Dicke-type energy level crossings in cavity-induced atom cooling: Another superradiant cooling

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This paper is devoted to energy-spectral analysis for the system of a two-level atom coupled with photons in a cavity. It is shown that the Dicke-type energy level crossings take place when the atom-cavity interaction of the system undergoes changes between the weak-coupling regime and the strong one. Using the phenomenon of the crossings, we develop the idea of cavity-induced atom cooling proposed by Horak *et al.*, and we lay mathematical foundations of a possible mechanism for another superradiant cooling in addition to that proposed by Domokos and Ritsch. The process of our superradiant cooling can function well by cavity decay and by control of the position of the atom, at least in (mathematical) theory, even if there is neither atomic absorption nor atomic emission of photons.

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I. INTRODUCTION

Laser cooling is one of attractive subjects in modern physics. It has been demonstrated with several experimental techniques such as the ion cooling [1], the Doppler cooling [2], and the Sisyphus cooling [3] (cooling wherein the atom spends more of its time climbing than descending the potential hills associated with the ac-Stark-shifted levels, and loses kinetic energy in doing so). Also it has enabled us to observe many fundamental phenomena in theoretical physics. One of typical instances of applications of the laser cooling is the observation of Bose-Einstein condensation (BEC) [4,5]. It has been about 10 years since another type of laser cooling was proposed using the strong atom-photon interaction. Such a strong interaction between atom and photons is realized in the so-called cavity quantum electrodynamics (QED) [6–11]. Thus, Horak *et al.* [12] investigated the system of a two-level atom coupled with a laser, and then they found a mechanism for cooling the atom, which is similar to that of the Sisyphus cooling. The cooling mechanism is called *cavity-induced atom cooling* (CIAC). In their CIAC process, the method for carrying away the energy from the atom coupled with photons is given not only by atomic decay (i.e., atomic spontaneous emission of photons) but also by cavity decay. Here, we note that the transition probability of atomic decay becomes small as the coupling strength grows large. It has experimentally been confirmed that the cavity decay works in the cooling system [13–15]. Concerning the cooling methods using cavity QED, Domokos and Ritsch [16,17] proposed a concept of superradiant cooling based on the atomic self-organization and cooperation among many atoms in a cavity. The atom-photon interaction in the strong-coupling regime brings the situation amazingly different from ordinary atomic decay [10]. We will adopt this difference into our arguments on the CIAC.

As well as the ensemble of many two-level atoms coupled with a laser has the possibility of making superradiance as a cooperative effect in optics [18,19], another superradiance may also appear in energy spectrum even for the system of a two-level atom coupled with a laser, provided that it is in the strong-coupling regime [20–24]. As far as atom-laser interaction in the cooling process for the BEC goes, superradiance has been experimentally observed under a certain physical condition [25–28], and this phenomenon has been theoretically shown [29–31]. It was pointed out that there is a possibility that superradiance causes the energy level crossing between the initial ground-state energy and an initial excited-state energy [20,23,24], namely, a kind of phase transition occurs. This is an optical phenomenon of light-induced phase transition, though it is not a cooperative effect in optics. The details of such an energy level crossing have precisely been studied, and this type of crossing is called the *Dicke-type* (*energy level*) *crossing* [32]. The term is abbreviated to DELC in this paper. We will strictly define its meaning in Sec. II. The reason why we call the energy level crossing this term is that it is basically caused by the mathematical mechanism [20,24] of the Dicke superradiance [33].

This paper is devoted to developing the CIAC. Namely, we will show that the DELC takes place when the system undergoes changes between the weak-coupling regime and the strong one. Using the crossings, we will propose the possibility of another superradiant cooling in terms of the energy spectrum from our point of view. In our proposal we will consider whether the following are possible in theory for cooling the atom in a cavity: (1) can we use a laser only for controlling the strength of the atom-cavity interaction without the need for another laser to drive the atom to an excited state? and (2) can we expect that the energy loss caused by cavity decay become much larger? Concerning question (1), we note that Maunz *et al.* [15] succeeded in performing such an experiment that the cooling process does not require atomic excitation as well as demonstrating the cavity decay in the process. The demonstration supplies an important technique for laser cooling because exciting the atom inside the cavity causes the heating effect [34]. To perform our research into the problems, we consider an ideal situation only to see the energy-spectral property for our system without considering, for example, the laser heating processes caused by diffusion of the atomic momentum. Some mathematical techniques to make the spectral analysis for such systems have been developed lately [32,35–38]. Thus, an-

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other purpose of this paper is to show the mechanism of the DELC in the CIAC as rigorously as in the works in Ref. [39] so that the process of our superradiant cooling can function well in (mathematical) theory.

Our paper is constructed as follows. In Sec. II we will give our Hamiltonian $H(\Omega,\alpha;d)$ by generalizing the Hamiltonian which Horak *et al.* [12] handled, where Ω is a function of space-time point and governs the atom-photon interaction, α is also a function of space-time and a generalization of the strength of the pump field, and d is a parameter for nonlinear coupling of the atom and photons. Moreover, we will define some notation to explain what the DELC is. We will make energy-spectral analysis for the generalized Hamiltonian $H(\Omega,\alpha;d)$ in and after Sec. III. In Sec. III we will show that the DELC takes place for $H(\Omega,\alpha;d)$ with $\alpha\equiv 0$. In Sec. IV we will show the existence of the superradiant ground-state energy for $H(\Omega,\alpha;d)$ with $\alpha\equiv 0$ and $d=1$ in the strong-coupling regime. In Sec. V we will argue the stability of the CIAC and the DELC in it the condition $\alpha\not\equiv 0$.

II. HAMILTONIAN AND SOME NOTATION

In Ref. [12] Horak *et al.* studied a Hamiltonian adopting dipole and rotating wave approximation. To write down their Hamiltonian, we define some operators: the atomic position (momentum) operator is denoted by x (p), the photon annihilation (creation) operator is denoted by a ($a^\dagger$), and the atomic operator is given by $\sigma_{ij}=|i\rangle\langle j|$, with $i,j=0,1$. Then, the Hamiltonian is

$$H=\frac{1}{2m}p^2-\Delta\sigma_{11}-\Delta_c a^\dagger a+i\Omega(x)(\sigma_{01}a^\dagger-\sigma_{10}a)+i\alpha(a-a^\dagger),$$

where two real numbers Δ and Δ_c with $-\infty<\Delta<+\infty$ and $\Delta_c<0$ are the atom-pump detuning and the detuning of the empty cavity relative to the pump frequency, respectively, and $\Omega(x)$ stands for the atom-cavity coupling constant, i.e., $\Omega(x)=\Omega_0\cos kx$ with the position x of the atom and the wave number k of photons of the laser. We note that in the case $\alpha=0$ the Hamiltonian H is used to argue the resonant interaction of an atom with a microwave field [40]. In Hamiltonian H, the part consisting of the first, the second, and the third terms [i.e., $(2m)^{-1}p^2-\Delta\sigma_{11}-\Delta_c a^\dagger a$] is the free Hamiltonian of our system. The fourth and fifth terms represent the Hamiltonian of interaction and the energy operator of the pump field, respectively. In this section we generalize the Hamiltonian H. Our generalization is the following: (1) we consider not only the linear coupling but also nonlinear coupling (see, for example, Sec. 8.1.3 of Ref. [11]); (2) we introduce the time dependence into the coupling constant $\Omega(x)$; and (3) we consider the general operator which represents not only the energy operator of the pump field but also the energy operator of the pump field plus some error potential coming from the environment of the experiment for testing the system. For instance, $W(x,t)$ may include the counter-rotating terms: $W(x,t)=i\{\alpha_1(a-a^\dagger)+\alpha_2(\sigma_{01}a-\sigma_{10}a^\dagger)\}$, where $0\le\alpha_1,\alpha_2\le 1$. Then, we note that $W(x,t)$ consists of the only counter-rotating terms in the case $\alpha_1=0$. Thus our Hamiltonian reads

$$H(\Omega,\alpha;d)=\frac{1}{2m}p^2-\Delta\sigma_{11}-\Delta_c a^\dagger a+i\Omega(x,t)(\sigma_{01}a^{\dagger d}-\sigma_{10}a^d)+\alpha(x,t)W(x,t)$$

for $d=1,2,\ldots$, where $\Omega(x,t)$ and $\alpha(x,t)$ are continuous, real-valued functions of (x,t) with $\Omega(x,0)=0=\alpha(x,0)$ for every position x of the atom, and $\alpha(x,t)W(x,t)$ is the generalization of the energy operator of the pump field. As an example of $\Omega(x,t)$, we often adopt $\Omega(x,t)=\Omega_0(t)\gamma(x)$ in this paper. Here $\Omega_0(t)$ is a continuous, real-valued function of time $t\ge 0$ with $\Omega_0(0)=0$, and $\gamma(x)$ is a bounded, continuous, and real-valued function of the position x of the atom. For instance, $\gamma(x)=\cos kx$.

In the case where $\Omega(x,t)\equiv 0$ and $\alpha(z,t)\equiv 0$, we denote eigenvalues of $H(0,0;d):=H(\Omega=0,\alpha=0;d)$ as $\mathcal{E}_0<\mathcal{E}_1<\cdots<\mathcal{E}_n<\cdots$. When either $\Omega(x,t)$ or $\alpha(t)$ is nonzero, we denote eigenvalues of $H(\Omega,\alpha;d)$ as $\mathcal{E}_n(\Omega,\alpha;d)$ for $n=0,1,\ldots$. If the interaction $H_{\rm int}:=i\Omega(t,x)(\sigma_{01}a^{\dagger d}-\sigma_{10}a^d)+\alpha(t)W(x,t)$ is a small perturbation for $H(0,0;d)$, then each eigenvalue $\mathcal{E}_n(\Omega,\alpha;d)$ sits near its original position $\mathcal{E}_n$, so that the primary order among eigenvalues is kept: $\mathcal{E}_0(\Omega,\alpha;d)<\mathcal{E}_1(\Omega,\alpha;d)<\cdots<\mathcal{E}_n(\Omega,\alpha;d)<\cdots$. On the other hand, the phase transition of the superradiance [20,23,24] tells us about the possibility that $\mathcal{E}_1(\Omega,\alpha;d)$ is less than $\mathcal{E}_0(\Omega,\alpha;d)$ and thus becomes a different ground-state energy provided that the interaction $H_{\rm int}$ has some strong strength. For our Hamiltonian, we can classify crossings into two types. One type is the crossing between an ascending eigenvalue $\mathcal{E}_n^+(\Omega,\alpha;d)$ and a descending one $\mathcal{E}_n^-(\Omega,\alpha;d)$ as the strength of the interaction $H_{\rm int}$ grows enough. Another type is the crossing only among descending eigenvalues $\mathcal{E}_n^-(\Omega,\alpha;d)$ [or ascending eigenvalues $\mathcal{E}_n^+(\Omega,\alpha;d)$]. We call the former type a *trivial* crossing, and the latter type a *nontrivial* crossing. The trivial crossing has a situation similar to that of the pseudocrossing in the Landau-Zenar theory. For the nontrivial crossing, as the strength of the interaction becomes much stronger, even many $\mathcal{E}_n^-(\Omega,\alpha;d)$'s may be less than $\mathcal{E}_0^-(\Omega,\alpha;d)$. We call such a nontrivial crossing the Dicke-type (energy level) crossing [32], i.e., DELC. Moreover, $\mathcal{E}_n^-(\Omega,\alpha;d)$'s are capable of usurping the position of the ground-state energy in turn. We call such a new ground-state energy the *superradiant ground-state energy*. The DELC and the appearance of the superradiant ground-state energy can be used, together with cavity decay, for carrying away the energy from the system. Based on this idea, we construct mathematical foundations of the concept of superradiant cooling different from that proposed in Ref. [16] in and after Sec. III.

III. DELC IN THE CASE $\alpha\equiv 0$

In this section we show how the DELC takes place for $H(\Omega,0;d)$, that is, for $H(\Omega,\alpha;d)$ in the case where $\alpha(x,t)\equiv 0$. As in Ref. [12], using the well-known identification so that the ground state $|0\rangle$ with the energy ε_0 and the first excited state $|1\rangle$ with the energy ε_1 are unitarily equivalent to $\binom{0}{1}$ and $\binom{1}{0}$, respectively, $H(\Omega,0;d)$ approximately reads

$$H_0(z,t;d) := \begin{pmatrix} -\Delta_c a^\dagger a + \varepsilon_1 - \Delta & -i\Omega(z,t)a^d \\ i\Omega(z,t)a^{\dagger d} & -\Delta_c a^\dagger a + \varepsilon_0 \end{pmatrix}. \quad (3.1)$$

Here we note $\sigma_{00}=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, $\sigma_{01}=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $\sigma_{10}=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, and $\sigma_{11}=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. In this paper we always assume that

$$\varepsilon_1 - \Delta > \varepsilon_0. \quad (3.2)$$

Therefore, the ground-state energy of $H_0(z,0;d)$ [i.e., $H_0(z,t;d)$ with $\Omega(z,t)\equiv 0$] is always ε_0.

As shown in Appendix A with the method which is a generalization of that in Refs. [11,23,41], since $H_0(z,t;d)$ is basically the Hamiltonian of the Jaynes-Cummings model [42], all the energy levels of $H_0(z,t;d)$ are perfectly determined. They are given by $-\Delta_c n+\varepsilon_0$ for non-negative integer n with $n<d$, and $\Xi_n(d)\pm\Upsilon_n(z,t;d)$ for non-negative integer n with $n\geq d$, where

$$\Xi_n(d) = -\Delta_c n + \frac{1}{2}(\varepsilon_0 + \varepsilon_1 + d\Delta_c - \Delta),$$

and $\Upsilon_n(z,t;d)$ is the generalized Rabi frequency [43]:

$$\Upsilon_n(z,t;d) = \frac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_0 + d\Delta_c - \Delta)^2 + 4|\Omega(z,t)|^2\frac{n!}{(n-d)!}}. \quad (3.3)$$

Here we note all the energy levels of $H_0(z,0;d)$ are $-\Delta_c n+\varepsilon_0$ and $-\Delta_c n+\varepsilon_1-\Delta$, with $n=0,1,\ldots$. Therefore, we can conclude that the energy levels of $H_0(z,t;d)$ are completely given by energies $E_n^0(z,t;d)$ and energies $E_n^\pm(z,t;d)$ of the generalized Jaynes-Cummings doublet [18], continuous functions of (z,t), for each $n=0,1,\ldots$: for non-negative integers n with $n<d$

$$E_n^0(z,t;d) = -\Delta_c n + \varepsilon_0.$$

For non-negative integers n with $n\geq d$, on the other hand,

$$E_n^-(z,t;d) = \begin{cases} \Xi_n(d) - \Upsilon_n(z,t;d) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c \\ \Xi_{n+d}(d) - \Upsilon_{n+d}(z,t;d) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c, \end{cases}$$

and

$$E_n^+(z,t;d) = \begin{cases} \Xi_{n+d}(d) + \Upsilon_{n+d}(z,t;d) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c \\ \Xi_n(d) + \Upsilon_n(z,t;d) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c. \end{cases}$$

It follows from these definitions that

$$E_n^-(z,0;d) = \begin{cases} -\Delta_c n + \varepsilon_0 & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c \\ -\Delta_c n + \varepsilon_1 - \Delta & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c, \end{cases}$$

and

$$E_n^+(z,t;d) \geq E_n^+(z,0;d) = \begin{cases} -\Delta_c n + \varepsilon_1 - \Delta & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c \\ -\Delta_c n + \varepsilon_0 & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c. \end{cases}$$

The latter inequality means that the candidates of superradiant ground-state energy are only $E_n^-(z,t;d)$'s.

We define two spatiotemporal domains $\mathbb{D}_{mn}^{\mathrm{wc}}(d)$ and $\mathbb{D}_{mn}^{\mathrm{sc}}(d)$ for non-negative integers m and n with $\max\{d,m\}<n$ as

$$\mathbb{D}_{mn}^{\mathrm{wc}}(d) := \{(z,t)|E_m^0(z,t;d) < E_n^-(z,t;d) \quad \text{if } m<d;$$

$$E_m^-(z,t;d) < E_n^-(z,t;d) \quad \text{if } m \geq d$$

and

$$\mathbb{D}_{mn}^{\mathrm{sc}}(d) := \{(z,t)|E_m^0(z,t;d) > E_n^-(z,t;d) \quad \text{if } m<d;$$

$$E_m^-(z,t;d) > E_n^-(z,t;d) \quad \text{if } m \geq d$$

respectively.

A. In the case $d=1$

In this subsection we investigate the behavior of the DELC in the case $d=1$. To do that, we introduce some positive numbers and some domains of the space-time. Then, we divide the whole space of the space-time into three classes, namely, the weak-coupling regime, the strong-coupling regime, and the critical regime.

For each natural number n, we define a positive number C_{0n}^0 as

$$C_{0n}^0 := \begin{cases} \Delta_c^2 n - \Delta_c(\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - \Delta_c \\ \Delta_c^2(n+1) - \Delta_c(\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - \Delta_c. \end{cases}$$

We define three domains $\mathcal{D}_{0n}^{\mathrm{wc}}(1)$, $\mathcal{D}_{0n}^{\mathrm{sc}}(1)$, and $\mathcal{D}_{0n}^{0}(1)$ of the space-time as follows: the spatiotemporal domain $\mathcal{D}_{0n}^{\mathrm{wc}}(1)$ for the weak-coupling regime is given by $\mathcal{D}_{0n}^{\mathrm{wc}}(1) := \{(z,t)||\Omega(z,t)|^2 < C_{0n}^0\}$, and the domain $\mathcal{D}_{0n}^{\mathrm{sc}}(1)$ for the strong-coupling regime by $\mathcal{D}_{0n}^{\mathrm{sc}}(1) := \{(z,t)||\Omega(z,t)|^2 > C_{0n}^0\}$. The domain $\mathcal{D}_{0n}^{\mathrm{cr}}(1)$ for the critical regime is defined by $\mathcal{D}_{0n}^{\mathrm{cr}}(1) := \{(z,t)||\Omega(z,t)|^2 = C_{0n}^0\}$.

The following theorem says that how the DELC takes place is completely determined: let us suppose $1<n$ now. Then, the spatiotemporal domain $\mathcal{D}_{0n}^{\mathrm{wc}}(1)$ is equal to the domain $\mathbb{D}_{0n}^{\mathrm{wc}}(1)$, and the spatiotemporal domain $\mathcal{D}_{0n}^{\mathrm{sc}}(1)$ to the domain $\mathbb{D}_{0n}^{\mathrm{sc}}(1)$, i.e., $\mathcal{D}_{0n}^{\mathrm{wc}}(1)=\mathbb{D}_{0n}^{\mathrm{wc}}(1)$ and $\mathcal{D}_{0n}^{\mathrm{sc}}(1)=\mathbb{D}_{0n}^{\mathrm{sc}}(1)$. Namely, the energy level crossing takes place as

$$E_0^0(z,t;1) < E_n^-(z,t;1) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}_{0n}^{\mathrm{wc}}(1), \quad (3.4)$$

$$E_0^0(z,t;1) = E_n^-(z,t;1) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}_{0n}^{\mathrm{cr}}(1), \quad (3.5)$$

$$E^0_0(z,t;1) > E^-_n(z,t;1) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}^{\mathrm{sc}}_{0n}(1). \tag{3.6}$$

These inequalities (3.4)–(3.6) guarantee the DELC because $E^0_0(z,t;1)$ and $E^-_n(z,t;1)$ are continuous functions of the space-time point (z,t). We will demonstrate this theorem in Appendix B 1 together with the proof of another theorem below.

When the above DELC between $E^0_0(z,t;1)$ and $E^-_n(z,t;1)$ takes place, there is certainly an energy level crossing between $E^-_m(z,t;1)$ and $E^-_n(z,t;1)$ for a natural number m with $1<m<n$. To show it, we introduce two positive constants and define two domains of the space-time.

For natural numbers m,n with $m<n$, we set positive constants C^{wc}_{mn} and C^{sc}_{mn} as

$$C^{\mathrm{wc}}_{mn} := \begin{cases} \Delta_c^2\left\{\dfrac{m+n}{2} + \sqrt{\left(\dfrac{m+n}{2}\right)^2 + \dfrac{K^2}{2\Delta_c^2}}\right\} & \text{if } \varepsilon_1-\varepsilon_0 \ge \Delta-\Delta_c \\ \Delta_c^2\left\{\dfrac{m+n+2}{2} + \sqrt{\left(\dfrac{m+n+2}{2}\right)^2 + \dfrac{K^2}{2\Delta_c^2}}\right\} & \text{if } \varepsilon_1-\varepsilon_0 < \Delta-\Delta_c, \end{cases}$$

and

$$C^{\mathrm{sc}}_{mn} := \begin{cases} \Delta_c^2\left\{m+n+2\sqrt{mn+\dfrac{K^2}{4\Delta_c^2}}\right\} & \text{if } \varepsilon_1-\varepsilon_0 \ge \Delta-\Delta_c \\ \Delta_c^2\left\{m+n+2+2\sqrt{m+n+mn+\dfrac{K^2}{4\Delta_c^2}}\right\} & \text{if } \varepsilon_1-\varepsilon_0 < \Delta-\Delta_c, \end{cases}$$

where $K := \varepsilon_1-\varepsilon_0+\Delta_c-\Delta$. We give two spatiotemporal domains $\mathcal{D}^{\mathrm{wc}}_{mn}(1)$ and $\mathcal{D}^{\mathrm{sc}}_{mn}(1)$ for non-negative integers m,n with $m<n$ in the following. The domain $\mathcal{D}^{\mathrm{wc}}_{mn}(1)$ for the weak-coupling regime is given by $\mathcal{D}^{\mathrm{wc}}_{mn}(1)=\{(z,t)\,|\,0\le|\Omega(z,t)|^2<C^{\mathrm{wc}}_{mn}\}$, and the domain $\mathcal{D}^{\mathrm{sc}}_{mn}(1)$ for the strong-coupling regime by $\mathcal{D}^{\mathrm{sc}}_{mn}(1)=\{(z,t)\,|\,|\Omega(z,t)|^2>C^{\mathrm{sc}}_{mn}\}$. Then, as far as such m goes, the following theorem gives a sufficient condition so that the crossing between $E^-_m(z,t;1)$ and $E^-_n(z,t;1)$ occurs: let $1<m<n$ now. Then, the spatiotemporal domain $\mathcal{D}^{\mathrm{wc}}_{mn}(1)$ is included in the domain $\mathbb{D}^{\mathrm{wc}}_{mn}(1)$, and the domain $\mathcal{D}^{\mathrm{sc}}_{mn}(1)$ in the domain $\mathbb{D}^{\mathrm{sc}}_{mn}(1)$, i.e., $\mathcal{D}^{\mathrm{wc}}_{mn}(1)\subset\mathbb{D}^{\mathrm{wc}}_{mn}(1)$ and $\mathcal{D}^{\mathrm{sc}}_{mn}(1)\subset\mathbb{D}^{\mathrm{sc}}_{mn}(1)$:

$$E^-_m(z,t;1) < E^-_n(z,t;1) \quad \text{for } (z,t) \text{ in } \mathcal{D}^{\mathrm{wc}}_{mn}(1), \tag{3.7}$$

$$E^-_m(z,t;1) > E^-_n(z,t;1) \quad \text{for } (z,t) \text{ in } \mathcal{D}^{\mathrm{sc}}_{mn}(1). \tag{3.8}$$

We will also demonstrate this theorem in Appendix B 1 because the proof is a little too long to put in this subsection. But, instead of proving them here, we concretely see some

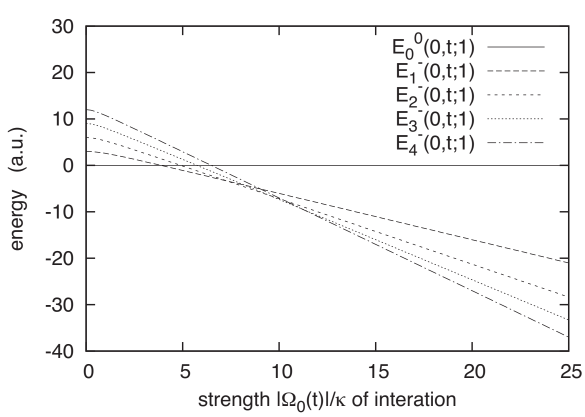

FIG. 1. DELC for $\gamma(z)=\cos 2\pi z$. Energies $E^0_0(0,t;1)$ (solid line), $E^-_1(0,t;1)$ (dashed line), $E^-_2(0,t;1)$ (short-dashed line), $E^-_3(0,t;1)$ (dotted line), and $E^-_4(0,t;1)$ (dashed-dotted line). The physical parameters are set as $\varepsilon_0=0$, $\varepsilon_1=6\kappa$, $\Delta=1\kappa$, and $\Delta_c=-3\kappa$ with a unit κ. The position z is fixed at $z=0$.

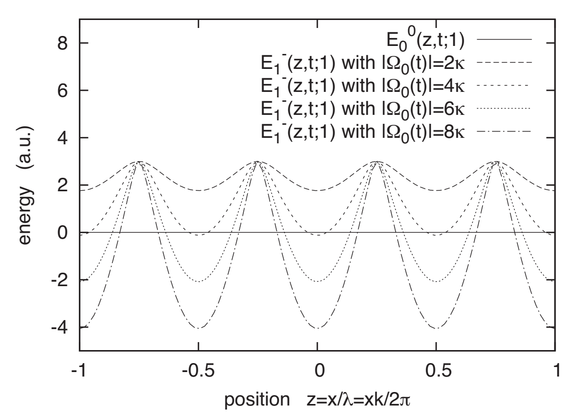

FIG. 2. DELC for $\gamma(z)=\cos 2\pi z$. Energies $E^0_0(z,t;1)$ (solid line), $E^-_1(0,t;1)$ with $|\Omega_0(t)|=2\kappa$ (dashed line), with $|\Omega_0(t)|=4\kappa$ (short-dashed line), with $|\Omega_0(t)|=6\kappa$ (dotted line), and with $|\Omega_0(t)|=8\kappa$ (dashed-dotted line). The time is fixed so that $|\Omega_0(t)|=2\kappa,4\kappa,6\kappa,8\kappa$.

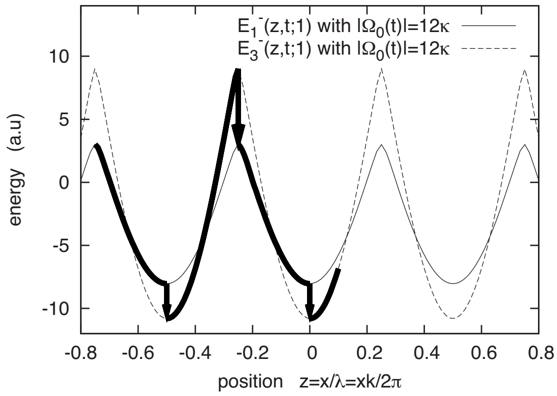

FIG. 3. Energy decay between $E_1^-(z,t;1)$ (solid line) and $E_3^-(z,t;1)$ (dashed line). The physical parameters are set as in Fig. 1 and the strength $|\Omega_0(t)|$ is fixed at 12κ. $\gamma(z)=\cos 2\pi z$ in $\Omega(z,t)$. The thick arrows stand for the temperature loss.

physical situation that the two theorems tell us. We employ $\cos 2\pi z$ as $\gamma(z)$. For the position fixed at $z=0$, the five energies $E_0^0(0,t;1)$ and $E_n^-(0,t;1)$, with $n=1,2,3,4$, are numerically calculated as in Fig. 1. In the case where the strength $|\Omega_0(t)|$ is given by $|\Omega_0(t)|=2\kappa,4\kappa,6\kappa,8\kappa$, the two energies $E_0^0(z,t;1)$ and $E_1^-(z,t;1)$ are in Fig. 2.

Suppose that $\varepsilon_1-\varepsilon_0\geq\Delta-\Delta_c$ now. Let our atom be in the state with the energy $E_m^-(z_0,t_0;1)$ at an initial space-time point (z_0,t_0) in the domain $\mathcal{D}_{mn}^{\mathrm{wc}}(1)$ for a non-negative integer m and a positive integer n with $0\leq m<n$. The atom is apt to sit in the state with energy $E_m^-(z_0,t_0;1)$ because being in the state with energy $E_m^-(z_0,t_0;1)$ is more stable than in the state with energy $E_n^-(z_0,t_0;1)$. Once, however, a space-time point (z,t) plunges into the domain $\mathcal{D}_{mn}^{\mathrm{sc}}(1)$, the energy level crossing takes place as shown in its process [Eqs. (3.4)–(3.6)]. Thus, being in the state with energy $E_m^-(z,t;1)$ is *not* stable any longer and, thus, it goes down to the state with energy $E_n^-(z,t;1)$. This descent can be caused by cavity decay as pointed out in Ref. [12], even if we cannot expect atomic decay because of the small transition probability. The circumstance for such a cooling is usually not in thermal equilibrium and, thus, the temperature which this system loses is

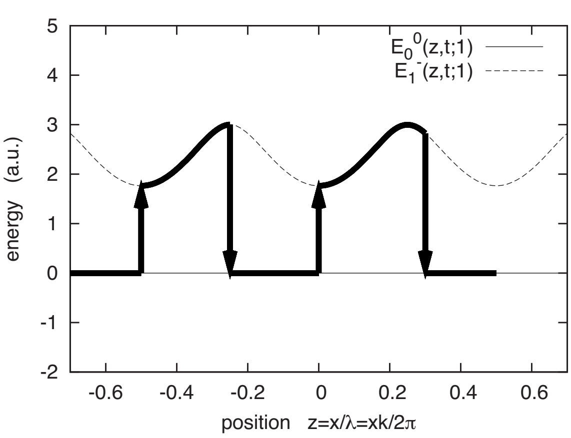

FIG. 4. CIAC (bold line) in $\mathbb{D}_{01}^{\mathrm{wc}}(1)$. Energies $E_0^0(t,z;1)$ (solid line) and $E_1^-(z,t;1)$ (dashed line). The physical parameters are set as in Fig. 1.

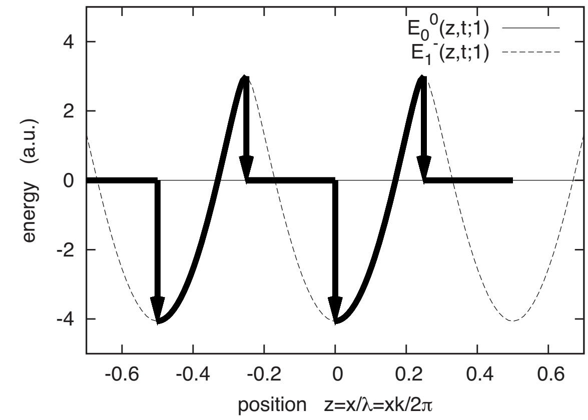

FIG. 5. DELC and CIAC (bold line). Energies $E_0^0(z,t;1)$ (solid line) and $E_1^-(z,t;1)$ (dashed line). The physical parameters are set as in Fig. 1.

not given by thermodynamic temperature. Nevertheless, according to the thermodynamics law as in Eq. (5.1) of Ref. [44] and Eq. (2.1) of Ref. [45], we roughly estimate the effective temperature $\Delta T_{m\to n}$ coming from the descent. Then, we can expect that the crossing between $E_m^-(z,t;1)$ and $E_n^-(z,t;1)$ makes the temperature go down at most $\Delta T_{m\to n}$ estimated at

$$\Delta T_{m\to n}\approx\frac{2(n-m)}{k_B}\left|\frac{|\Omega(z,t)|^2}{\Upsilon_m(z_0,t_0;1)+\Upsilon_n(z,t;1)}-|\Delta_c|\right| + \frac{2m}{k_B}\left|\frac{|\Omega(z,t)|^2-|\Omega(z_0,t_0)|^2}{\Upsilon_m(z_0,t_0;1)+\Upsilon_n(z,t;1)}\right| \tag{3.9}$$

for the point (z_0,t_0) in the spatiotemporal domain $\mathcal{D}_{mn}^{\mathrm{wc}}(1)$ and the point (z,t) in the domain $\mathcal{D}_{mn}^{\mathrm{sc}}(1)$, of course, provided that there is nothing to obstruct the temperature loss. Here k_B is the Boltzmann constant and $\Upsilon_\ell(z,t;1)=(1/2)\sqrt{K^2+4|\Omega(z,t)|^2\ell}$ is the generalized Rabi frequency (3.3) for each natural number ℓ.

Conversely, let our atom be in the state with the energy $E_n^-(z_0,t_0;1)$ at an initial space-time point (z_0,t_0) in the domain $\mathcal{D}_{mn}^{\mathrm{sc}}(1)$. Then, since a process reverse to the above energy level crossing takes place if the coupling regime recoils from the spatiotemporal domain $\mathcal{D}_{mn}^{\mathrm{sc}}(1)$ to the domain $\mathcal{D}_{mn}^{\mathrm{wc}}(1)$, this reverse crossing can also carry away the temperature $\Delta T_{n\to m}$ of the atom so that $\Delta T_{n\to m}=\Delta T_{m\to n}$ for the

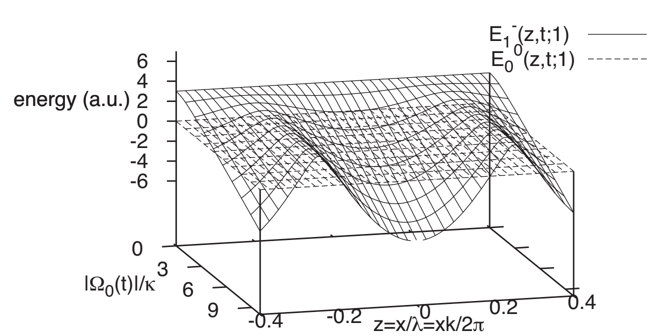

FIG. 6. Surfaces of energies $E_0^0(z,t;1)=0$ (dashed line) and $E_1^-(z,t;1)$ (solid line). The physical parameters are set as in Fig. 1.

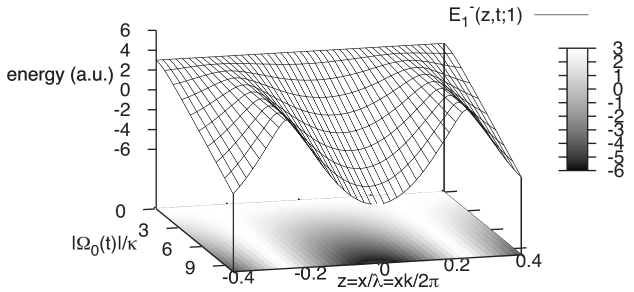

FIG. 7. Surface of energy $E_1^-(z,t;1)$ (solid line), where $E_0^0(z,t;1)=0$. The physical parameters are set as in Fig. 1.

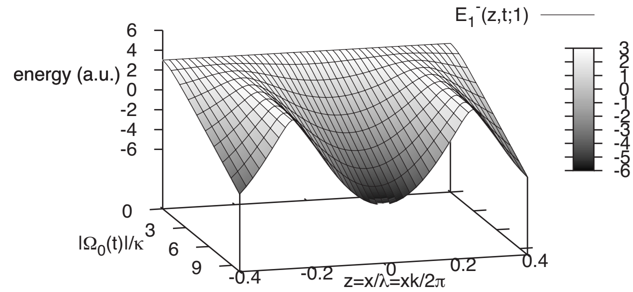

FIG. 8. Surface of energy $E_1^-(z,t;1)$ (solid line), where $E_0^0(z,t;1)=0$. The physical parameters are set as in Fig. 1.

point (z_0,t_0) in the spatiotemporal domain $\mathcal{D}_{mn}^{\rm sc}(1)$ and the point (z,t) in the domain $\mathcal{D}_{mn}^{\rm wc}(1)$. We illustrate this situation for $E_1^-(z,t;1)$ and $E_3^-(z,t;1)$ in Fig. 3. The diagrammatic illustration about temperature loss is represented by the thick arrows in Fig. 3. Therefore, our arguments say that there is a possibility of the following mechanism for superradiant cooling. Let a space-time point $(z_{2\ell+1},t_{2\ell+1})$ be in the domain $\mathcal{D}_{01}^{\rm wc}(1)$ and a space-time point $(z_{2\ell+2},t_{2\ell+2})$ be in the domain $\mathcal{D}_{01}^{\rm sc}(1)$ for each $\ell=0,1,\ldots,N-1$ with a natural number N. The DELC and the reverse crossings may carry away the temperature of the atom,

$$\frac{2}{k_B}\sum_{\nu=1}^{2N}\frac{|\Omega(z_\nu,t_\nu)|^2}{|K|/2+\Upsilon_1(z_\nu,t_\nu;1)}-\frac{4N}{k_B}|\Delta_c| \tag{3.10}$$

at most by Eq. (3.9) if $K\geq 0$. We can make similar argument when $K<0$.

When the whole space of the space-time is $\mathbb{D}_{01}^{\rm wc}(1)$, we find the Sisyphus-type mechanism in the energy spectrum as Horak *et al.* pointed out in Ref. [12]. To see the diagrammatic representation we consider a concrete example now. For the strength $|\Omega_0(t)|=2\kappa$ we have the CIAC as in Fig. 4. Up arrows in Fig. 4 represent the energy that the atom coupled with photon gains by photon absorption; namely, we have to throw a driving laser to the atom for its excitement. Down arrows mean the energy loss caused by cavity decay. In this domain $\mathbb{D}_{01}^{\rm wc}(1)$ of the space-time we cannot find such a sequence $\{(z_\nu,t_\nu)\}_{\nu=1}^{2L}$ of the space-time points. Thus, Eq. (3.10) does not work. As shown in the following concrete example, on the other hand, we can expect the sequence $\{(z_\nu,t_\nu)\}_{\nu=1}^{2N}$. If we take a strength as $|\Omega_0|=8\kappa$, then Fig. 4 changes to Fig. 5, and then we can find the sequence $\{(z_\nu,t_\nu)\}_{\nu=1}^{2N}$ in Fig. 5. Compare Figs. 4 and 5. In Fig. 5 we can make only down arrows without any up arrow. It means that the system loses energy only because of cavity decay. Namely, *we do not have to throw the driving laser to the atom in the cavity for its excitement*. We note this type emission comes from a kind of superradiance [20,23,24]. Thus, this energy-spectral property may give a mechanism for another superradiant cooling for a two-level atom in the strong-coupling regime, as well as self-organized, cooperative atoms [16,17]. As we can realize it from Figs. 2 and 5, the energy loss by cavity decay gets large as the strength $|\Omega_0(t)|$ of the coupling grows large. We give the surfaces of $E_0^0(z,t;1)$ and $E_1^-(z,t;1)$ as functions of (z,t) in Figs. 6–8.

B. In the case $d\geq 2$

We assume $d\geq 2$ in this subsection. Let m and n be nonnegative integers satisfying $m<d\leq n$. We set a positive number $C_{mn}^0(d)$ as

$$C_{mn}^0(d):=\begin{cases}\dfrac{(n-d)!}{n!}(n-m)|\Delta_c|\{|\Delta_c|(n-m)+K_d\} & \text{if } \varepsilon_1-\varepsilon_0\geq\Delta-d\Delta_c\\[2ex] \dfrac{n!}{(n+d)!}(n+d-m)|\Delta_c|\{|\Delta_c|(n+d-m)+K_d\} & \text{if } \varepsilon_1-\varepsilon_0<\Delta-d\Delta_c,\end{cases}$$

where $K_d:=\varepsilon_1-\varepsilon_0+d\Delta_c-\Delta$. Then, we define three domains $\mathcal{D}_{mn}^{\rm wc}(d)$, $\mathcal{D}_{mn}^{\rm sc}(d)$, and $\mathcal{D}_{mn}^{\rm cr}(d)$ of the space-time in the following: the domain $\mathcal{D}_{mn}^{\rm wc}(d)$ for the weak-coupling regime is given by $\mathcal{D}_{mn}^{\rm wc}(d):=\{(z,t)\,|\,|\Omega(z,t)|^2<C_{mn}^0\}$, and the domain $\mathcal{D}_{mn}^{\rm sc}(d)$ for the strong-coupling regime by $\mathcal{D}_{mn}^{\rm sc}(d):=\{(z,t)\,|\,|\Omega(z,t)|^2>C_{mn}^0(d)\}$. The domain $\mathcal{D}_{mn}^{\rm cr}(d)$ for the critical regime is defined by $\mathcal{D}_{mn}^{\rm cr}(d):=\{(z,t)\,|\,|\Omega(z,t)|^2=C_{mn}^0(d)\}$.

In the case $d\geq 2$ we can show the following theorem: the domain $\mathbb{D}_{mn}^{\rm wc}(d)$ is equal to the domain $\mathcal{D}_{mn}^{\rm wc}(d)$, and the domain $\mathbb{D}_{mn}^{\rm sc}(d)$ is equal to the domain $\mathcal{D}_{mn}^{\rm sc}(d)$, i.e., $\mathbb{D}_{mn}^{\rm wc}(d)=\mathcal{D}_{mn}^{\rm wc}(d)$ and $\mathbb{D}_{mn}^{\rm sc}(d)=\mathcal{D}_{mn}^{\rm sc}(d)$, for every m and n with m

$<d\leq n$. Namely, the energy level crossing takes place as

$$E^0_m(z,t;d) < E^-_n(z,t;d) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}^{\rm wc}_{mn}(d), \tag{3.11}$$

$$E^0_m(z,t;d) = E^-_n(z,t;d) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}^{\rm cr}_{mn}(d), \tag{3.12}$$

$$E^0_m(z,t;d) > E^-_n(z,t;d) \quad \text{if and only if } (z,t) \text{ in } \mathcal{D}^{\rm sc}_{mn}(d). \tag{3.13}$$

We will give a proof of this theorem in Appendix B 2. Instead, here, we note some facts.

We have the inequality

$$C^0_{0n}(d) = \Delta_c^2 \frac{n}{(n-1)\cdots(n-d+1)} + |\Delta_c| \frac{K_d}{(n-1)\cdots(n-d+1)} < \Delta_c^2 n + |\Delta_c| K \leq C^0_{0n},$$

because $d\Delta_c < \Delta_c < 0$. Thus, it follows from this inequality that $\mathcal{D}^{\rm sc}_{0n}(1) \subset \mathcal{D}^{\rm sc}_{0n}(d)$. It means that *nonlinear coupling* (i.e., $d\geq 2$) gives rise to the DELC *more easily* than *linear coupling* (i.e., $d=1$) does.

We have to note that in the case where $d\geq 2$, there is not always a ground state of $H_0(z,t;d)$. Namely, there is a case that the minimum energy of $H_0(z,t;d)$ does *not* exist. To see this fact, we assume $\varepsilon_1-\varepsilon_0\approx\Delta-d\Delta_c$ for simplicity. Then, we have

$$E^-_n(z,t;d) - E^-_{n+1}(z,t;d) = \Delta_c + |\Omega(z,t)|\sqrt{n(n-1)\cdots(n-d+2)} \times \{\sqrt{n+1} - \sqrt{n-d+1}\} > \Delta_c + |\Omega(z,t)|\sqrt{n(n-1)\cdots(n-d+2)} \times \{\sqrt{n+1} - \sqrt{n-1}\} = \Delta_c + 2|\Omega(z,t)| \frac{\sqrt{(n-1)\cdots(n-d+2)}}{\sqrt{1+1/n} + \sqrt{1-1/n}} > \Delta_c + |\Omega(z,t)|\sqrt{(d-1)!},$$

where we used the inequalities $n>d\geq 2$ and $\sqrt{1+1/n}+\sqrt{1-1/n}\geq 2$. The last inequality says that the Hamiltonian $H_0(z,t;d)$ does *not* have a ground state for (z,t) satisfying $|\Omega(z,t)|>|\Delta_c|/\sqrt{(d-1)!}$ because

$$\cdots < E^-_{n+1}(z,t;d) < E^-_n(z,t;d) < \cdots < E^-_{d+1}(z,t;d)$$

in the case where $d\geq 2$. Here we note $\Delta_c<0$.

IV. SUPERRADIANT GROUND-STATE ENERGY IN THE CASE $\alpha\equiv 0$

As we knew at the end of Sec. III, once the DELC takes place in the case $d\geq 2$, there is *every* possibility that $H_0(z,t;d)$ is not bounded from below. Thus, to make sure of the existence of the superradiant ground-state energy, we consider only the case $d=1$ in this section.

Set a positive number Θ_n for $n=1,2,\ldots$ as

$$\Theta_n := \begin{cases} 2n\Delta_c^2 + |\Delta_c|\sqrt{4n^2\Delta_c^2 + K^2} & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - \Delta_c \\ 2(n+1)\Delta_c^2 + |\Delta_c|\sqrt{4(n+1)^2\Delta_c^2 + K^2} & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - \Delta_c, \end{cases}$$

where $K=\varepsilon_1-\varepsilon_0+\Delta_c-\Delta$. Using this number, we define a domain $\mathcal{G}_n(1)$ of the space-time by

$$\mathcal{G}_n(1) := \{(z,t) | \sqrt{\Theta_n} < |\Omega(z,t)| < \sqrt{\Theta_{n+1}}\}$$

for each $n=1,2,\ldots$. We can prove the following theorem:

(i) When a space-time point (z,t) is in $\mathcal{G}_n(1)$, the point (z,t) is always in $\mathcal{D}^{\rm sc}_{0n}(1)$. Namely, the DELC takes place for that point (z,t) in $\mathcal{G}_n(1)$.

(ii) The superradiant ground-state energy inf Spec$[H_0(z,t;1)]$ appears as

$$\inf \mathrm{Spec}[H_0(z,t;1)] = \min\{E^-_n(z,t;1), E^-_{n+1}(z,t;1)\} \tag{4.1}$$

for (z,t) in $\mathcal{G}_n(1)$.

Since we have already obtained the theorems on the DELC in Sec. III, it is easy to demonstrate the above theorem on the existence of superradiant ground state. Part (i) follows from the following easy inequalities: $C^0_{0n}<2n\Delta_c^2+|\Delta_c|K\leq\Theta_n$ if $K\geq 0$ and $C^0_{0n}<2(n+1)\Delta_c^2+|\Delta_c|K\leq\Theta_n$ if $K<0$.

Before proving part (ii) let the symbol $\sharp$ denote either $>$, $=$, or $<$. We can demonstrate part (ii) as follows. For a nonnegative number r, define a function $g(r)$ by $g(r):=|\Delta_c|r+L-(1/2)\sqrt{K^2+4|\Omega(z,t)|^2 r}$, where $L=(\varepsilon_0+\varepsilon_1+\Delta_c-\Delta)$. Then, for every positive number r the expression $g'(r)\sharp 0$ is equivalent to the expression $\Delta_c^2[K^2+4|\Omega(z,t)|^2 r] \sharp |\Omega(z,t)|^4$. Hence it follows from this that

$$g'(r) \sharp 0 \Leftrightarrow r \sharp r_0 := \frac{|\Omega(z,t)|^2}{4\Delta_c^2} - \frac{K^2}{4|\Omega(z,t)|^2}, \qquad (4.2)$$

provided that $r_0 > 0$. Here "statement A" $\Leftrightarrow$ "statement B" means that statement A *holds if and only if* statement B *holds*. Namely, statement A $\Leftrightarrow$ statement B stands for the *equivalence* of statement A and statement B. It is evident that the number r_0 is positive if and only if the inequality $|\Omega(z,t)|^2 > |\Delta_c K|$ holds. By equivalence (4.2) together with this fact, we have the implication

$$|\Omega(z,t)|^2 > |\Delta_c K| \Rightarrow \inf_{r\geq 0} g(r) = g(r_0), \qquad (4.3)$$

where "hypothesis" $\Rightarrow$ "conclusion" means that *if* hypothesis *holds, then* conclusion *holds*. That is, hypothesis *implies* conclusion. Simple calculation leads to the equivalence

$$n \leq \frac{|\Omega(z,t)|^2}{4\Delta_c^2} - \frac{K^2}{4|\Omega(z,t)|^2} < n+1$$
$$\Leftrightarrow \begin{cases} 0 \leq |\Omega(z,t)|^4 - 4n\Delta_c^2|\Omega(z,t)|^2 - \Delta_c^2 K^2 \\ |\Omega(z,t)|^4 - 4(n+1)\Delta_c^2|\Omega(z,t)|^2 - \Delta_c^2 K^2 < 0. \end{cases} \qquad (4.4)$$

We define two functions $g_j(r)$, with $j=1,2$, for positive number r by $g_1(r) := r^2 - 4n\Delta_c^2 r - \Delta_c^2 K^2$ and $g_2(r) := r^2 - 4(n+1)\Delta_c^2 r - \Delta_c^2 K^2$. Then, setting r_{j0} as $r_{10} := 2n\Delta_c^2 + |\Delta_c|\sqrt{K^2+4n^2\Delta_c^2}$ and $r_{20} := 2(n+1)\Delta_c^2 + |\Delta_c|\sqrt{K^2+4(n+1)^2\Delta_c^2}$, respectively, we have $g_1(r) > g_1(r_{10}) = 0$ if $r > r_{10}$ and $g_2(r) < g_2(r_{20}) = 0$ if $0 < r < r_{20}$. Set a non-negative number θ_n for each $n=1,2,\ldots$ as $\theta_n := 2n\Delta_c^2 + |\Delta_c|\sqrt{K^2+4n^2\Delta_c^2}$. Then, we have $\theta_n \geq |\Delta_c K|$. Thus, inserting $|\Omega(z,t)|^2$ into r of the above inequalities, we reach the implication

$$\theta_n < |\Omega(z,t)|^2 < \theta_{n+1} \Rightarrow \text{RHS of equivalence (4.4) is satisfied.} \qquad (4.5)$$

Combining results (4.3)–(4.5), we obtain the implication

$$\theta_n < |\Omega(z,t)|^2 < \theta_{n+1} \Rightarrow g(n) \text{ or } g(n+1) \text{ is located nearest } \inf_{r\geq 0} g(r). \qquad (4.6)$$

Noting that $E_n^-(z,t;1) = g(n)$ if $K \geq 0$ and $E_n^-(z,t;1) = g(n+1)$ if $K < 0$, and $\Theta_n = \theta_n$ if $K \geq 0$ and $\Theta_n = \theta_{n+1}$ if $K < 0$, we obtain part (ii).

V. STABILITY OF CIAC AND DELC IN THE CASE $\alpha \not\equiv 0$

In this section we consider the case where $d=1$ too. We show the stability of the DELC under the effect of the generalized energy operator $\alpha(z,t)W(z,t)$ of the pump field for sufficiently small strength $|\alpha(z,t)|$. Our $\alpha(z,t)W(z,t)$ includes the energy operator of the pump field, $i\alpha(a-a^\dagger)$, of course. Moreover, as stated in Sec. II, it may also include the energy operator of the pump field plus the counter-rotating terms:

$$\alpha(z,t)W(z,t) = i\alpha(z,t)\{\alpha_1(a-a^\dagger) + \alpha_2(\sigma_{01}a - \sigma_{10}a^\dagger)\},$$

where $0 \leq \alpha_1,\ \ \alpha_2 \leq 1$.

In the same way we followed in Sec. III, through the well-known identification as in Ref. [12], $H(\Omega,\alpha;1)$ reads $H_\alpha(z,t;1) := H_0(z,t;1) + \alpha(z,t)W(z,t)$. We recall $\alpha(z,0)=0$ for all the position z.

For the Hilbert space representing the state space in which $H_\alpha(z,t;d)$ acts, we denote the inner product of the Hilbert space by $\langle\Psi|\Phi\rangle$.

For every positive number ϵ, we denote $C_\epsilon(z,t)$ as

$$C_\epsilon(z,t) := (1+\epsilon)\left\{1 + \left(\frac{\epsilon}{1+\epsilon}\right)^2(\varepsilon_0 + |\varepsilon_1 - \Delta|) + \left(\frac{1+\epsilon}{\epsilon}\right)^2\frac{|\Omega(z,t)|^2}{4\Delta_c^2}\right\}^{1/2}.$$

Set a positive number $C_{0n}^0[\theta]$ for each natural number n and a non-negative number θ as

$$C_{0n}^0[\theta] := \begin{cases} (\theta-\Delta_c)^2 n + (\theta-\Delta_c)(\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - \Delta_c \\ (\theta-\Delta_c)^2(n+1) + (\theta-\Delta_c)(\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - \Delta_c, \end{cases}$$

and set a positive number $C[n]$ for each natural number n as

$$C[n] := \begin{cases} \Delta_c^2 n + \dfrac{\varepsilon_0(\varepsilon_1+\Delta_c-\Delta)}{n} - \Delta_c(\varepsilon_0+\varepsilon_1+\Delta_c-\Delta) & \text{if } \varepsilon_1-\varepsilon_0 \geq \Delta-\Delta_c \\ \Delta_c^2(n+1) + \dfrac{\varepsilon_0(\varepsilon_1+\Delta_c-\Delta)}{n+1} - \Delta_c(\varepsilon_0+\varepsilon_1+\Delta_c-\Delta) & \text{if } \varepsilon_1-\varepsilon_0 < \Delta-\Delta_c. \end{cases}$$

Moreover, for every positive function $f(z,t)$ and positive numbers b and ϵ', we define a domain $\mathcal{D}(\epsilon,\epsilon';b,f)$ of the space-time as

$$\mathcal{D}(\epsilon,\epsilon';b,f) := \left\{(z,t)\middle||\alpha(z,t)| < \frac{\epsilon'}{2b} \text{ and } |\alpha(z,t)|[bC_\epsilon(z,t)+f(z,t)] < \frac{\epsilon'(1+\epsilon)}{2}\right\}.$$

Also the domain $\mathcal{D}_{0n}^{\rm sc}(1;\theta)$ of the space-time is defined by

$$\mathcal{D}_{0n}^{\rm sc}(1;\theta) := \{(z,t)||\Omega(z,t)|^2 > C_{0n}^0[\theta] \text{ and } |\Omega(z,t)|^2 > C[n]\}.$$

We prove the following theorem concerning the stability of the DELC: let our $W(z,t)$ satisfy conditions (A1)–(A3):

(A1) For every space-time point (z,t), $W(z,t)$ is a symmetric operator so that it can act on all states on which $H_0(z,0;1)$ acts.

(A2) There is a positive constant b_1 so that

$$\langle W(z,t)\Psi|W(z,t)\Psi\rangle^{1/2} \leq b_1\langle H_0(z,0;1)\Psi|H_0(z,0;1)\Psi\rangle^{1/2} + b_2(z,t)\langle\Psi|\Psi\rangle^{1/2} \qquad (5.1)$$

for all states Ψ on which $H_0(z,0;1)$ acts and every space-time point (z,t), where $b_2(z,t)$ is some positive function of (z,t).

(A3) There is a constant ϵ with $0<\epsilon<1$ so that $|\alpha(z,t)| < \{b_1(1+\epsilon)\}^{-1}$ for all the space-time points (z,t).

Then, the Hamiltonian $H_\alpha(z,t;1)$ has an eigenvalue $\mathcal{E}_n^\natural(z,t;1)$ near $E_n^\natural(z,t;1)$ for each non-negative integer n, where $\natural=0,\pm$. Moreover, there is a constant κ_0 with $0<\kappa_0<1/4$ so that the DELC takes place between the eigenvalues $\mathcal{E}_0^0(z,t;1)$ and $\mathcal{E}_n^-(z,t;1)$ in the process from the space-time point $(z,0)$ to the space-time point (z_*,t_*), provided that the latter point (z_*,t_*) is in $\mathcal{D}(\epsilon,\kappa_0;b_1,b_2)\cap\mathcal{D}_{0n}(1;\theta)$ for a number θ with $\theta>0$. Therefore, $H_\alpha(z_*,t_*;1)$ has the superradiant ground-state energy.

This theorem says that the CIAC by Horak *et al.* [12] is stable under the effect of the pump field without and with an error term such as the counter-rotating terms, provided that the strength $|\alpha(z,t)|$ is sufficiently small compared with the coupling strength $|\Omega(z,t)|$. Moreover, the DELC is also stable.

We will give a proof of the theorem in Appendix C because we use the mathematical techniques of regular perturbation theory and the proof gets a little too long.

VI. CONCLUSION

We have shown that the system of a two-level atom coupled with a laser in a cavity has the DELC in the process that the atom-cavity interaction of the system undergoes changes between the weak-coupling regime and the strong one. We have also shown that, even if there is an error term such as counter-rotating terms, the DELC as well as the CIAC itself is stable when the strength of the energy of the generalized pump field with an error term is small compared with the coupling strength. By using the DELC, we have found the following two possibilities in (mathematical) theory for the CIAC. We can use a laser only for controlling the strength of the atom-cavity interaction without the need for another laser for driving the atom to the excited state. Moreover, we can obtain much larger energy loss caused by cavity decay, if we obtain the cavity that implements the domain $\mathcal{D}_{0n}^{\rm sc}(1)$ of the space-time. Based on these results, we can say that we lay mathematical foundations for the concept of another superradiant cooling in addition to that proposed by Domokos and Ritsch [16,17]. Adding the mathematical foundations to the idea of the CIAC by Horak *et al.* [12], we can also say that the process of our superradiant cooling requires only cavity decay and control of the position of the atom, without atomic absorption and emission of photons [46]. Therefore, whether the mechanism of our superradiant cooling can primarily be demonstrated or not depends on whether we can make such a fine cavity that the spatiotemporal domain $\mathcal{D}_{01}^{\rm sc}(1)$ for the strong-coupling regime can be implemented or not [6–9,15,34,47–53].

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APPENDIX A: THE EIGENVALUE PROBLEM FOR $H_0(z,t;d)$

In this appendix, to solve the eigenvalue problem $H_0(z,t;1)\Psi(g)=E\Psi(g)$, we adopt the way that we followed in Secs. 3 and 6 of Ref. [32] into our calculations. Set $\lambda := i\Omega(z,t)$ for simplicity.

Let $n<d$. For a complex constant g set $\Psi(g) := \binom{0}{ga^{\dagger n}\psi_0}$, where ψ_0 is the vacuum state of the photon field of our laser. It follows immediately that the condition $H(d;\Omega,0)\Psi(g) = E\Psi(g)$ is equivalent to the condition $E=\omega n$.

Let $n\geq d$ now. Set $\Psi(g) := \binom{a^{\dagger n-d}\psi_0}{ga^{\dagger n}\psi_0}$ this time. In the same way as in Ref. [32], we conclude that the condition $H(d;\Omega,0)\Psi(g)=E\Psi(g)$ is equivalent to the conditions

$$\begin{cases} (n-d)\omega + \lambda^* \binom{n}{d} d\,!\, g = E - \mu, \\ \lambda + ng\omega = Eg. \end{cases}$$

Solving these equations, we obtain

$$g = \frac{d\omega - \mu \pm \sqrt{(\mu - d\omega)^2 + 4\binom{n}{d} d\,!\,|\lambda|^2}}{2\lambda^* \binom{n}{d} d!}$$

and

$$E = n\omega + \frac{\mu - d\omega}{2} \pm \frac{1}{2}\sqrt{(\mu - d\omega)^2 + 4\binom{n}{d} d\,!\,\lambda^2}.$$

APPENDIX B: PROOFS OF THEOREMS IN SEC. III

We prove our two theorems in Sec. III on the DELC in this appendix.

1. In the case $d=1$

To make the calculations below simple, we recall that $K = \varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta$.

We give a proof of crossings (3.4)–(3.6) first. Let us suppose $K \geq 0$ now. It is easy to show the equation

$$E_0^0(z,t;1) - E_n^-(z,t;1) = \Delta_c n - \frac{1}{2}K + \Upsilon_n(z,t;1), \quad \text{(B1)}$$

where $\Upsilon_n(z,t;1)$ is the generalized Rabi frequency (3.3). After multiplying both sides of the expression $|\Omega(z,t)|^2 \sharp \Delta_c^2 n - \Delta_c K$ by $4n$, add the term K^2 to both sides of the multiplied expression. Then, we know that the expression $|\Omega(z,t)|^2 \sharp \Delta_c^2 n - \Delta_c K$ is equivalent to the expression $K^2 + 4n\Omega(z,t)^2 \sharp 4(\Delta_c^2 n^2 - \Delta_c K n + K^2/4)$. Since $-\Delta_c n + K/2 \geq 0$, we can take the square root of the last expression; thus, it is equivalent to $2\Upsilon_n(z,t;1) \sharp 2(-\Delta_c n + K/2)$. Therefore, Eq. (B1) says that

$$E_0^0(z,t;1) - E_n^-(z,t;1) \sharp 0 \Leftrightarrow |\Omega(z,t)|^2 \sharp \Delta_c^2 n - \Delta_c K. \quad \text{(B2)}$$

Equivalence (B2) brings crossings (3.4)–(3.6).

Let us suppose $K<0$ now. Then, in the same way we followed in the case $K>0$, we have the equivalence

$$E_0^0(z,t;1) - E_n^-(z,t;1) \sharp 0 \Leftrightarrow \Upsilon_{n+1}(z,t;1) \sharp -\Delta_c(n+1) + \frac{1}{2}K. \quad \text{(B3)}$$

The term $-\Delta_c(n+1)+K/2$ is positive because we assumed condition (3.2) and $\Delta_c<0$. Thus, by taking the square of both sides of the right expression of equivalence (B3) and following what we did in the case $K\geq 0$, we know that the expression $E_0^0(z,t;1)-E_n^-(z,t;1)\sharp 0$ is equivalent to the expression $|\Omega(z,t)|^2 \sharp \Delta_c^2(n+1)-\Delta_c K$, which implies our desired result.

We consider the proofs of crossings (3.7) and (3.8) next. Let us suppose $d \leq m < n$. A direct calculation leads to

$$\begin{aligned} E_m^-(z,t;1) - E_n^-(z,t;1) &= \Delta_c(n-m) + \Upsilon_n(z,t;1) - \Upsilon_m(z,t;1) \\ &= (n-m)\left[\Delta_c + \frac{|\Omega(z,t)|^2}{\Upsilon_n(z,t;1)+\Upsilon_m(z,t;1)}\right]. \end{aligned} \quad \text{(B4)}$$

Since $n>m$ and $|\Delta_c| = -\Delta_c$, Eq. (B4) says that the expression $E_m^-(z,t;1)-E_n^-(z,t;1)\sharp 0$ is equivalent to the expression $|\Omega(z,t)|^2[\Upsilon_n(z,t;1)+\Upsilon_m(z,t;1)]^{-1}\sharp|\Delta_c|$. Multiplying both sides of this by $2|\Delta_c|^{-1}|\Omega(z,t)|^{-1}[\Upsilon_n(z,t;1)+\Upsilon_m(z,t;1)]$, we realize that the latter expression is equivalent to the expression $2|\Omega(z,t)|^{-1}/|\Delta_c| \sharp \sqrt{[K/|\Omega(z,t)|]^2+4n} + \sqrt{[K/|\Omega(z,t)|]^2+4m}$. Hence it follows from these equivalences that

$$E_m^-(z,t;1) - E_n^-(z,t;1) \sharp 0 \Leftrightarrow 2\frac{|\Omega(z,t)|}{|\Delta_c|} - \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m} \sharp \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4n}. \quad \text{(B5)}$$

Suppose $K\geq 0$ now. Here we note a simple inequality for non-negative numbers A, B, and C: $(\sqrt{A+B}+\sqrt{A+C})^2 \geq A+B+A+C$. Using this, we can show that the expression $4[|\Omega(z,t)|/|\Delta_c]^2 < [K/|\Omega(z,t)|]^2+4n+[K/|\Omega(z,t)|]^2+4m$ implies the expression $4[|\Omega(z,t)|/|\Delta_c]^2 < \{\sqrt{[K/|\Omega(z,t)|]^2+4n}+\sqrt{[K/|\Omega(z,t)|]^2+4m}\}^2$. Taking the square root of both sides of this inequality implies the expression, $2[|\Omega(z,t)|/|\Delta_c|] < \sqrt{[K/|\Omega(z,t)|]^2+4n}+\sqrt{[K/|\Omega(z,t)|]^2+4m}$. Therefore, we obtain the following:

$$\begin{aligned} &4\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 < \left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4n + \left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m \\ &\Rightarrow 2\frac{|\Omega(z,t)|}{|\Delta_c|} - \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2+4m} < \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2+4n}. \end{aligned} \quad \text{(B6)}$$

We define a polynomial $g_{\mathrm{wc}}(r)$ as $g_{\mathrm{wc}}(r) := 2r^2 - 2(m+n)r - K^2/|\Delta_c|^2$. Multiplying both sides of this by 2, the inequality $g_{\mathrm{wc}}([|\Omega(z,t)|/|\Delta_c|]^2)<0$ is equivalent to the inequality $4[|\Omega(z,t)|/|\Delta_c|]^4 < 4(m+n)[|\Omega(z,t)|/|\Delta_c|]^2 + 2(K/|\Delta_c|)^2$. Multiplying both sides of this inequality by $[|\Delta_c|/|\Omega(z,t)|]^2$, we reach the following:

$$g_{\mathrm{wc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) < 0 \Leftrightarrow 4\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 < \left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m + \left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4n. \quad \text{(B7)}$$

It follows from implication (B6) and equivalences (B5) and (B7) that

$$g_{\mathrm{wc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) < 0 \Rightarrow E_m^-(z,t;1) - E_n^-(z,t;1) < 0. \quad \text{(B8)}$$

Since the point r_0^{wc} defined by $r_0^{\mathrm{wc}} := (m+n)/2 + \sqrt{[(m+n)/2]^2 + K^2/(2\Delta_c^2)}$ satisfies $g_{\mathrm{wc}}(r_0^{\mathrm{wc}})=0$, we have the inequality $g_{\mathrm{wc}}(r) < g_{\mathrm{wc}}(r_0^{\mathrm{wc}})=0$ provided that $0 \le r < r_0^{\mathrm{wc}}$. This fact tells us that

$$\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 < \frac{m+n}{2} + \sqrt{\left(\frac{m+n}{2}\right)^2 + \frac{K^2}{2\Delta_c^2}} \Rightarrow g_{\mathrm{wc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) < 0. \tag{B9}$$

Therefore, we can conclude inequality (3.7) from implications (B8) and (B9).

We prove inequality (3.8) next. Multiply both sides of the inequality $[|\Omega(z,t)|/|\Delta_c|]^2 - (n-m) > [|\Omega(z,t)|/|\Delta_c|]\sqrt{[K/|\Omega(z,t)|]^2 + 4m}$ by 4, and add $[K/|\Omega(z,t)|]^2$ to both sides of the multiplied inequality. Then, we know that the inequality $[|\Omega(z,t)|/|\Delta_c|]^2 - (n-m) > [|\Omega(z,t)|/|\Delta_c|]\sqrt{[K/|\Omega(z,t)|]^2 + 4m}$ implies the inequality $4[|\Omega(z,t)|/|\Delta_c|]^2 + [K/|\Omega(z,t)|]^2 - 4(n-m) > [K/|\Omega(z,t)|]^2 + 4[|\Omega(z,t)|/|\Delta_c|]\sqrt{[K/|\Omega(z,t)|]^2 + 4m}$, which is equivalent to the inequality

$$4\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 - 4\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)\sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m} + \left\{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m\right\} > \left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4n.$$

Taking the square root of both sides of the last inequality, we reach the implication

$$\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 - (n-m) > \left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)\sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m} \Rightarrow 2\frac{|\Omega(z,t)|}{|\Delta_c|} - \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4m} > \sqrt{\left(\frac{K}{|\Omega(z,t)|}\right)^2 + 4n}. \tag{B10}$$

We define a polynomial $g_{\mathrm{sc}}(r)$ as $g_{\mathrm{sc}}(r) := r^2 - 2(m+n)r + (n-m)^2 - K^2/|\Delta_c|^2$ this time. We note that the inequality $g_{\mathrm{sc}}([|\Omega(z,t)|/|\Delta_c|]^2) > 0$ is equivalent to the inequality $[|\Omega(z,t)|/|\Delta_c|]^4 - 2(n-m)[|\Omega(z,t)|/|\Delta_c|]^2 + (n-m)^2 > [|\Omega(z,t)|/|\Delta_c|]^2\{[K/|\Omega(z,t)|]^2 + 4m\}$. Here we added $4m[|\Omega(z,t)|/|\Delta_c|]^2 + (K/|\Delta_c|)^2$ to both sides of $g_{\mathrm{sc}}([|\Omega(z,t)|/|\Delta_c|]^2) > 0$, and then the right-hand side was factorized with the factor $[|\Omega(z,t)|/|\Delta_c|]^2$. Thus, taking the square root of both sides of this inequality, we obtain the following implication:

$$g_{\mathrm{sc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) > 0 \Rightarrow \left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 - (n-m) > \frac{|\Omega(z,t)|}{|\Delta_c|}\sqrt{\left(\frac{K}{|\Delta_c|}\right)^2 + 4m}. \tag{B11}$$

It follows from implications (B10) and (B11) and equivalence (B5) that

$$g_{\mathrm{sc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) > 0 \Rightarrow E_m^-(z,t;1) - E_n^-(z,t;1) > 0. \tag{B12}$$

Since $r_0^{\mathrm{sc}} := m+n+\sqrt{4mn + K^2/\Delta_c^2}$ satisfies $g_{\mathrm{sc}}(r_0^{\mathrm{sc}})=0$, we have the inequality $g_{\mathrm{sc}}(r) > g_{\mathrm{sc}}(r_0^{\mathrm{sc}})=0$ provided that $r_0^{\mathrm{sc}} < r$. This fact tells us that

$$\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2 > m+n+2\sqrt{mn + \frac{K^2}{4\Delta_c^2}} \Rightarrow g_{\mathrm{sc}}\left(\left(\frac{|\Omega(z,t)|}{|\Delta_c|}\right)^2\right) > 0. \tag{B13}$$

Therefore, we can conclude inequality (3.8) from implications (B12) and (B13).

In the case where $K < 0$, we have

$$E_m^-(z,t;1) - E_n^-(z,t;1) = (n-m)\left[\Delta_c + \frac{\Omega(z,t)^2}{\Upsilon_{n+1}(z,t;;1) + \Upsilon_{m+1}(z,t;;1)}\right].$$

Hence it follows from this that we obtain crossings (3.7) and (3.8) by using $m+1$ and $n+1$ instead of m and n, respectively, in the above argument for the case where $K \ge 0$.

2. In the case $d \ge 2$

Before proving our desired statements, we recall that $K_d = \varepsilon_1 - \varepsilon_0 + d\Delta_c - \Delta$ for simplicity of calculation. It is easy to check that

$$E_m^0(z,t;d) - E_n^-(z,t;d) = \begin{cases} -\Delta_c(m-n) - \dfrac{K_d}{2} + \Upsilon_n(z,t;d) & \text{if } K_d \ge 0 \\ -\Delta_c(m-n-d) - \dfrac{K_d}{2} + \Upsilon_{n+d}(z,t;d) & \text{if } K_d < 0, \end{cases}$$

which implies the following equivalence:

$$E_m^0(z,t;d) - E_n^-(z,t;d) \,\sharp\, 0 \Leftrightarrow \begin{cases} \Upsilon_n(z,t;d) \,\sharp\, -\Delta_c(n-m) + \dfrac{K_d}{2} & \text{if } K_d \ge 0 \\ \Upsilon_{n+d}(z,t;d) \,\sharp\, -\Delta_c(n+d-m) + \dfrac{K_d}{2} & \text{if } K_d < 0. \end{cases} \tag{B14}$$

We see the energy level crossing in the case where $K_d \ge 0$ first. Thus, suppose $K_d \ge 0$. Since $|\Delta_c| = -\Delta_c$, multiplying both sides of the expression $|\Omega(z,t)|^2 \,\sharp\, C_{mn}^0(d)$ by $n!/(n-d)!$, we know that the expression is equivalent to the expression $|\Omega(z,t)|^2 n!/(n-d)! \,\sharp\, \Delta_c^2(n-m)^2 - \Delta_c(n-m)K_d$. Adding $K_d^2/4$ to both sides of the multiplied expression and factorizing the left-hand side with 1/4, we know that the expression is equivalent to the expression $[K_d^2 + 4|\Omega(z,t)|^2 n!/(n-d)!]/4 \,\sharp\, \Delta_c^2(n-m)^2 - \Delta_c(n-m)K_d + K_d^2/4$. Taking the square root of both sides of this expression, we obtain the equivalence

$$|\Omega(z,t)|^2 \sharp\, C_{mn}^0(d) \Leftrightarrow \Upsilon_n(z,t;1) \sharp -\Delta_c(n-m)+\frac{K_d}{2} \tag{B15}$$

since $\Delta_c<0$, $m<n$, and $0\leq K_d$. Finally, by equivalences (B14) and (B15), we reach the equivalence

$$E_m^0(z,t;d)-E_n^-(z,t;d) \sharp\, 0 \Leftrightarrow |\Omega(z,t)|^2 \sharp\, C_{mn}^0(d), \tag{B16}$$

which says that the desired energy level crossing takes place when $K_d\geq 0$.

We see the energy level crossing in the case where $K_d<0$ next. Let us suppose $K_d<0$ now. Note the inequalities $-\Delta_c(n+d-m)+K_d/2\geq(\varepsilon_1-\varepsilon_0-d\Delta_c-\Delta)/2 >(\varepsilon_1-\varepsilon_0-\Delta_c-\Delta)/2>(\varepsilon_1-\varepsilon_0-\Delta)/2>0$ because of assumption (3.2) and the condition $-\Delta_c>0$. Thus, in the same way we had equivalence (B15), we obtain the equivalence

$$|\Omega(z,t)|^2 \sharp\, C_{mn}^0(d) \Leftrightarrow \Upsilon_{n+d}(z,t;d) \sharp -\Delta_c(n+d-m)+\frac{K_d}{2}. \tag{B17}$$

Then, equivalences (B14) and (B17) bring equivalence (B16), which secures our statement in the case $K_d<0$.

APPENDIX C: A PROOF OF THEOREM IN SEC. V

In this appendix we demonstrate the theorem on the stability of the CIAC and the DELC, stated in Sec. V. To show the stability we employ the method for proving Theorem 4.3(v) of Ref. [32]. Set H_I as $H_\mathrm{I}:=H_0(z,t;1)-H_0(z,0;1)$ for simplicity. The symbol H_0 stands for $H_0(z,0;1)$ from now on. Here we recall $\Omega(z,0)=0$ for every position z.

In the same way that we proved Lemma 4.2 of Ref. [32], we can estimate $\langle W(z,t)\Psi|W(z,t)\Psi\rangle$ from above by using $\langle H_0(z,t;d)\Psi|H_0(z,t;d)\Psi\rangle$ and $\langle\Psi|\Psi\rangle$. Using the canonical commutation relation $[a,a^\dagger]=1$ leads to the equation

$$\langle H_\mathrm{I}\Psi|H_\mathrm{I}\Psi\rangle=|\Omega(z,t)|^2(\langle\Psi|a^\dagger a|\Psi\rangle+\langle\sigma_{11}\Psi|\sigma_{11}\Psi\rangle),$$

which easily implies the inequality

$$\langle H_\mathrm{I}\Psi|H_\mathrm{I}\Psi\rangle\leq|\Omega(z,t)|^2(\langle\Psi|a^\dagger a|\Psi\rangle+\langle\Psi|\Psi\rangle).$$

By using the Schwarz inequality and the inequality $XY \leq(\eta X+Y/4\eta)^2$ for every $X,Y\geq 0$ and arbitrary $\eta>0$, we obtain the inequality

$$\langle\Psi|a^\dagger a|\Psi\rangle\leq[\eta\langle a^\dagger a\Psi|a^\dagger a\Psi\rangle^{1/2}+\langle\Psi|\Psi\rangle^{1/2}/(4\eta)]$$

for arbitrary $\eta>0$. Simple calculation yields

$$\langle H_0\Psi|H_0\Psi\rangle=\Delta_c^2\langle a^\dagger a\Psi|a^\dagger a\Psi\rangle+(\varepsilon_1-\Delta)\langle\sigma_{11}\Psi|\sigma_{11}\Psi\rangle +\varepsilon_0\langle\sigma_{00}\Psi|\sigma_{00}\Psi\rangle,$$

so that

$$\langle a^\dagger a\Psi|a^\dagger a\Psi\rangle=(1/\Delta_c^2)\{\langle H_0\Psi|H_0\Psi\rangle-(\varepsilon_1-\Delta)\langle\sigma_{11}\Psi|\sigma_{11}\Psi\rangle -\varepsilon_0\langle\sigma_{00}\Psi|\sigma_{00}\Psi\rangle\}.$$

Combining these inequalities and setting η as $\eta :=|\Delta_c|\delta/[\sqrt{2}|\Omega(z,t)|]$ for arbitrary $\delta>0$, we reach the inequality

$$\begin{aligned}\langle H_\mathrm{I}\Psi|H_\mathrm{I}\Psi\rangle^{1/2} &\leq|\Omega(z,t)|\left\{2\eta^2\langle a^\dagger a\Psi|a^\dagger a\Psi\rangle+\left(\frac{1}{8\eta^2}+1\right)\langle\Psi|\Psi\rangle\right\}^{1/2}\\ &\leq\{\delta^2\langle H_0\Psi|H_0\Psi\rangle+\Theta(\delta;\Omega)^2\langle\Psi|\Psi\rangle\}^{1/2}\\ &\leq\delta\langle H_0\Psi|H_0\Psi\rangle^{1/2}+\Theta(\delta;\Omega)\langle\Psi|\Psi\rangle^{1/2},\end{aligned} \tag{C1}$$

where

$$\Theta(\delta;\Omega)=\sqrt{1+\delta^2(\varepsilon_0+|\varepsilon_1-\Delta|)+\frac{|\Omega(z,t)|^2}{4\Delta_c^2\delta^2}}.$$

Applying the triangle inequality $\langle(A+B)\Psi|(A+B)\Psi\rangle^{1/2} \leq\langle A\Psi|A\Psi\rangle^{1/2}+\langle B\Psi|B\Psi\rangle^{1/2}$ to the term $\langle[H_0(z,t;1) -H_\mathrm{I}]\Psi|[H_0(z,t;1)-H_\mathrm{I}]\Psi\rangle^{1/2}$, we have the inequality

$$\langle H_0\Psi|H_0\Psi\rangle^{1/2}\leq\langle H_0(z,t;1)\Psi|H_0(z,t;1)\Psi\rangle^{1/2} +\langle H_\mathrm{I}\Psi|H_\mathrm{I}\Psi\rangle^{1/2}. \tag{C2}$$

Inequalities (C1) and (C2) tell us that the term $\langle H_0\Psi|H_0\Psi\rangle^{1/2}$ is bounded from above as

$$\langle H_0\Psi|H_0\Psi\rangle^{1/2}\leq\langle H_0(z,t;1)\Psi|H_0(z,t;1)\Psi\rangle^{1/2} +\delta\langle H_0\Psi|H_0\Psi\rangle^{1/2}+\Theta(\delta;\Omega)\langle\Psi|\Psi\rangle^{1/2},$$

which implies the inequality

$$(1-\delta)\langle H_0\Psi|H_0\Psi\rangle^{1/2}\leq\langle H_0(z,t;1)\Psi|H_0(z,t;1)\Psi\rangle^{1/2} +\Theta(\delta;\Omega)\langle\Psi|\Psi\rangle^{1/2}. \tag{C3}$$

Set δ as $0<\delta:=\epsilon(1+\epsilon)^{-1}<1$ for arbitrary number ϵ with $0<\epsilon<1$ now. Then, combining inequalities (5.1) and (C3), we can conclude that

$$\begin{aligned}\langle W(z,t)\Psi|W(z,t)\Psi\rangle^{1/2} &\leq b_1(1+\epsilon)\langle H_0(z,t;1)\Psi|H_0(z,t;1)\Psi\rangle^{1/2}\\ &\quad+[b_1C_\epsilon(z,t)+b_2(z,t)]\langle\Psi|\Psi\rangle^{1/2}.\end{aligned} \tag{C4}$$

Thus, applying Lemma 4.1 of Ref. [32] and Theorem 6.29 in Sec. III6 of Ref. [54] to our Hamiltonian $H_\alpha(z,t;1) =H_0+H_\mathrm{I}$, we know that the Hamiltonian $H_\alpha(z,t;1)$ has an eigenvalue $\mathcal{E}_n^\natural(z,t;1)$ for each $n=0,1,2,\ldots$.

Define an operator $T(\kappa)$ as the closure of $H_0(z,t;1)+\kappa W(z,t)$ for every complex number κ. By applying Theorem 2.6 and Remark 2.7 in Sec. VII 2 of Ref. [54] to inequality (C4), the operator $T(\kappa)$ is an analytic family of type (A) for every $\kappa\in\mathbb{C}$ with $|\kappa|<\{b_1(1+\epsilon)\}^{-1}$. Thus, taking $\alpha(z,t)$ as this κ, i.e., $H_\alpha(z,t;1)=T(\alpha(z,t))$, Theorem 3.9 in Sec. VII3 of Ref. [54] says that $\mathcal{E}_n^\natural(z,t;1)$ is a continuous function of (z,t) by assumption (A3), and it sits near $E_n^\natural(z,t;1)$. So, as shown in Sec. III, candidates which make the DELC are $E_n^-(z,t;1)$'s. Namely, for $H_\alpha(z,t;1)$, candidates which make the DELC are $\mathcal{E}_n^-(z,t;1)$'s.

Let E_n and $\mathcal{E}_n(\kappa)$ be eigenvalues of $H_0(z,t;1)$ and $T(\kappa)$, respectively, satisfying $\mathcal{E}_n(0)=E_n$. Following the regular perturbation theory (see Sec. XII of Ref. [55]), the eigenvalue $\mathcal{E}_n(\kappa)$ has the expression

$$\mathcal{E}_n(\kappa) = \frac{\langle \Phi_n | T(\kappa) P_n(\kappa) \Phi_n \rangle}{\langle \Phi_n | P_n(\kappa) \Phi_n \rangle} = E_n + \kappa \frac{\langle \Phi_n | W(z,t) P_n(\kappa) \Phi_n \rangle}{\langle \Phi_n | P_n(\kappa) \Phi_n \rangle} \tag{C5}$$

where $P_n(\kappa)$ is the orthogonal projection given by

$$P_n(\kappa) = -\frac{1}{2\pi i} \oint_{|E-E_n|=\tilde{\epsilon}} [T(\kappa) - E]^{-1} dE$$

with a sufficiently small $\tilde{\epsilon} > 0$, and Φ_n is a normalized eigenvector of $H_0(z,t;1)$ satisfying $H_0(z,t;1)\Phi_n = E_n \Phi_n$.

Since $W(z,t)$ is symmetric, use of the Schwarz inequality makes the inequality

$$\left| \frac{\langle \Phi_n | W(z,t) P_n(\kappa) \Phi_n \rangle}{\langle \Phi_n | P_n(\kappa) \Phi_n \rangle} \right| \le \frac{\langle W(z,t)\Phi_n | W(z,t)\Phi_n \rangle^{1/2}}{\langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle}. \tag{C6}$$

To get the right-hand side of the above inequality, we used the equations $P_n(\kappa)^2 = P_n(\kappa) = P_n(\kappa)^*$ in the denominator, and the inequality $\langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle \le 1$ in the numerator. By inequality (C4) we can estimate $\langle W(z,t)\Phi_n | W(z,t)\Phi_n \rangle^{1/2}$ as $\langle W(z,t)\Phi_n | W(z,t)\Phi_n \rangle^{1/2} \le b_1(1+\epsilon)E_n + b_1 C_\epsilon(z,t) + b_2(z,t)$. Here we note that $0 \le \langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle \to 1$ as $|\kappa| \to 0$, so that we have $1/2 < \langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle$ for sufficiently small $|\kappa|$. Thus, combining these with inequality (C6), there is a positive constant κ_0 so that

$$|\mathcal{E}_n(\kappa) - E_n| \le 2|\kappa| \{ b_1(1+\epsilon)E_n + b_1 C_\epsilon(z,t) + b_2(z,t) \} \tag{C7}$$

if $|\kappa| \le \kappa_0$.

Now we take the coupling strength $|\alpha(z,t)|$ for space-time point $(z,t) \in \mathcal{D}(\epsilon,\kappa_0;b_1,b_2)$ as the coupling parameter κ. Let us define a positive number κ_1 as $\kappa_1 := \kappa_0(1+\epsilon)$. Then, it is easy to check that $2|\alpha(z,t)| b_1(1+\epsilon) < \kappa_1$ and that $2|\alpha(z,t)| \{ b_1 C_\epsilon(z,t) + b_2(z,t) \} < \kappa_1$. Combining these inequalities with inequality (C7), we obtain the inequality

$$(1-\kappa_1)E_n - \kappa_1 \le \mathcal{E}_n(\alpha(z,t)) \le (1+\kappa_1)E_n + \kappa_1 \tag{C8}$$

for space-time point $(z,t) \in \mathcal{D}(\epsilon,\kappa_0;b_1,b_2)$. From now on, we take the κ_0 and ϵ so that $0 < \kappa_1 < 1/2$.

Let us set a number L as $L := \varepsilon_0 + \varepsilon_1 + \Delta_c - \Delta$ again. It is easy to show that $\Omega_n(z,t;1)^2 - \Xi_n(1)^2 = -\Delta_c^2 n^2 + [|\Omega(z,t)|^2 + \Delta_c^2 L] n - \varepsilon_0(\varepsilon_1 + \Delta_c - \Delta)$; thus, we obtain the equivalence

$$\Omega_n(z,t;1) = |\Omega_n(z,t;1)| > |\Xi_n(1)| \ge \Xi_n(1)$$
$$\Leftrightarrow |\Omega(z,t)|^2 > \Delta_c^2 n - \Delta_c L + \frac{\varepsilon_0(\varepsilon_1 + \Delta_c - \Delta)}{n}.$$

Hence it follows from this that $E_n^-(z,t;1)$ is negative for the point $(z,t) \in \mathcal{D}_{0n}^{\rm sc}(1;\theta)$.

In the same way we obtained equivalence (B2), we obtain the equivalence in the following. In the case $K \equiv \varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta$ is non-negative, for every $\theta \ge 0$ and each natural number n we have

$$E_0^0(z,t;1) \sharp E_n^-(z,t;1) + n\theta$$
$$\Leftrightarrow |\Omega(z,t)|^2 \sharp (\theta - \Delta_c)^2 n + (\theta - \Delta_c)K \equiv C_{0n}^0[\theta]. \tag{C9}$$

Let K be negative now. We note the inequality $(\theta-\Delta_c)\{(\theta-\Delta_c)(n+1)+K\} \ge (\theta-\Delta_c)\{-\Delta_c+K\} \ge 0$ because of condition (3.2). Thus, in the same way we followed to get equivalence (B2), for every $\theta \ge 0$ and each natural number n we have

$$E_0^0(z,t;1) \sharp E_n^-(z,t;1) + (n+1)\theta$$
$$\Leftrightarrow |\Omega(z,t)|^2 \sharp (\theta - \Delta_c)^2 (n+1) + (\theta - \Delta_c)K \equiv C_{0n}^0[\theta]. \tag{C10}$$

Thus, we obtain that $E_0^0(z,t;1) > (1+\theta)E_n^-(z,t;1) + \theta$ since $1 < 1+\theta$ and $E_n^-(z,t;1) < 0$ for every point $(z,t) \in \mathcal{D}_{0n}^{\rm sc}(1;\theta)$. We take the θ defined by the equation $\kappa_1 = \theta/(2+\theta)$ now. Then, the inequality $E_0^0(z,t;1) > (1+\kappa_1)E_n^-(z,t;1)/(1-\kappa_1) + 2\kappa_1/(1-\kappa_1)$ holds, which implies

$$(1-\kappa_1)E_0^0(z,t;1) - \kappa_1 > (1+\kappa_1)E_n^-(z,t;1) + \kappa_1 \tag{C11}$$

for every $(z,t) \in \mathcal{D}_{0n}^{\rm sc}(1;\theta)$.

Combining inequalities (C8) and (C11) leads to the inequality

$$\mathcal{E}_n^-(z,t;1) \le (1+\kappa_1)E_n^-(z,t;1) + \kappa_1 < (1-\kappa_1)E_0^0(z,t;1) - \kappa_1$$
$$< \mathcal{E}_0^0(z,t;1) \tag{C12}$$

for every $(z,t) \in \mathcal{D}(\epsilon,\kappa_0;b_1,b_2) \cap \mathcal{D}_{0n}^{\rm sc}(1;\theta)$.

Since $\mathcal{E}_n^\natural(z,t;1)$ is a continuous function of (z,t) and $\mathcal{E}_0^0(z,0;1) = E_0^0(z,0;1) < E_n^-(z,0;1) = \mathcal{E}_n^-(z,0;1)$, inequality (C12) means that the DELC takes place.

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