eta(x-x_0)\right), \quad (16)$$

in which the constant $\eta>0$ is the amplitude and $x_0 \in (-\infty,\infty)$ is an arbitrary position and the kink boundary condition is $\varphi(-\infty,t)=-\varphi(\infty,t)$. The kink has the fundamental period $\tau(\eta)=2\pi/(m|1-\eta^2|)$. Next we compute explicitly the classical action $S=\int dtdxL(\Psi)$ for a large box of length L and over a period $T=\tau l$, $l=1,2,3,\ldots$,

$$\begin{aligned} S_{\mathrm{cl}} &= m^2\eta^4\int dtdx\Bigg[\tanh^2\left(\sqrt{\frac{m}{2}}\eta x\right) \\ &\quad - \tanh^2\left(\sqrt{\frac{m}{2}}\eta x\right)\mathrm{sech}^2\left(\sqrt{\frac{m}{2}}\eta x\right)\Bigg] \\ &= m^2\eta^4\int dtdx\Bigg[1 - 2\,\mathrm{sech}^2\left(\sqrt{\frac{m}{2}}\eta x\right) \\ &\quad + \mathrm{sech}^4\left(\sqrt{\frac{m}{2}}\eta x\right)\Bigg] \\ &= Tm^2\eta^4\left(L - \frac{8}{3|\eta|}\sqrt{\frac{2}{m}}\right), \end{aligned} \quad (17)$$

where, for convenience, we had set $x_0=0$.

We review briefly our tool of choice, the semiclassical quantization method [9,16,17]. As noted in Sec. I the original theory by DHN is for manifestly relativistic systems whereas we will be concerned here with a nonrelativistic system. Hence we will follow Nohl [16] more closely than DHN. For a generic system, the semiclassical procedure is employed to compute the trace of the propagator,

$$G(E) = \mathrm{tr}\frac{1}{H-E} = i\,\mathrm{tr}\int_0^\infty dTe^{iET/\hbar}\int_{\varphi_i=\varphi}^{\varphi_f=\varphi}[d\varphi]e^{iS/\hbar}, \quad (18)$$

in which H and E are the Hamiltonian and energy and the trace is taken over *all* field configurations $\varphi(x,t)$ with identical initial and final configurations. In the semiclassical method the functional integral is evaluated for paths close to the classically allowed orbits (retaining only terms quadratic in the deviation from the classical orbits) since these yield the dominant contribution. Because this expansion is per-formed on the exponent, the semiclassical method is not a perturbative expansion [9,16]. This evaluation of the integral by stationary phases effectively restricts us solely to all periodic orbits of the system. If τ is the fundamental period of the classical solution, then the time is given by $T=\tau l$, l being an integer as noted above. The stationary phase point is obtained through the condition

$$\frac{\partial}{\partial \eta}(E\tau + S) = 0. \tag{19}$$

Shifting by the solution of Eq. (19) yields a factor $e^{iS_{\text{cl}}}$, where S_{cl} is the action for classical paths. We may then write schematically in the stationary phase approximation,

$$G(E) \approx i \sum_{\substack{\text{classical}\\ \text{periodic}\\ \text{orbits}}} \int_0^\infty dT e^{iET} e^{iS_{\text{cl}}(T)} \int dq \left| \frac{\partial S(q',q'')}{\partial q' \, \partial q''} \right|^{1/2}_{q'=q''=q}, \tag{20}$$

where $\hbar$ is set to unity, and the last integral is due to the singular part of the action. When we gather all the results together we have

$$G(E) \propto i\left(\frac{2\pi i}{W''}\right) f(\eta) \sum_{l=1}^{\infty} e^{iWl}, \tag{21}$$

in which the exact form of $f(\eta)$ is not required except that it is independent of l. We have obtained a geometric series in which the sum is over l, the discrete index for the periodic orbits. W represents the expression for $(E\tau + S_{\text{cl}})$ at the stationary phase point and a prime denotes differentiation with respect to η. A factor $\hbar$ is implicit in W. (It can be verified for the particular system that W'' does not vanish at the stationary points.) The poles of $G(E)$ occur at

$$W = 2\pi n, \quad n = \text{integer}. \tag{22}$$

Equation (22), the principal goal of a semiclassical quantization calculation, is the generalization of the Bohr-Sommerfeld quantization into our field-theoretic system: for the Bohr-Sommerfeld case n characterizes the energy levels. (Even nondiscrete levels can be found by putting the system in a box with periodic boundary conditions and allowing the box size to tend to infinity.) As we will discuss in the next paragraph, for our case n refers to the number of particles, not energy levels.

To understand Eq. (22) and hence to go a step further in forming a more precise picture of what quantization implies in our context, let us examine what the stationary phase approximation means in nonrelativistic physics. Consider first a collection of nonrelativistic particles described by a set of N Cartesian coordinates $\{x_i\}$. We assume the presence of a potential $V(\{x_i\})$ whose expansion to second-order derivatives about the minimum at $\{\bar{x}_i\}$ is sufficiently accurate. The energy eigenstates correspond to the oscillator-ensemble result $E_{\{n_i\}} = V(\{\bar{x}_i\}) + \sum_{i=1}^{N}(n_i + \frac{1}{2})\hbar\omega_i$, where the frequencies are found by diagonalizing $(\partial^2 V/\partial x_i \partial x_j)_{\{\bar{x}\}}$. In field theory the field variables at each point $\phi_i(x)$ replace the x_i and the potential $V[\phi_i(x)]$ is a functional of the fields. If $V[\phi_i]$ has a local minimum at the static classical solution $\bar{\phi}_i(x)$, then we can expand the potential about $\bar{\phi}_i(x)$ and obtain a set of low lying energy levels. The first (lowest) level is conventionally interpreted as the "extended particle" and the rest are its excitations. When we apply this to the kink below [see Eq. (32) below] we will find energy eigenstates of the form corresponding to the oscillator ensemble for which this interpretation of Eq. (22) is natural, but instead of interpreting n as the *energy level* quantum number, we take n to represent the *number of particles* with some frequency ω. Thus the particles will all be assumed to lie at the ground state and n gives their number (for simplicity we ignore the zero-point energy). Moreover we will also discover that for the kink, another set of energies emerges which does not have the oscillator form. We find in this case that it corresponds to a system of bosons in their bound state. For this second case the eigenenergies are not excitations of the extended particle; n' (in general different from n) is the number of bosons of this bound state and there is only one bound state for each value of n'. Therefore for both cases describing the kink, n, n' represent the *number of particles in the ground state* of some minimum configuration. In Secs. III B and III C we work out the explicit details of this interpretation.

B. Application of DHN to the one-kink soliton

1. Basic approach

Except for our interpretation of the quantum number, the above procedure describes the standard approach for finding the energy levels of a one-dimensional system. Let us recall once more that in the semiclassical approximation, the quantum fluctuations are calculated over an underlying stable classical configuration. We will argue in the following that, as anticipated in Sec. II, two phenomena are encapsulated in the NLSE for the dark soliton: a cw stable background over which harmonic fluctuations are taking place and a topological kink somewhere in the background which interpolates between one side of the background and the other (with opposite phase to the former) (see also Ref. [15], where this is treated briefly). We will interpret the fluctuations over the kink as massive bosons with attractive point-particle interaction.

As Eq. (15) stands, it contains only one natural scale, namely, m. As a nonlinear equation, it really supports an infinite number of solutions [24] and moreover, these solutions possess an amplitude dependence, a fact that is exploited precisely in the Kerr effect. Thus, another scale becomes available once a particular solution with given amplitude is chosen [see, e.g., Eq. (17)]. But these dynamical scales must now be augmented by a new feature characteristic of a nonlinear differential equation, namely, that the solution may also be a topological object.

The kink is, in fact, a localized phenomenon with topological character. The locality aspect is borne out in the manner by which it is created in the laboratory: for instance, dark optical solitons can be created using grids and stripes [25] while BEC dark solitons by phase imprinting [26]. Theoretically this fact of locality was already employed extensively

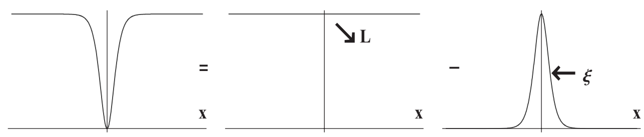

FIG. 3. The absolute square of one-kink soliton envelope φ can be thought of as two terms with different length scales, as shown above (see text for discussion).

by Callan and Coleman [27] in their treatment of the double well via a "dilute gas approximation applied to an instanton gas." We can see this for the one-kink solution by focusing on its Lagrangian which has the generic shape shown in Fig. 3. Except at the region close to the kink, it varies very slowly (over the scale of the total length, L) so the fluctuations are small. The semiclassical theory as described in Sec. III A predicts that a system described by a static and stable configuration (such as the flat line in Fig. 3) corresponds to an ensemble of harmonic oscillators ([7], especially p. 132ff). Clearly in this case the quartic term may be ignored and the Lagrangian is indeed that of an oscillator field. For the kink, however, the field changes drastically over a scale $\xi = \sqrt{2/m\eta^2}$ dictated by m and the amplitude η. Although the quadratic and quartic terms operate together, it is the latter that is dominant. Under this circumstance, it is safe to ignore the quadratic term so the theory reduces effectively to a ϕ^4 theory whose nonrelativistic version is a system of massive particles undergoing point interaction. We will see in Sec. III C that this interaction becomes, unexpectedly, attractive instead of remaining repulsive. This means that at the kink the repulsive character of the interaction undergoes a local change operating only in the region of the kink. It will then not surprise us that the field theory corresponding to this situation is exactly that of massive bosons with attractive δ interaction. We conclude then that the one-kink (and later, the two-kink) dark soliton displays two distinct features: a global harmonic oscillator-type feature (for the background) and a system of bosons with δ interaction around the kink(s) (for the notch). These features are not at all evident by inspection of the original NLSE but they are necessary elements for analyzing the dark soliton. To summarize, the NLSE along with the solution φ provided us with two dynamical scales. Then deeper scrutiny of the topological character of the dark soliton implied that a kink exists in which a local dynamical change must also occur. It is not possible for us to suppose that the system can only remain with the oscillator-type fluctuations because the solution has to undergo a phase change at some place and this is the kink, or more precisely, the topological aspect of the dark soliton. The transition from a repulsive to an attractive and then back to repulsive interaction is another way of understanding the locality of the dark soliton kink.

Because the locality of the kink is seen only in the Lagrangian $L(\varphi)$ computed for a specific solution φ, the separation of the dynamics in terms of two subsystems cannot be made prior to having $L(\varphi)$ (as there are an infinite number of solutions there would be also an infinite number of Lagrangians). A putative separation can be effected by noticing terms in $L(\varphi)$ that are governed by different dynamical scales [cf. Eq. (17)]. In the case of the one-kink and two-kink solutions this separation will turn out to be unambiguous and unique. Once the separation has been made, the quantization procedure must now verify that the interpretation in terms of the two types of fluctuations can be made. (For the one-kink dark soliton this is carried out in Sec. III C.) In the next paragraph we explain how this separation is effected for the one-kink dark soliton. We will call this the *dynamical separation* because it is clearly rooted in the recognition of distinct dynamical phenomena in the system. Then in the succeeding paragraph we will show that the separated subsystems can be quantized separately and independently. This will be termed the *ideal-gas approximation*, because, like a mixture of ideal gases, the two subsystems interact negligibly and may thus be treated separately.

2. Dynamical separation

For the dark soliton let us then note from Eq. (17) that the Lagrangian consists of two terms, denoted here as $L_{\rm cl}^{(n)}(\eta,\xi)$ and $L_{\rm cl}^{(b)}(\eta,L)$. The first is the contribution from the notch which is characterized by a length scale $\xi=\sqrt{2/m\eta^2}$ [i.e., this is due to the sech terms in Eq. (17)] and the second is the contribution from the background [unity in Eq. (17)] whose length scale is the size of the system, that is, L [see also Eq. (31) below]. We explained at the end of Sec. II that the particle-chain model manifests dynamical behavior near the kink which differs markedly from the rest of the chain, a situation analogous to our present case. We augment the discussion in Sec. III B 1 with insights from the chain model. From Eq. (11), the origin of the length scale ξ is the ratio of the force constants $(K_2/K_4)^{1/2}$ which is an indication of the interplay between the harmonic and nonlinear dynamics as the displacements grow from zero and where the nonlinear force dominates or at least is important. On the other hand, physically we can trace the background spectrum to a previous observation: far away from the kink when the displacements satisfy $|\psi|^2 \cong \text{const}$, i.e., $|\psi_0|^2 \ll 1$, then Eq. (7) reduces to the harmonic equation with the harmonic force modified very slightly by the term $\frac{3}{4}K_4|\psi_0|^2$, i.e., the effective force is a linear combination of the harmonic and nonlinear forces with the harmonic force dominant. [This picture is in line with Sec. III B 1 and will be borne out in Eq. (32)]. In fact, $(K_2/K_4)^{1/2}$ and a linear combination of the K's are the only physically relevant scales of the theory.

To be consistent with the above we may also identify energy with the two underlying phenomena by writing $E_b = \int_0^L dx(dE_b/dx)$ and $E_n = \int_0^L dx(dE_n/dx)$ for the energies associated with the background and the notch, respectively. This means that, for example, $dE_{(b)}/dx$ is an energy density associated with the background. Naturally these densities share the same scales as $L_{\rm cl}^{(n)}(\eta,\xi)$ and $L_{\rm cl}^{(b)}(\eta,L)$ because the energy is an extensive quantity. Hence the identification of the energies is exact. We may then cast Eq. (19) as

$$\frac{\partial}{\partial\eta}(E_b\tau + S_{\rm cl}^{(b)}) + \frac{\partial}{\partial\eta}(E_n\tau + S_{\rm cl}^{(n)}) = 0, \qquad (23)$$

where $S_{\rm cl}^{(b)}(\eta)$ and $S_{\rm cl}^{(b)}(\eta)$ are the actions obtained from $L_{\rm cl}^{(n)}(\eta,\xi)$ and $L_{\rm cl}^{(b)}(\eta,L)$. (Since the actions have been integrated over space, the scales do not appear explicitly.) In the semiclassical approximation η is an amplitude which takes quantized values. However there is nothing to exclude the possibility that this amplitude takes more than one quantized value in the whole span of the system. Such as generalized outcome would, presumably, require extra formal development of the semiclassical approximation. In view of the developments leading to this discussion it is reasonable to take instead a more direct *phenomenological* approach and assume that two quantized values of the amplitude are appropriate for the background and the notch. Then we may replace Eq. (23) by

$$\frac{\partial}{\partial\eta}(E_b\tau+S_{\rm cl}^{(b)}) = -\frac{\partial}{\partial\eta}(E_n\tau+S_{\rm cl}^{(n)}) = K,$$

where K represents two numbers, in general, not zero. But now by a suitable redefinition of the energies $E_{(b)}$ and $E_{(n)}$ we may, however, replace K with zero. We justify this step in Sec. III B 3.

3. Ideal-gas approximation

In what we refer to as the ideal-gas approximation we now replace Eq. (23) by two separate equations,

$$\frac{\partial}{\partial\eta}(E_b\tau+S_{\rm cl}^{(b)}) = 0, \quad \frac{\partial}{\partial\eta}(E_n\tau+S_{\rm cl}^{(n)}) = 0. \tag{24}$$

Several observations already point to this separation. The discussions of Secs. II and III B 1 indicate that the amplitudes of the fluctuations about the notch are much larger than those of the background. This is also implied by the different phenomena operating in the background and kink and encapsulated in the scales identified above. The fact that the kink is a local object allows us to consider it as having an entity distinct from the background. Because the kink width is inversely proportional to the amplitude, a more intense background implies also a more concentrated kink. By analogy with the ideal gas we can assume that interactions between the background and kink are small and the above separation follows. We neglect interactions altogether. Equation (24) expresses the Bohr-Sommerfeld quantization condition for the background and notch, respectively.

The semiclassical approximation itself can be invoked to place the redefinition of energies on a firmer basis as follows. Each equation in Eq. (24) gives a quantization condition [cf. Eq. (22)], namely,

$$W_{(b)} = 2\pi n_1, \quad W_{(n)} = 2\pi n_2, \tag{25}$$

where $n_{1,2}$ are integers. The procedure of DHN stipulates that we enumerate all the periodic orbits which means all possible $n_{1,2}$, that is, all the integers. Hence the absorption of constants into the $E_{(b),(n)}$ introduces no change in this enumeration of values. (Even if the redefinition of energy gave a noninteger $n_{1,2}$, we only need take the integer value because, in general, the $n_{1,2}$ are large and the error incurred in simply ignoring the fractional part is small.) The price we have to pay for this boon is that we do not know *a priori* which values of n_1 go with which values of n_2. By invoking particle conservation and initial conditions one is able to circumvent this difficulty. While the redefinition argument works well for obtaining the energy levels, it would lead to an overcounting of states if we were to use it to evaluate the path integral. Fortunately we will not do this here. As an aside, we observe that among the examples worked out by DHN [9], they considered *separable* systems for which the action could be separated into two distinct parts unambiguously; although our discussion above is not strictly of this type, we might also refer to it as a *quasiseparable system*. We have thus given an explicit unambiguous albeit phenomenological procedure for quantizing the dark soliton semiclassically provided the interaction between the kink and background is neglected.

We close this section with a word on interactions. The most relevant interaction results from the depletion of the notch and can be included in the NLSE by appending a term of the form $i\Gamma(\varphi-\varphi_d)$ (where φ_d is the driving amplitude) in Eq. (15) [11]. Here the cavity decay rate Γ is proportional to the transmission coefficient of the cavity mirrors and, in principle, can be made very small.

C. Quantization

We now carry out the quantization initially for the second term of Eq. (17) which is the action attributed to the notch. Then we do a similar and independent calculation for the first term but with the *opposite* sign because we assume that this background contribution "balances" that coming from the L-dependent term of $S_{\rm cl}$. We also obtain expressions for the energy in terms of the quantum number. We can infer in a handwaving way from Eq. (15) that the kink corresponds to an attractive interaction: given tanh solution (16), we split $|\phi|^2$ into 1 and $-\text{sech}^2$ (Fig. 3). Assigning the unit term to the background and concentrating only on the kink, hence $-\text{sech}^2$, we see that Eq. (15) becomes a NLSE with the sign of the nonlinear term reversed, that is, an attractive interaction (see also Sec. III C).

The contribution from the notch is then

$$G^{(\rm notch)}(E) \cong i\sum_{l=1}^{\infty}\int d\eta\int l d\tau\int d^2q\left|\frac{\partial^2 S}{\partial q'\,\partial q''}\right|^{1/2}_{q=q'=q''} \times e^{iE\tau l}e^{-i\tau l(8m^2\eta^3/3)\sqrt{2/m}}\left|\frac{\partial\tau}{\partial\eta}\right|\delta\!\left(\tau-\frac{2\pi}{m|1-\eta^2|}\right). \tag{26}$$

The stationary phase approximation requires that we locate the stationary point, i.e., $\frac{\partial}{\partial\eta}(E-8\sqrt{2}m^{3/2}\eta^3/3)\ \tau(\eta)=0$. As with Eq. (21), our final result for $G^{(\rm notch)}(E)$ can be cast as a geometric series (a prime denotes $\partial/\partial\eta$),

$$G^{(\rm notch)}(E) \propto i\left(\frac{2\pi i}{\tilde{W}''}\right)^{1/2} J(\cdot)\sum_l e^{-i8\pi m\eta\sqrt{2/m}l}, \tag{27}$$

where $J(\cdot)$ is independent of l and whose detailed form is not required, $\tilde{W}$ is the coefficient of l in the exponent of $G(E)$, and the quantities had been evaluated at the stationary point. (It can be verified that $\tilde{W}''=8\sqrt{2}m^{3/2}\eta[\frac{1}{\pi}\eta^2 m\tau(\tau-1)-1]$ does not vanish at the stationary point.) The poles of $G^{(\rm notch)}(E)$ occur when

$$4\sqrt{2m}\,\eta = n, \tag{28}$$

where n is an integer. Note the dependence of n on the length scale ξ, as noted in Eq. (23). From the condition on the stationary point we also obtain the notch's bound energies,

$$E^{(\mathrm{notch})} = 4\sqrt{2}m^{3/2}\left(\eta - \frac{\eta^3}{3}\right). \tag{29}$$

The detuning m is unspecified. Following Nohl [16] we choose it such that $E^{(\mathrm{notch})}$ has the characteristic $E \propto -\mathrm{const}(n^3 - n)$ form of a system of n bosons interacting via an attractive delta potential [18,28]. Thus choosing $m=1/96$ and invoking Eq. (28), the above reduces to $E^{(\mathrm{notch})}=(1/96)(n-n^3)$, which is the bound energy of an n-particle bound state of bosons with an *attractive* delta function potential [16,17]. Hence n represents the number of notch particles. (We have also verified that m is positive.) This discussion may now be connected with the notch particles introduced in Sec. II. In fact, the n-boson bound system is known to be just the bright soliton with attractive interaction [13]. The choice of m does not imply that the detuning is locked to the driving frequency; it simply means that the above choice is what is compatible with the quantization condition. Except for $n=1$, $E^{(\mathrm{notch})}<0$. If the center-of-mass motion is also considered ($v \neq 0$), one finds the bound state acquiring additionally the correct free-particle de Broglie momenta k in a box of length L [12]. It can be shown that the approximate n-particle bound state wave function has a sech form and this can be related once again to the discussion following Eq. (1) [15].

A parallel and independent semiclassical calculation for the background (b) now gives us the L-dependent result

$$G^{(\mathrm{background})}(E) \propto i\sum_l e^{-i4\pi m\eta^2 Ll} \tag{30}$$

whose poles occur when

$$\eta^2 = \frac{r}{2mL}, \quad r = 1,2,3,\ldots, \tag{31}$$

with the corresponding bound energies given by

$$E^{(b)} = mr - \frac{r^2}{4L}. \tag{32}$$

Clearly η^2 here depends on the length scale L but not on ξ. In general $\xi \ll L$. (Thus, as noted already, we have verified explicitly that the notch and background are characterized by two distinct length scales.) If we refer to $m-r/4L$ as the background frequency then the small shift $r/4L$ must be associated with Ω in the particle-chain model of Sec. II and according to Eq. (31) it is proportional to the square of the amplitude as also noted in Sec. II [Eq. (13)]. We interpret r as the number of background particles with the frequency $m-r/4L$ (again we ignore the zero-point energy). Thus Eq. (32) gives the ground energy of a collection of r harmonic oscillators with a small nonlinear repulsive anharmonicity due to $-r/4L$ (as in the chain model of Sec. II). By analogy with elastic phonons which are oscillators, we refer to the background particles as phonons (which do not undergo excitations above the ground level). We do not know which n [cf. Eq. (28)] corresponds to which r [cf. Eq. (31)]. This is the price we pay for separately quantizing the notch and background. However they can be related by the fact that, for instance for black solitons, the notch completely cancels the background intensity at one point. For both notch and background the quantum number obtained is interpreted as the number of particles in the ground state of a stable minimum of the system.

D. Notch particles and topological nature of the dark soliton

We take another look at the notch. It is reasonable to assume that the notch particles originate from the particles comprising the background. If we imagine the notch to be at the middle of the background and try to draw the displacement pattern as shown in Fig. 1 we find that there is always a shift by one step as one crosses $x=0$. Figure 1 shows one possibility in which the tanh envelope connects equally spaced particles. By focusing on the particle types one sees that at $x=0$ a jump in the up–down sequence has occurred. Because of the phase relation [Eq. (6)], this implies that one side of the background (right of the notch, say) must have an extra phase of π relative to the other side. But also by virtue of the staggered displacement field (6) this is really a global phase requirement (not merely a local one) so the extra phase is truly a topological quantity. This then is the origin of the topological character of the dark soliton (see also [23]). In Sec. V we will extend this to the case of two dark solitons and to the conservation of topological charge.

E. Dark soliton notch as a superposition phenomenon

Although we are studying a truly nonlinear phenomenon we can consider the point of minimum intensity in the dark soliton as the *linear superposition* of the notch and background envelopes. This point was first mentioned at the end of Sec. II. To see this we return to Eq. (16) and observe that the field envelope $\varphi(x,t)$ is a tanh function which, outside the notch, is either very nearly $+1$ or very nearly -1. From studies of the NLSE for attractive interactions [i.e., Eq. (15) but with a positive sign for the last term since the notch corresponds to an attractive nonlinearity interaction], the field envelope is $\tilde{\phi}(x,t)=2\sqrt{m}\,\eta e^{im(1+\eta^2)t}\,\mathrm{sech}(\sqrt{m}\,\eta x)$. By properly choosing the background and notch quantum numbers (or numbers of particles) n and r we can make the amplitudes of φ and $\tilde{\varphi}$ equal. Since $1-\tanh\chi=\mathrm{sech}\,\chi e^{-\chi}$ (where the argument χ is proportional to distance from the origin), we find that very close to the origin, where χ is almost zero, the background (represented by unity) superimposed with the notch (represented by the sech function) yields the kink (represented by the tanh function). Of course this no longer holds as we move away from the origin [see also Figs. 2(c) and 2(d)].

An objection could arise about our argument above: if the zero intensity is an interference phenomenon would this not destroy the topological character of the dark soliton since the parts to the right and left of this zero-intensity point differ by a phase of π? To answer this objection we observe that the

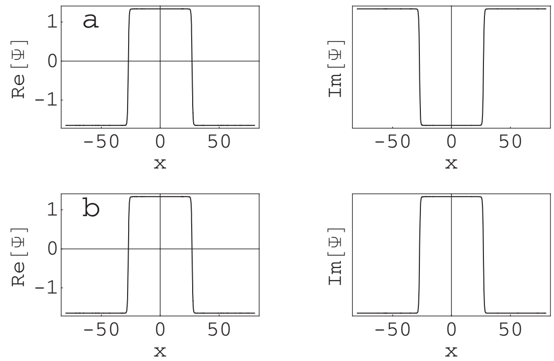

FIG. 4. Plots of Re(Ψ) and Im(Ψ) versus x for $q=3$, (a) $t=9$ and (b) $t=-9$. Comparing (a) and (b) shows a phase change due to the collision event at $t=0$.

interference phenomenon is a local one whereas the topological nature of the soliton is global: it requires that the phases at $\pm\infty$ (i.e., very far from the notch) differ by π. Exactly where the transition takes place is a local issue which needs not to destroy the global character of the soliton. It is clear, however, that the interference has to occur at some point. A similar argument (regarding Kramer's theorem) is given in [23].

We can see that the analogy drawn up at the end of Sec. II for the particle-chain model is qualitatively consistent with what has just been explained here. Thus there is a connection between the kink of the particle-chain model and the intensity minimum of the dark soliton: the minimum intensity is due to the superposition of the notch and background envelopes.

IV. TWO-KINK DARK SOLITONS

We extend our study to a pair of dark solitons by introducing the Lagrangian,

$$L(\Psi) = \frac{i}{2}(\Psi_t^*\Psi - \Psi^*\Psi_t) - |\Psi_x|^2 - |\Psi|^4 + m|\Psi|^2, \quad (33)$$

in which subscripts denote derivatives with respect to time and space. This Lagrangian is the same as Eq. (14) with numerical changes in some coefficients so that our results will conform with the literature on the two-kink soliton. The NLSE is

$$i\Psi_t - \Psi_{xx} + 2|\Psi|^2\Psi - m\Psi = 0. \quad (34)$$

General solutions of Eq. (34) had been studied previously [29]; in this section and in Sec. V we will only be interested in a particular solution describing the approach, collision, and subsequent propagation of two dark solitons. Hence we consider the following particular solution [29]:

$$\Psi(x,t) = \frac{q}{\sqrt{2}}\frac{-\cosh qx + i\sqrt{2}\sinh q^2 t}{\sqrt{2}\cosh q^2 t + \cosh qx}e^{i(q^2-m)t}, \quad (35)$$

where q is arbitrary, its square being proportional to the maximum intensity of the field (see Fig. 4). By analogy with the one-kink soliton, this maximum intensity field corresponds to the background. The minimum intensity turns out to be half of the maximum. Solution (35) is symmetric about $x=0$ and describes two dark solitons each with a propagation velocity $q>0$ and passing each other in the vicinity of the origin at the time $t=0$ in an elastic collision event. If we ignore the overall phase from the exponential in Eq. (35), we can fix the time and examine the phase of $\Psi(x,t)$ as we move from $x=-\infty$ to $x=+\infty$. If the phase at $x=-\infty$ is adjusted so it is at $-\pi$, we find that it increases toward positive phase (in the first quadrant) and attains its highest positive value when one is halfway between the dark solitons; then it decreases along the same path back to $-\pi$ as we move to $x=+\infty$. Referring to Fig. 4 we see a phase change as a result of the collision at $t=0$. The phase of the background outside the kinks is constant. After the collision, each soliton is shifted by the amount $q^{-1}\ln 2$ relative to the place where it would have been if no collision had occurred [see also Eq. (55)]. Taking advantage of the symmetry about $x=0$, we have drawn the square of the mod$|\Psi|^2$ in Fig. 5 for a typical value $q=3$ as a function of time at fixed position and as a function of position at fixed time. The plots show a two-soliton collision event occurring close to the origin at $t=0$ and subsequent (positive time) propagation of the dark solitons.

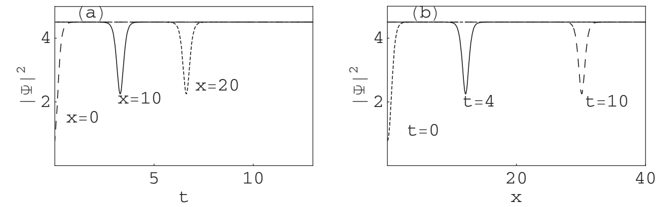

FIG. 5. Plots of $|\Psi|^2$ for $q=3$ (a) as functions of time for $x=0$ (dashed), $x=10$ (full), and $x=20$ (dotted) and (b) as functions of space for $t=0$ (dotted), $t=4$ (full), and $t=10$ (dashed). A collision takes place around $t=0$, $x=0$. Reflection of the graphs about $x=0$ yields plots for negative x values. Thus the second soliton is a mirror reflection of the soliton shown above and is moving to the left. Units for the vertical axis are arbitrary.

We next evaluate the action for particular solution (35) in parallel with Eq. (17). Figure 6 displays the typical situation for the various terms of the Lagrangian. When carrying out the integration over space we will separate the terms extend-

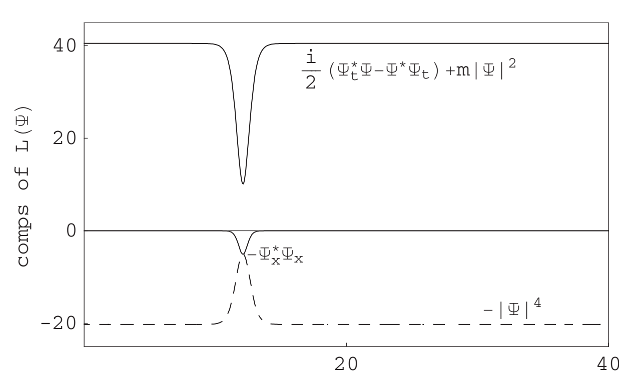

FIG. 6. Plots of components of the Lagrangian for $q=3$, $t=4$. Reflection about $x=0$ gives the result for the other soliton. Units for vertical axis are arbitrary.

ing throughout the entire range of integration from those that quickly decay to zero. As in Sec. III, we associate the former with the background and the latter with the notch. We display below the various terms of the action. As with the one-kink case, a careful examination shows that $S_{\rm cl}$ is even in q. (In Sec. III, this fact was not mentioned since the equations were transparent. This is no longer the case here.) There are the contribution from the kinetic energy [30],

$$S^1_{\rm cl} \equiv \int dtdx\left\{\frac{i}{2}(\Psi^*_t\Psi - \Psi^*\Psi_t) + m|\Psi|^2\right\} = \int dt\{q^4L - 2(\beta^2-1)|q|^3[-3\sqrt{2}Q^1_1(\beta)\cosh(q^2t) + 2Q_1(\beta)]\}, \tag{36}$$

the contribution from the space derivative part,

$$S^2_{\rm cl} \equiv -\int dtdx|\Psi_x|^2 = -\int dt\left\{|q|^3(\beta^2-1)^2\cosh(2q^2t)\left[\frac{1}{6}Q^2_3(\beta) - Q_3(\beta)\right]\right\}, \tag{37}$$

and the contribution from the nonlinear interaction,

$$S^3_{\rm cl} \equiv -\int dtdx|\Psi|^4 = -\int dt\left\{\frac{q^4}{4}L - 2|q|^3(\beta^2-1)\left[Q_1(\beta) - \sqrt{2}\cosh(q^2t)Q^1_1(\beta) - (\beta^2-1)\left\{[1+\cosh^2(q^2t)]Q_3(\beta) - \frac{2\sqrt{2}}{3}\cosh(q^2t)Q^1_3(\beta) + \frac{1}{6}\cosh^2(q^2t)Q^2_3(\beta)\right\}\right]\right\}. \tag{38}$$

In writing these, we had introduced the parameter β,

$$\frac{\beta}{(\beta^2-1)^{1/2}} \equiv \sqrt{2}\cosh q^2t, \tag{39}$$

and $Q_\nu(z)$ and $Q^\mu_\nu(z)$, which are the Legendre and associated Legendre functions of the second kind, respectively,

$$Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}F\left(\frac{\nu+2}{2},\frac{\nu+1}{2},\frac{2\nu+3}{2};\frac{1}{z^2}\right), \tag{40}$$

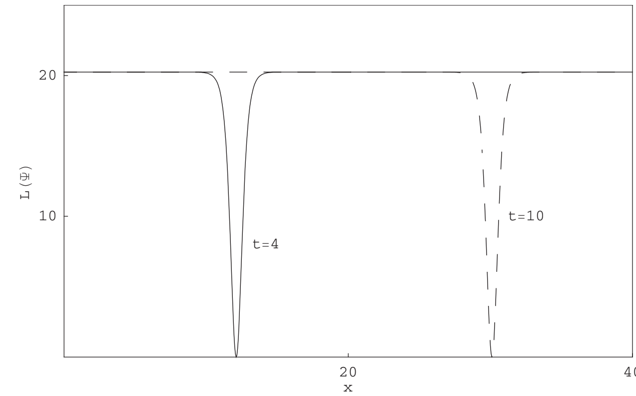

FIG. 7. Plot of the Lagrangian versus x for $q=3$ and $t=4$ (full) and $t=10$ (dashed). Vertical units are arbitrary. This shows only the result for the soliton propagating to the right.

$$Q^\mu_\nu(z) = e^{i\mu\pi}\frac{\Gamma(\nu+\mu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}(z^2-1)^{\mu/2}z^{-\nu-\mu-1} \times F\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2},\frac{2\nu+3}{2};\frac{1}{z^2}\right). \tag{41}$$

As in Sec. III, L represents the large but finite size of the system. The background contribution (i.e., the L-dependent term) is thus a constant throughout the system. We confine our subsequent calculation to the case of large (positive as well as negative) times and large values of q. Then $\beta \cong 1 + e^{-2q^2|t|}$ and the final result is simply

$$S_{\rm cl} \cong \left(\tfrac{1}{4}q^4L - \tfrac{10}{3}|q|^3 + (O|q|)\right)T, \tag{42}$$

where T is the total time. The above result is strikingly similar to Eq. (17). Note that the integration over time excludes the small time interval when the collision is taking place, i.e., when $2q^2|t| \lesssim 1$.

The contribution of the background to the action is constant over space and time (if we exclude the short time interval representing the collision of the solitons) and has a characteristic quartic dependence of q. The contribution due to the notch has a negative sign relative to the background contribution, a cubic dependence on the absolute value of q, and decays exponentially at the region of the notch as can be seen in Figs. 6 and 7. One can trace the numerical factor for the notch to the length scale q^{-1}. Provided we stay outside the relatively insignificant collision zone, the contribution from the notch is almost time independent. These characteristics are general and are not dependent on the number of dark solitons and we already saw these same characteristics in the one-kink case (see also [15]). The observations given at the end of Sec. II have now been verified.

Working within the same large q and t limit, the total classical energy of the system can be written as

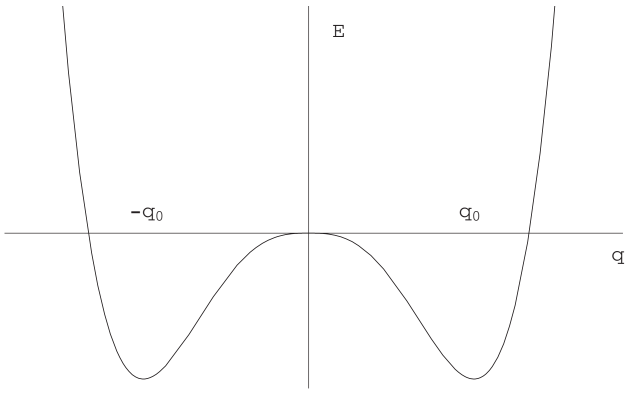

FIG. 8. Total energy of the two-kink dark soliton system. Two stable minima at $q=\pm q_0$ correspond to doubly degenerate ground states associated with these values of q in $\Psi(x,t)$ [see Eq. (35)].

$$E=\tfrac{1}{4}q^4L-\tfrac{8}{3}|q|^3, \tag{43}$$

which has two minima $q=\pm q_0$ corresponding to doubly degenerate ground states, while $q=0$ is a local (unstable) maximum (see Fig. 8). The twofold degeneracy of the ground state implies that topological excitations exist and these act as boundaries between domains having different ground states [31]. We will discuss this further in Sec. V.

V. QUANTIZATION OF THE TWO-KINK SOLITON

The quantization of the two-soliton system now proceeds as in Sec. III C since all the preliminary results are available. We consider the notch (represented by the superscript n) first. The period of the system is $\tau=2\pi/|q^2-m|$ and if we first take the second term of Eq. (42), the stationary point is found from (we assume positive q)

$$\frac{\partial}{\partial q}\left(E^{(n)}-\frac{5}{3}q^3\right)\frac{2\pi}{|q^2-m|}=0. \tag{44}$$

The reason why only half of the full contribution is used here is that there are two dark solitons sharing the full notch action. The quantization condition turns out to be

$$q=\frac{2n}{5}, \tag{45}$$

n being an integer. Following the argument after Eq. (29), we choose for the detuning $m=4/75$ and find that the bound state energies for each dark soliton notch are

$$E^{(n)}=\tfrac{4}{75}n(1-n^2). \tag{46}$$

As in Sec. III n represents the number of notch particles. Once again we have a notch which is an n-particle bound state of bosons with attractive delta interaction. There are, of course, $2n$ notch particles for the two dark solitons. Similarly the stationary point for the background (represented by the superscript b) is obtained from the L-dependent term

$$\frac{\partial}{\partial q}\left(E^{(b)}-\frac{q^4}{4}L\right)\frac{2\pi}{|q^2-m|}=0 \tag{47}$$

from which we find $q^2=2r/L$, r=integer, and the background energies are

$$E^{(b)}=mr-\frac{r^2}{L}. \tag{48}$$

These results parallel Eqs. (28), (29), (31), and (32) and the remarks on them apply in the present case as well. As with the one-kink case we refer to the background as phonons.

Along with the motivation given in Sec. III B, the fact that the two notches of this system are quantized separately can also be justified since from Fig. 7, for instance, we observe that these notches are widely separated (on either side of the origin) and are moving in opposite directions. Moreover since these notches propagate over a background that remains stable (almost featureless except close to the notches) we conclude that the background also has a separate identity of its own. Of course, this conclusion could be questionable during the collision of the solitons but it is clear that the effect of this collision on the *full* action is not significant. Thus the separation of the action into the background and notch terms as in Eq. (25) is justified over the much longer period of the soliton.

We had seen in Sec. III D that the topological nature of the dark soliton is due to the fact that the kink introduces a phase change in π. This remains true for two and more dark solitons. Clearly two notches will give rise to a total phase of 2π. This is exactly what we see in Fig. 4: Ψ has the same phase for large positive and negative x. If we "split" Fig. 4 at the origin we have two one-kink dark solitons each with their topological phase. They are referred to as kink and antikink [7]. Note that dark soliton (35) is *complex* as opposed to the *real* tanh function generally used to describe a one-kink soliton. Therefore the actual structure of the two-kink is not as simple as that of a one-kink soliton. A glimpse of that complexity will be seen in Sec. VI. This simply illustrates the point we made earlier: *distinguishing* the components of the dark soliton does not necessarily allow us to separate them. If the total phase *between* $x=-\infty$ and $x=\infty$ is observed we find that it does not change on account of a collision event; this is just the conservation of topological charge despite collisions [7].

The semiclassical analysis given above can now be complemented by appealing to the Bogoliubov–de Gennes (BG) equations which we derive from Eq. (34). In matrix form these equations are [15,32]

$$i\frac{\partial}{\partial t}\begin{pmatrix}\delta\varphi\\ \delta\varphi^*\end{pmatrix}=\begin{pmatrix}\mathcal{L}-2|\Psi|^2 & -2\Psi^2\\ 2\Psi^{*2} & -\mathcal{L}+2|\Psi|^2\end{pmatrix}\begin{pmatrix}\delta\varphi\\ \delta\varphi^*\end{pmatrix}, \tag{49}$$

in which $\mathcal{L}=\partial^2/\partial x^2+m-2|\Psi|^2$ and Ψ is given by Eq. (35). The 2×2 matrix operator on the right-hand side is non-Hermitian. We verify that there are two zero modes of this system, namely,

$$\begin{pmatrix}u_\theta\\ v_\theta\end{pmatrix}\propto -i\frac{\partial}{\partial\theta}\begin{pmatrix}\Psi(x,t)e^{i\theta}\\ \Psi^*(x,t)e^{-i\theta}\end{pmatrix},$$

$$\xi_0 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix} \propto \frac{\partial}{\partial x}\begin{pmatrix} \Psi(x,t)e^{i\theta} \\ \Psi^*(x,t)e^{-i\theta} \end{pmatrix}. \tag{50}$$

The first zero mode (u_θ, v_θ) is due to the breaking of the U(1) gauge symmetry $\Psi e^{i\theta} \to \Psi e^{i(\theta+\varepsilon)}$, whereas the second zero mode $\xi_0(x)$ is from the breaking of the translational symmetry, $x \to x+\varepsilon$. This latter mode is localized about the positions of the two notches. Although we are unable to provide explicit forms for the continuum (nonzero energy) solutions of Eq. (49) we can verify that if a pair (a,b) is a solution for positive energy, then (b^*, a^*) is a solution for negative energy. In this instance the sign of the energy can be associated with the direction of propagation (to the right or to the left). Thus the solutions are time-reversal symmetric. If the solutions are normalized, it is clear that this remains unchanged for given positive and negatives times; however the phase relation between the components a and b changes. Thus the absolute square of the wave function for given large positive and large negative times should be the same. Stated differently, the soliton does not scatter phonons but shifts their phase and so the soliton belongs to a class of potentials that enjoy the property of being reflectionless, i.e., complete transmission. We do not have explicit expressions of the solutions for the two-kink case, but the one-kink case is discussed in detail along with explicit solutions of the BG equations in Refs. [15,32] and a general discussion is given in Snyder *et al.* [5]. By analogy with Ref. [15]. these continuum solutions correspond to the background excitations while the zero mode $\xi_0(x)$ relates to the notch.

We can quantify the effect of the solitons on the continuum solutions of the BG equations by reverting to the particle-chain model of Sec. II [33]. The completeness of the suitably normalized eigenstates $\xi_\nu(x)$ (ν is a state label) at each site implies that the energy integral of the local particle density $\rho_{nn}(E)$ at any site n is unity. Formally we can define $\rho_{nn}(E)$ as

$$\rho_{nn}(E) = \sum_\nu |\xi_\nu(n)|^2 \delta(E - E_\nu), \quad \rho(E) = \sum_{n=1}^{N} \rho_{nn}(E). \tag{51}$$

Moreover by virtue of our interpretation above, $\rho_{nn}(E) = \rho_{nn}(-E)$. We can understand this relation in the following way. Both the notch and background have continuum states because of the large size of L. This is clear for the background as noted in the previous paragraph and for the notch in remarks on center-of-mass motion after Eq. (29). By momentum conservation, a notch particle moving in one direction implies a background particle moving in the other. This establishes the symmetry relation. Returning now to Eq. (51), if we break up the spectral integral into contributions from the continuum energy states and the zero-energy states and we have

$$\int_{-\infty}^{\infty} dE \rho'_{nn}(E) + |\xi_0(n)|^2 = \int_{-\infty}^{\infty} dE \rho_{nn}(E), \tag{52}$$

where the primed quantity is the local density in the presence of the soliton; the unprimed refers to the case without the soliton. Invoking the symmetry relation $\rho_{nn}(E) = \rho_{nn}(-E)$, we find

$$\int_{-\infty}^{0} dE[\rho'_{nn}(E) - \rho_{nn}(E)] = -\frac{1}{2}|\xi_0(n)|^2. \tag{53}$$

Summing over all N sites (equivalently, integrating over space) we see that the total deficit is one-half. That is, the deficit in the number of notch particles (interpreted as the sum over negative energies) as a result of the soliton is one-half.

The occurrence of half numbers here appears puzzling at first glance [33,34]. This can be resolved as follows. The soliton and antisoliton are created simultaneously. According to Eq. (53), one-half of a state is missing from a notch in the vicinity of the soliton and similarly for the other notch around the antisoliton, so a total of one state is missing from the pair of notches. By virtue of the symmetry relation above a total of one state is also missing from the background. These two states form a pair of states above and below $E=0$, their energy-difference becoming vanishingly small as the soliton–antisoliton separation d increases. As $d \to \infty$ the two states become independent. These are precisely the zero modes $\xi_0(x)$.

VI. REPULSIVE POTENTIAL

In this section we study the interaction of two dark solitons with equal intensities but opposite phase jumps. Although it is known that such dark solitons repel each other weakly we are not aware of any effort to explicitly derive this fact from an exact solution. Early studies were numerical [35]. More recently approximate results were given by Kivshar and Krolikowski [36] and by Martinez *et al.* [15]. One reason why this repulsive interaction is important derives from the fact that this interaction does not lend us flexible control over dark solitons thereby imposing a fundamental limit on the applicability of dark solitons [37]. Although the calculation following is classical in nature it is not unrelated to the machinery already developed since the semiclassical quantization method becomes “quantum” only after quantization condition (22) is invoked. Hence the classical underpinnings of the theory remain clear and present. For instance, the importance of the nonlinearity will become manifest in Eq. (58). Moreover a two-kink soliton is complex, unlike a one-kink soliton which is real: hence the former is really more complicated than the latter. We believe our result is of great and immediate utility [38].

Since we are interested in an exact analysis we will make use of the more general exact two-parameter solution of the nonlinear Schrödinger equation [Eq. (34)],

$$\tilde{\Psi}(x,t) = \frac{2(a_3 - 2a_1)\cosh(\mu t) - 2\sqrt{a_3 a_1}\cosh(2px) + i\mu \sinh(\mu t)}{2\sqrt{a_3}\cosh(\mu t) + 2\sqrt{a_1}\cosh(2px)} \times e^{2ia_3 t}, \tag{54}$$

where the parameters a_3 and a_1 are real and positive and

$a_3 > a_1$, $p=\sqrt{a_3-a_1}$, $\mu=4p\sqrt{a_1}$ [29]. We distinguish this more general solution from Eq. (35) by a tilde. The maximum intensity corresponding to $\tilde{\Psi}(x,t)$ is a_3 which is the background intensity and the minimum intensity is a_1. The above solution encapsulates the approach from $x=\pm\infty$, $t=-\infty$, the collision (around the origin) and the continued propagation of two dark solitons. Solution (35) in Sec. IV is a special case of the above two-parameter solution with $q=\sqrt{2a_3}$ and $a_3=2a_1$. Therefore our results apply to the general two-kink collision. It will be convenient to identify the points where the intensity is a minimum. One finds that the (x,t) values corresponding to minimum intensity are given by

$$\exp[2p(\pm x - 2\sqrt{a_1}t)] = \left(\frac{a_3}{a_1}\right)^{1/2}. \quad (55)$$

It follows that the velocity of each soliton is $2\sqrt{a_1}$ and the shift per soliton as a result of the elastic collision event is $\frac{1}{2p}\ln(a_3/a_1)$. Next we expand Eq. (54) about (x,t) satisfying relation (55). Choosing positive x and t, and designating $x_0>0$ as the solution of Eq. (55) for a fixed value of (positive) time t, we obtain the expansion for $x\geq 0$,

$$\tilde{\Psi}(x,t) = -\sqrt{a_3} + 2pe^{i\theta}\frac{1}{1+e^{2py}} + e^{-4px_0}\left(\frac{2p(a_3/a_1)e^{-i\theta} - \sqrt{a_3}e^{-2py}}{1+e^{2py}} - \frac{2p(a_3/a_1)e^{i\theta}}{(1+e^{2py})^2}\right) + \cdots, \quad (56)$$

where $\tan\theta=\sqrt{a_1}/p$ and the unwritten terms are of order $O(e^{-8px_0})$ and higher. In writing Eq. (56) we had also introduced $x=x_0+y$ so the interval of y is $-x_0\leq y\leq\infty$. The expansion is valid provided we stay clear of the collision zone and this is generally the desired situation. A similar expansion can be given for $x\leq 0$. Corresponding to Lagrangian (33) the potential energy functional for fixed time is

$$V[\tilde{\Psi}] = \int dx(|\tilde{\Psi}_x|^2 + |\tilde{\Psi}|^4). \quad (57)$$

The potential can now be computed directly from Eqs. (57) and (56),

$$V(\tilde{\Psi}) = 8pa_3e^{-pz} + O(e^{-2pz}), \quad (58)$$

in which $z=2x_0$ is the separation between solitons. This is clearly a repulsive potential. The entire lowest order contribution to the potential came from the quartic interaction term $|\tilde{\Psi}|^4$. For the two-kink soliton of Sec. V, $p=q/2=n/5$, so the potential is very short ranged when there are more notch particles. Because the background has few distinguishing features the repulsive interaction is mainly seen with the notch particles.

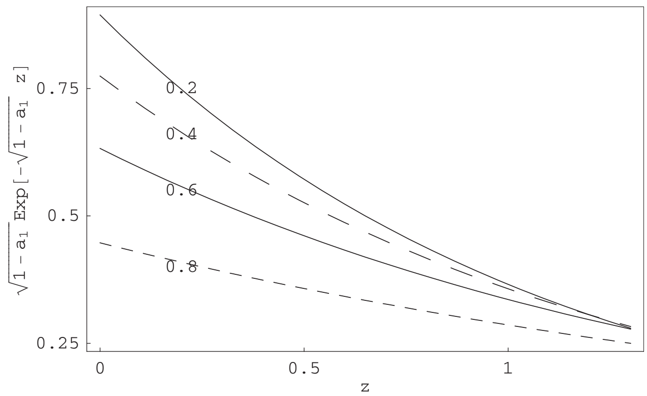

FIG. 9. Plots of the repulsive potential $V(\tilde{\Psi})$ for $a_3=1$ and several values of $a_1=0.2, 0.4, 0.6, 0.8$.

Plots of $V(\tilde{\Psi})$ versus z for $a_3=1$ and several values of a_1 are shown in Fig. 9. We observe that for fixed a_3 the repulsive strength of the potential can be varied significantly as we change the second parameter a_1. This effect of changing a_1 is evidently stronger for lower z values. Along with the repulsive interaction there is also the possibility of an *attractive* interaction that arises from the Casimir effect. The interplay between these two potentials in the context of creating bound pairs of dark solitons and the possible control over them is an interesting question [38].

VII. OUTLOOK

We had explored an analogy with a linear particle-chain model and obtained a picture of the dark optical soliton as a composite of a background and notch. Noting that the action of the dark soliton is made up of two terms with different length scales, we modified the semiclassical quantization procedure of Dashen *et al.* [9] to allow for the separate quantization of these two entities. We showed how the quantization of these entities is equivalent to the quantization of the original system in the semiclassical approximation. Applying this to the one-kink and two-kink dark solitons, we found that the notch could be understood as a bound state of bosons bound by an attractive delta potential while the background is an ensemble of oscillators (phonons) with a slightly repulsive interaction. Outside the collision zone the two-kink soliton can be seen as two separate notches. The quantum numbers emerging from this analysis are interpreted in terms of number of particles. We also showed how the topological character of the kink can be incorporated into our description. We speculated on the possibility of fractional quantum numbers and derived the repulsive interaction of two dark solitons directly from our framework. An interesting question that merits attention is whether an attractive force to bind dark solitons might be found [38].

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