Quantum-information storage: A Schrödinger-picture approach

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A theory is developed to obtain the state vector for an atom-field system in which a single-photon pulse propagates in an optically dense ensemble of two-level or three-level atoms. The manner in which the field and the atoms become entangled is described, as is the entanglement of the atoms produced by the radiation pulse. Two protocols for quantum-information storage are studied. In the first protocol, the single-photon pulse is totally absorbed in an ensemble of two-level atoms that are embedded in a dielectric host. To arrive at analytical expressions for the state vector, fluctuations of atomic position are neglected and it is assumed that the atoms' transition frequencies are inhomogeneously broadened with a width that is much larger than the bandwidth of the single-photon pulse. In the second protocol, a single-photon pulse is sent into a semi-infinite medium of three-level atoms under the conditions where electromagnetic-induced transparency can occur. It is shown that the usual results of pulse compression and reduced group velocity are recovered. Moreover, an explicit expression for the state vector of the atom-field system is obtained.

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I. INTRODUCTION

The storage of quantum information in an atomic ensemble has received a great deal of attention over the past 10 years or so. Among the many protocols that have been proposed to reach this goal, three main trends can be identified. In protocols based on electromagnetic-induced transparency (EIT), an input signal propagates with reduced group velocity in a medium of three-level atoms. As long as the transparency window associated with EIT is much larger than the bandwidth of the incident pulse, the pulse propagates with negligible absorption and is adiabatically coupled to a collective atomic state coherence [1]. This class of protocols has been investigated intensively both theoretically [2] and experimentally, leading to the storage and retrieval of both discrete [3,4] and continuous [5,6] quantum variables. A second class of protocols involves the complete absorption of a single-photon pulse in an optically dense medium [7–12]. Schemes involving electric-dipole transitions [8–11] and Raman transitions [7,12] have been proposed. The idea underlying these schemes is the creation of an entangled state in the medium resulting from the complete absorption of the single-photon pulse. In effect, the excitation is shared by the atoms and stored in the medium. Although there are several proofs of principle experiments of these absorption-type protocols [11–14], all such experiments have involved classical input fields. The third class of protocols is based on a proposal of Duan, Lukin, Cirac, and Zoller (DLCZ) [15]. Unlike the others, this protocol involves postselection. By measuring a spontaneous Raman photon, one insures that the medium has been prepared in a collective excitation of ground-state coherence. This storage procedure has been demonstrated experimentally [16,17] and can also serve as a source of narrow bandwidth single photons [3].

In the present paper, we focus on the first two classes of protocols, in the limit that the input field consists of a single-photon pulse. The theory of single-photon pulse propagation is usually formulated using a quantized field version of the Maxwell-Bloch equations [(2), (8), and (15)]. This is a Heisenberg picture approach for the atomic and field operators. To arrive at a viable theory, a macroscopic average over a slice of atoms is taken, assuming that there is a sufficiently large number of atoms in the slice. In effect, this approach neglects fluctuations in atomic positions. Such fluctuations are important, for example, in Rayleigh scattering from an ensemble of randomly spaced atoms or oscillators. In this problem, however, it is often a good approximation to neglect fluctuations, for reasons that are discussed below.

The Heisenberg picture approach is especially useful when one is concerned with *either* field *or* atomic operators. The reason for this is that the field operators contain a trace over atomic state variables while the atomic operators contain a trace over field states. Often it is possible to get the equations of motion for either the field or atomic operators, separately. Thus if one wishes to follow either the evolution of the field or atomic operators separately, the Heisenberg approach is usually the most efficient way to proceed. On the other hand, if one is interested in the *correlations* between the atoms and the fields, this correlation is most easily seen in the state vector for the atom-field system. As a consequence, the Schrödinger equation approach is a natural one for understanding the atom-field correlations [18].

The Heisenberg approach leads to Maxwell's equations for the field operators, equations which contain partial derivatives of the field operators with respect to both time and space coordinates [2]. In other words, with such an approach, one is led to a solution of the problem that is local in both time and space. It is not easy to identify the global entanglement of the atoms and the field in the Heisenberg approach, although polariton states are meant to convey such entanglement on a macroscopic level. The Schrödinger-picture that we adopt allows us to follow the global entanglement on a

microscopic level. Moreover, it does not necessitate the use of an approach in which the absorbing material slices are modeled as beam splitters [19].

We consider first the case where a single-photon pulse is absorbed in a storage medium consisting of an ensemble of two-level atoms embedded in a dielectric host. The calculation is carried out for an optically allowed dipole transition, but the method works equally well for Raman transitions. To simplify the problem, we restrict the discussion to an effective one-dimensional problem. That is, we neglect all the modes of the radiation field having propagation vectors in other than the $\hat{x}$ direction; we do not consider diffraction and reradiation of the incident field into transverse field modes. The neglect of reradiation into transverse modes is valid provided that the calculation is limited to times for which $\gamma_e t \ll 1$, where γ_e is the excited-state decay rate [20]. In this limit, the energy in the field reradiated by the atoms is much less than the energy absorbed by the atoms. In other words, the process we consider is truly absorption by the atoms and not scattering. The energy lost by the field at any time is converted into excitation energy of the atoms. In addition, we require that any relaxation resulting from homogeneous broadening during the time t is negligible.

We wish to show the manner in which the atomic states become entangled with the field state and, eventually, with each other. To this end, we analyze two problems involving the absorption of the single-photon pulse. In the first example, the host medium is infinite and a single-photon pulse propagates in the medium. In the second example, we consider the more realistic case of a single-photon pulse incident on a semi-infinite dielectric. The dielectric host does not absorb the radiation; it simply provides a background medium having a constant index of refraction that also provides inhomogeneous broadening for the ensemble of two-level atoms.

It turns out that the inhomogeneous broadening of the atomic transition frequencies provides an important simplification in these problems provided that the inhomogeneous width is much larger than the single-photon pulse bandwidth. In this limit, all frequency components of the incident pulse are absorbed in an identical manner, resulting in a pulse that propagates without distortion. In the case of the homogeneous broadening, there could be significant reshaping of the radiation pulse as it propagates in the medium. Fluctuations in atomic position and transition frequency are neglected to arrive at analytical expressions for the state vector.

The need for large inhomogeneous broadening provides a more serious constraint than the need to excite the system on a short-time scale. From an experimental perspective, it is not difficult to excite atoms on time scales for which decay and homogeneous broadening can be neglected. In atomic vapors, such times are on the order of tens to hundreds of nanoseconds while, in solids, on the order of picoseconds to nanoseconds. They may even be as large as hundreds of microseconds in solid dilute systems such as rare-earth ion-doped crystals. One cannot choose an arbitrarily short pulse duration, however, since the pulse bandwidth must be much less than the inhomogeneous width of the sample, which ranges from gigahertz in thermal vapors to tetrahertz in solids. Thus, to be able to simultaneously satisfy both the short excitation time and large inhomogeneous broadening requirements, it is necessary that the inhomogeneous width be much larger than the spontaneous decay rate and any homogeneous broadening rate, a condition that often holds in both vapors and solids. If one uses Raman transitions between ground-state sublevels rather than optical transitions, these time and bandwidth constraints can all but disappear, owing to the long lifetime of the Raman coherence (assuming the Raman transitions are driven by off-resonant optical pulses).

We next turn our attention to EIT and consider a single-photon pulse incident on an ensemble of three-level atoms. The system consists of the incoming single-photon-pulsed field and a semi-infinite medium consisting solely of three-level atoms (no background dielectric) in a Λ configuration (ground-state levels 1 and 2 and excited-state level 3). A classical control field drives the 2–3 transition, the signal field drives the 1–2 transition, and the atoms are all initially in state 1. No inhomogeneous broadening need be present—the EIT transparency window ensures that each atom has the same response to the incident field. Moreover, for EIT it is no longer necessary to assume that the pulse has a temporal width that is short compared with the excited-state lifetime. In EIT, there is negligible excited-state population (i.e., negligible absorption) as the pulse propagates in the medium. All that is needed is a sufficiently large transparency window to guarantee that adiabatic conditions hold for pulse propagation.

All cooperative decay processes are ignored based on the assumption that the average separation of the two-level atoms is much larger than an optical wavelength. One might think that cooperative decay could play a role, since the requirement that the medium be optically thick coincides with a general condition for cooperative decay. We return to this topic in Sec. V. Moreover we indicate how our approach might be extended to the DLCZ protocol.

II. ABSORPTION PROTOCOL: INFINITE DIELECTRIC HOST

The first problem we consider is that of a single-photon pulse propagating in an infinite dielectric host. Embedded in the host is an ensemble of two-level atoms, each atom having ground state $|g\rangle$ and excited state $|e\rangle$ (Fig. 1). Owing to inhomogeneous broadening, these atoms have a distribution of atomic transition frequencies centered about some central frequency ω_0. That is, atom m has frequency ω_m which is detuned from the central transition frequency by an amount

$$\delta_m = \omega_m - \omega_0. \tag{1}$$

The two-level atoms are uniformly distributed in the dielectric host with the position of atom m denoted by $\mathbf{R}_m$. The (real) index of refraction of the dielectric host is denoted by n and dispersion in the dielectric host is neglected.

Initially, all the atoms are in their ground states and the field is in a one-photon state. Admittedly, this initial state is somewhat unphysical since we do not indicate the manner in which the field was injected into the medium. On the other hand, the calculation in this section introduces many of the concepts that are used in Sec. III, where the more realistic

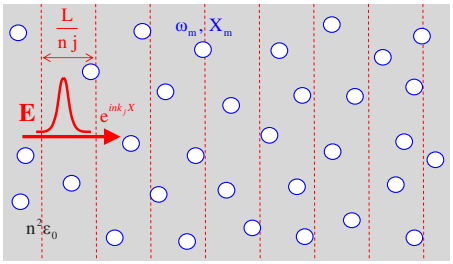

FIG. 1. (Color online) A single-photon pulse propagates to the right in an infinite dielectric host having index of refraction n and permittivity $\epsilon=n^2\epsilon_0$. Embedded in the dielectric is an ensemble of inhomogeneously broadened two-level atoms whose density is sufficiently high to ensure that many atoms are contained in a slice of wavelength $\lambda_j=L/nj$ for those wavelengths λ_j contained in the one-photon pulse. Periodic boundary conditions are used such that $nk_j=2\pi/\lambda_j=2\pi nj/L$. The pulse is totally absorbed by the medium, leading to an entangled state of the atoms.

model of a pulse incident on a semi-infinite dielectric slab is discussed.

To express the initial field state and the atom-field interaction, it is necessary to use a quantized description for the field. There exist many prescriptions for quantizing the radiation field in a dielectric [21]. Consistent with these approaches, we write the positive frequency component of the electric field as

$$\mathbf{E}^+(X)=i\sum_{j=-\infty}^{\infty}\left(\frac{\hbar\omega_j}{2\epsilon_0 AL}\right)^{1/2}\frac{\hat{\mathbf{z}}}{\sqrt{n}}e^{ink_jX}a_j, \tag{2}$$

where

$$\omega_j=|k_j|/c, \tag{3}$$

A is the cross-sectional area of the field pulse and a_j is a destruction operator for field mode j. As noted in Sec. I, we consider an effective one-dimensional problem in which the field propagates in the $\pm\hat{\mathbf{x}}$ direction with polarization $\hat{\mathbf{z}}$. We have chosen to define the k_j in terms of *vacuum* field modes, since this will prove convenient for our purposes. The quantization scheme is one in which the field is periodic over a distance $L_c=L/n$, implying that

$$k_j=2\pi j/L, \tag{4}$$

where j is an integer (positive, negative, or zero). With this definition, the quantization volume is equal to AL/n and the field modes are defined such that [21]

$$2A\int_{-L/2n}^{L/2n}\mathbf{E}_j^-(X)\cdot\mathbf{E}_{j'}^+(X)\epsilon(X)dX=\hbar\omega_j a_j^\dagger a_j\delta_{j,j'}, \tag{5}$$

where

$$\epsilon(X)=\epsilon_0 n^2(X) \tag{6}$$

is the permittivity of the medium [in this section $\epsilon(X)=\epsilon_0 n^2$, independent of X],

$$\mathbf{E}_j^+(X)=i\left(\frac{\hbar\omega_j}{2\epsilon_0 AL}\right)^{1/2}\frac{\hat{\mathbf{z}}}{\sqrt{n}}e^{ink_jX}a_j=[\mathbf{E}_j^-(X)]^\dagger, \tag{7}$$

is the positive frequency component of the electric field operator associated with mode j, and $\delta_{j,j'}$ is a Kronecker delta. For future reference, we note that the prescription for transforming from discrete to continuum modes of the field is

$$\sum_j\rightarrow\frac{L}{2\pi}\int_{-\infty}^{\infty}dk. \tag{8}$$

In dipole and rotating-wave approximation, the Hamiltonian appropriate to our atom-field system is then given by

$$H=\sum_m\frac{\hbar\omega_m}{2}\sigma_z^m+\sum_j\hbar\omega_j a_j^\dagger a_j+\sum_{j,m}\hbar g_j[e^{ink_jX_m}\sigma_+^m a_j - a_j^\dagger e^{-ink_jX_m}\sigma_-^m], \tag{9}$$

where

$$g_j=-i\left(\frac{\omega_j}{2\hbar\epsilon_0 AL}\right)^{1/2}\mu \tag{10}$$

is a coupling constant, $\sigma_\pm^m$ are raising (+) and lowering (−) operators for atom m; σ_z^m is the population difference operator, $(|e\rangle\langle e|-|g\rangle\langle g|)$, for atom m, and μ is the atomic dipole moment matrix element (assumed to be real). We stress again that we neglect any coupling to initially unoccupied field modes based on our assumption that radiation by the atoms is negligible on the time scale of the excitation dynamics.

Since there can be at most one atomic excitation starting from a state in which all atoms are in their ground states and the field is in a one-photon state, the state vector for the atom-field system can be written as

$$|\psi(t)\rangle=\sum_j b_j(t)e^{-i\omega_j t}|G;j\rangle+\sum_m c_m(t)e^{-i\omega_m t}|m;0\rangle, \tag{11}$$

where $b_j(t)$ is the probability amplitude for the field to be in mode j and all the atoms to be in their ground states, while $c_m(t)$ is the probability amplitude for the field to be in the vacuum mode and the atoms to be in a state where atom m is in its excited state and all the other atoms in their ground states. The initial state for the system is taken as

$$|\psi(0)\rangle=\sum_j b_j(0)|G;j\rangle, \tag{12}$$

a single-photon state of the field with all atoms in their ground states, normalized such that $\Sigma_j|b_j(0)|^2=1$. In practice, the $|b_j(0)|$ are chosen to correspond to a pulsed field propagating in the $\hat{x}$ direction with $k_j\approx k_0=\omega_0/c$ (recall that the field modes are defined to ensure that a mode k_j actually corresponds to a propagation constant nk_j in the medium).

It is now a simple matter to write the equations for the state amplitudes using the Schrödinger equation. One finds that the equations of motion for the relevant state amplitudes are given by

$$\dot{b}_j=ig_j\frac{1}{\sqrt{n}}\sum_m e^{-ink_jX_m}e^{i(\delta_j-\delta_m)t}c_m, \tag{13a}$$

$$\dot{c}_m = -i\frac{1}{\sqrt{n}}\sum_{j'} g_{j'} e^{ink_{j'}X_m} e^{-i(\delta_{j'}-\delta_m)t} b_{j'}, \tag{13b}$$

where

$$\delta_j = \omega_j - \omega_0 \tag{14}$$

and δ_m is given by Eq. (1). By formally solving Eq. (13b) and substituting the result in Eq. (13a), we find

$$\dot{b}_j = n^{-1} g_j \sum_{m,j'} g_{j'} e^{-in(k_j-k_{j'})X_m} e^{i(\delta_j-\delta_m)t} \int_0^t dt' e^{-i(\delta_{j'}-\delta_m)t'} b_{j'}(t'). \tag{15}$$

Equations (13) and (15) are exact, giving rise to a rather complicated entangled state of the atoms and the field. Note that, in general, field mode j is coupled to all other field modes at *earlier* times. The equations take on a simplified form only if one neglects fluctuations in both particle position and frequency. Normally, the average over particle position and frequency is carried out *after* one obtains expressions for the expectation values of quantum-mechanical operators. However, if fluctuations are neglected, one can carry out this average directly in Eq. (15). This type of average is implicit in approaches based on the Maxwell-Bloch equations. In general, for the geometry considered in this paper, fluctuations can be neglected if the number of two-level atoms in a slice of length $\lambda_0 = 2\pi/nk_0$ is much greater than unity; that is, if

$$\mathfrak{N}A\lambda_0/n \gg 1, \tag{16}$$

where $\mathfrak{N}$ is the atomic density. In the remainder of this work, we shall assume that fluctuations in particle position and frequency can be neglected.

If fluctuations in particle position and frequency are neglected, we can replace the summation over m in Eq. (15) by $\mathfrak{N}A\int_{-L/2n}^{L/2n} dX \int_{-\infty}^{\infty} W(\delta_m) d\delta_m$, where $W(\delta_m)$ represents the inhomogeneous distribution of atomic transition frequencies. As a consequence of the quantization condition (4), the integral gives a result proportional to $\delta_{j,j'}$ and one obtains

$$\dot{b}_j = -\mathfrak{N}\left(\frac{\omega_j \mu^2}{2\hbar\epsilon}\right) \int_{-\infty}^{\infty} W(\delta_m) d\delta_m \int_0^t dt' e^{i(\delta_j-\delta_m)(t-t')} b_j(t'), \tag{17}$$

where $\epsilon = n^2\epsilon_0$. The exact form of the distribution is not of critical importance; for the sake of definiteness, we choose the Gaussian distribution,

$$W(\delta_m) = \left(\frac{1}{\pi\delta_0^2}\right)^{1/2} e^{-\delta_m^2/\delta_0^2}. \tag{18}$$

Performing the integral over δ_m in Eq. (17), we find

$$\dot{b}_j = -\mathfrak{N}\left(\frac{\omega_j \mu^2}{2\hbar\epsilon}\right) \int_0^t dt' e^{-\delta_0^2(t-t')^2/4} e^{i\delta_j(t-t')} b_j(t'). \tag{19}$$

This equation is still complicated since the temporal dependence of the field amplitude is not local, a consequence of the fact that different frequency components of the field are absorbed differently by the atoms. If the inhomogeneous width δ_0 is much larger than the pulse bandwidth τ^{-1}, however, then all frequency components in the field are affected in an identical manner by the atoms. We shall assume this to be the case, $\delta_0\tau \gg 1$, allowing us to approximate

$$e^{-\delta_0^2\tau^2/4} \sim \frac{2\sqrt{\pi}}{\delta_0}\delta_D(\tau), \tag{20}$$

where δ_D is a Dirac delta function. As a consequence, each field mode decays exponentially,

$$b_j(t) = e^{-\alpha_j ct/2n} b_j(0), \tag{21}$$

with

$$\alpha_j = \frac{\sqrt{\pi}\mathfrak{N}\omega_j\mu^2 n}{\hbar\epsilon c\,\delta_0}. \tag{22}$$

[In arriving at Eq. (21), we used $\int_0^t dt'\,\delta_D(t-t') = 1/2$.]

It is important to remember that the simplified dependence given in Eq. (21) follows only if (a) fluctuations are neglected and (b) if the inhomogeneous width is much larger than the bandwidth of the one-photon pulse. The result is somewhat remarkable in that the absorption coefficient α (with $\omega_j = \omega_0$) is identical to the one would obtain in considering the steady-state *scattering* of a cw radiation field by an inhomogeneously broadened medium whose natural width is much less than the inhomogeneous width. In other words, in both the scattering of cw radiation and the absorption of a pulse of radiation, the same absorption coefficient enters. This would not be the case for a homogeneously broadened sample.

By combining Eqs. (13b) and (21), one finds that the atomic state amplitudes evolve according to

$$\dot{c}_m = -i\frac{1}{\sqrt{n}}\sum_j g_j e^{ink_jX_m} e^{-i(\delta_j-\delta_m)t} e^{-\alpha_j ct/2n} b_j(0). \tag{23}$$

To gain some insight into this equation, we go over to a continuum of field modes by replacing $b_j(0)$ with

$$b_j(t) \to \sqrt{\frac{2\pi}{L}} b(k,t), \tag{24}$$

k_j by k, and ω_j by $\omega_k = kc$. With these substitutions and the prescription (8), Eq. (23) is transformed into

$$\dot{c}_m = -i\sqrt{\frac{L}{2\pi n}} \int_{-\infty}^{\infty} dk g_k e^{inkX_m} e^{-i(\delta_k-\delta_m)t} e^{-\alpha_k ct/2n} b(k,0), \tag{25}$$

where

$$\delta_k = \omega_k - \omega_0, \tag{26}$$

$$g_k = -i\left(\frac{\omega_k}{2\hbar\epsilon_0 AL}\right)^{1/2}\mu, \tag{27}$$

and

$$\alpha_k = \frac{\sqrt{\pi}\mathfrak{N}\omega_k \mu^2 n}{\hbar \epsilon c \delta_0}. \tag{28}$$

The $b(k,0)$ are normalized such that

$$\int_{-\infty}^{\infty} dk |b(k,0)|^2 = 1. \tag{29}$$

We choose an initial frequency distribution that is peaked sharply about $k=k_0=\omega_0/c$ with characteristic width Δk, namely,

$$b(k,0) = \left(\frac{1}{\pi(\Delta k)^2}\right)^{1/4} e^{-(k-k_0)^2/2(\Delta k)^2} \tag{30}$$

and evaluate both g_k and α_k at $\omega_k=\omega_0$ in Eq. (25). When Eq. (30) is substituted into Eq. (25) and the integral over k is carried out, one finds

$$\dot{c}_m(X_m,t) = -e^{ink_0X_m} e^{i\delta_m t} \sqrt{\frac{\Delta k}{\sqrt{\pi}}} \left(\frac{\omega_0 \mu^2}{2\hbar n A \epsilon_0}\right)^{1/2} \times e^{-\alpha ct/2n} e^{-(nX_m - ct)^2(\Delta k)^2/2}, \tag{31}$$

where

$$\alpha = \frac{\sqrt{\pi}\mathfrak{N}\omega_0 \mu^2 n}{\hbar \epsilon c \delta_0}. \tag{32}$$

The spatial nonuniformity of atomic excitation, previously concealed in the time dependence, emerges explicitly in Eq. (31) through the attenuation factor $e^{-\alpha ct/2n}$ and the envelope function $e^{-(nX_m - ct)^2(\Delta k)^2/2}$. The field pulse arrives at a site X_m with reduced amplitude owing to the absorption for all times $t \lesssim nX_m/c$ and atoms located at position X_m are excited by this reduced amplitude pulse. Explicitly, one finds

$$c_m(X_m,t) = -\pi^{1/4} \left(\frac{\omega_0 \mu^2 \tau}{4\hbar n c A \epsilon_0}\right)^{1/2} e^{in(k_0+\delta_m/c)X_m} e^{-\alpha X_m/2} \times \exp\left[\frac{1}{8}(\alpha \Delta X - 2i\delta_m \tau)^2\right] \times \left[\Phi\left(\frac{t}{\sqrt{2}\tau} - u\right) + \Phi(u)\right], \tag{33}$$

where

$$u = \frac{1}{\sqrt{2}}\left[\frac{X_m}{\Delta X} - \frac{(\alpha \Delta X - 2i\delta_m \tau)}{2}\right], \tag{34}$$

$$\Delta X = 1/n\Delta k, \quad \tau = 1/c\Delta k,$$

and Φ is an error function. The pulse propagates with speed c/n in the medium and decays in a time of order $n/\alpha c$. Consequently, only those atoms having $X_m \lesssim 1/\alpha$ can be excited by the field. Moreover, even for $X_m < 1/\alpha$, only those atoms having $\delta_m \tau \lesssim 1$ are in resonance with the field; atoms having transition frequencies corresponding to $\delta_m \tau > 1$ adiabatically follow the field and return to their ground state following the passage of the pulse.

For sufficiently long times $\alpha ct/n \gg 1$, all the energy initially in the single-photon pulse has been transferred to the atoms. The state vector following absorption of the single-photon pulse is

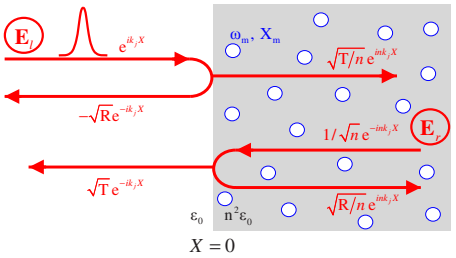

FIG. 2. (Color online) A single-photon pulse propagates to the right toward a semi-infinite dielectric host having the same properties as those in Fig. 1. Mode functions for pulses incident from both the left E_l and right E_r are shown; each of these mode functions has an incident, transmitted, and reflected branch. The two-level atoms embedded in the dielectric couple the transmitted part of E_l and the reflected part of E_r as the pulse is absorbed in the medium. Once the transmitted pulse is totally absorbed, there is left only the reflected pulse and an entangled state of the atoms.

$$|\psi(t)\rangle \sim \sum_m c_m(X_m, t \gg n\alpha/c) e^{-i\omega_m t} |m;0\rangle, \tag{35}$$

with c_m given by Eq. (33). This is the key result of this section; the atoms are left in an in an entangled state after the pulse has been totally absorbed in the medium. Using Eq. (31), it is not difficult to show that

$$\mathfrak{N}A \int_{-\infty}^{\infty} dX_m \int_{-\infty}^{\infty} d\delta_m W(\delta_m) |c_m(X_m, t \gg n\alpha/c)|^2 = 1, \tag{36}$$

all the original energy has been transferred to the atoms, within the approximations of the theory.

If $\alpha c\tau/n \ll 1$, that is, if the pulse travels several pulse widths before being absorbed, then the solution of Eq. (31) can be approximated by

$$c_m(X_m,t) = -i\left(\frac{\omega_0 \mu^2 \tau \sqrt{\pi}}{\hbar n c A \epsilon_0}\right)^{1/2} e^{in(k_0+\delta_m/c)X_m} \times \exp(-\alpha X_m/2) \exp(-\delta_m^2 \tau^2/2), \tag{37}$$

for $X_m \geq 0$. The atoms are in an entangled state with a spatial phase factor $e^{ink_0X_m}$ that has been imprinted by the field, along with an exponentially decaying factor $e^{-\alpha X_m/2}$ resulting from the absorption in the medium. It is easy to show that Eq. (37) is consistent with Eq. (36) provided $\delta_0 \tau \gg 1$.

III. ABSORPTION PROTOCOL: SEMI-INFINITE DIELECTRIC HOST

We now turn our attention to the more realistic problem of a single-photon pulse incident on a semi-infinite dielectric (Fig. 2). The quantization volume we choose has cross-sectional area A and an X-dependent permittivity given by

$$\epsilon(X) = \begin{cases} \epsilon_0 & -L/2 \le X \le 0 \\ n^2\epsilon_0 & 0 < X \le L/2n, \end{cases} \tag{38}$$

which is meant to model a uniform dielectric in the positive halfspace $X>0$. There are two sets of field modes associated with this problem, one for fields incident from the left,

$$\mathbf{E}_l^+(X) = i\sum_{j=0}^{\infty}\left(\frac{\hbar\omega_j}{2\epsilon_0 AL}\right)^{1/2} \times \hat{\mathbf{z}}a_j^l \begin{cases} e^{ik_jX} - \sqrt{R}e^{-ik_jX} & -L/2 \le X \le 0 \\ \sqrt{\dfrac{T}{n}}e^{ink_jX} & 0 < X \le L/2n, \end{cases} \tag{39}$$

and one for fields incident from the right,

$$\mathbf{E}_r^+(X) = i\sum_{j=0}^{\infty}\left(\frac{\hbar\omega_j}{2\epsilon_0 AL}\right)^{1/2} \times \hat{\mathbf{z}}a_j^r \begin{cases} \sqrt{T}e^{-ik_jX} & -L/2 \le X \le 0 \\ \sqrt{\dfrac{1}{n}}(e^{-ink_jX} + \sqrt{R}e^{ink_jX}) & 0 < X \le L/2n. \end{cases} \tag{40}$$

The subscripts l and r refer to the fields incident from the left and right, respectively, while the reflection and transmission coefficients are given by

$$\sqrt{R} = \frac{n-1}{n+1}, \quad \sqrt{T} = \frac{2\sqrt{n}}{n+1}, \tag{41}$$

with $R+T=1$. These mode functions satisfy the quantization condition (5) provided that $k_j = 2\pi j/L$, where j is a positive integer.

As in Sec. II, the dielectric host contains an ensemble of inhomogeneously broadened two-level atoms. A one-photon pulsed field is incident on the dielectric from the left. Part of the field is reflected and part is transmitted into the dielectric and absorbed by the two-level atoms. After a sufficiently long time, the only field present is the one that has been reflected from the dielectric surface. This *final field pattern* cannot be expanded solely in terms of the l modes. In other words, although the r modes were initially unpopulated, they must be populated at a later time. The two-level atoms couple the l and r modes.

The calculation proceeds as in Sec. II, except that an additional sum over $\beta=l,r$ is needed. In other words, the state vector is expanded as

$$|\psi(t)\rangle = \sum_{\beta=l,r}\sum_j b_j^\beta(t)e^{-i\omega_j t}|G;\beta,j\rangle + \sum_m c_m(t)e^{-i\omega_m t}|m;0\rangle, \tag{42}$$

while the initial-state vector is

$$|\psi(0)\rangle = \sum_j b_j^l(0)|G;l,j\rangle, \tag{43}$$

corresponding to a single-photon state of the field incident from the left and all atoms in their ground states.

The equations of motion for the relevant state amplitudes are given by

$$\dot{b}_j^l = ig_j\sqrt{\frac{T}{n}}\sum_m e^{-ink_jX_m}e^{i(\delta_j-\delta_m)t}c_m, \tag{44a}$$

$$\dot{b}_j^r = ig_j\sqrt{\frac{R}{n}}\sum_m e^{-ink_jX_m}e^{i(\delta_j-\delta_m)t}c_m, \tag{44b}$$

$$\dot{c}_m = -i\sum_{j'} g_{j'}e^{ink_{j'}X_m}e^{-i(\delta_{j'}-\delta_m)t}\left[\sqrt{\frac{T}{n}}b_{j'}^l + \sqrt{\frac{R}{n}}b_{j'}^r\right]. \tag{44c}$$

Only the terms corresponding to a wave propagating to the right in the dielectric have been retained, as the other terms are negligibly small.

By formally solving Eq. (44c) and substituting the result in Eqs. (44a) and (44b), we find the coupled equations for the field mode amplitudes

$$\dot{b}_j^l = n^{-1}g_j\sum_{m,j'} g_{j'}e^{-in(k_j-k_{j'})X_m}e^{i(\delta_j-\delta_m)t}\int_0^t dt'\,e^{-i(\delta_{j'}-\delta_m)t'} \times [Tb_{j'}^l(t') + \sqrt{RT}b_{j'}^r(t')], \tag{45a}$$

$$\dot{b}_j^r = n^{-1}g_j\sum_{m,j'} g_{j'}e^{-in(k_j-k_{j'})X_m}e^{i(\delta_j-\delta_m)t}\int_0^t dt'\,e^{-i(\delta_{j'}-\delta_m)t'} \times [Rb_{j'}^r(t') + \sqrt{RT}b_{j'}^l(t')]. \tag{45b}$$

As in Sec. II, we neglect the fluctuations and replace the sums over m by

$$\mathfrak{N}A\int_0^{L/2n} dX_m \int_{-\infty}^{\infty} d\delta_m W(\delta_m).$$

In contrast to the case of an infinite dielectric, however, integrating over X_m does not lead to a decoupling of the modes. Instead, we first integrate over δ_m and use Eq. (20) to obtain

$$\dot{b}_j^l = \frac{\sqrt{\pi}}{n\delta_0}g_j\mathfrak{N}A\int_0^{L/2n} dX\sum_{j'} g_{j'}e^{-in(k_j-k_{j'})X}e^{i(\delta_j-\delta_{j'})t} \times [Tb_{j'}^l + \sqrt{RT}b_{j'}^r], \tag{46a}$$

$$\dot{b}_j^r = \frac{\sqrt{\pi}}{n\delta_0}g_j\mathfrak{N}A\int_0^{L/2n} dX\sum_{j'} g_{j'}e^{-in(k_j-k_{j'})X}e^{i(\delta_j-\delta_{j'})t} \times [Rb_{j'}^r + \sqrt{RT}b_{j'}^l]. \tag{46b}$$

To make further progress, we go over to continuum states by using Eqs. (8) and (24) to transform Eq. (46) into

$$\dot{b}^l(k,t) = \frac{\sqrt{\pi}Lg^2}{2\pi n\delta_0}\mathfrak{N}A\int_0^{L/2n} dX\int_{-\infty}^{\infty} dk'\,e^{-in(k-k')X}e^{i(\delta_k-\delta_{k'})t} \times [Tb^l(k',t) + \sqrt{RT}b^r(k',t)], \tag{47a}$$

$$\dot{b}^r(k,t) = \frac{\sqrt{\pi}Lg^2}{2\pi n\delta_0}\mathfrak{N}A\int_0^{L/2n} dX\int_{-\infty}^{\infty} dk' e^{-in(k-k')X}e^{i(\delta_k-\delta_{k'})t} \times[Rb^r(k',t)+\sqrt{RT}b^l(k',t)]. \quad (47b)$$

In writing these equations, we have set $g=g_k(\omega_k=\omega_0)$ and extended the k integral to $-\infty$, since the major contributions from both k and k' are in the vicinity of k_0.

These equations are integral equations with a kernel that is a function of $(k-k')$. As such, they can be solved using Fourier transform techniques. Defining

$$b(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\rho e^{-ik\rho}B(\rho,t), \quad (48)$$

one can convert Eqs. (47) into the form

$$\dot{B}^l(\rho,t) = \frac{\sqrt{\pi}Lg^2}{n\delta_0}\mathfrak{N}A\int_0^{L/2n} dX\delta_D(nX-ct-\rho) \times[TB^l(\rho,t)+\sqrt{RT}B^r(\rho,t)], \quad (49a)$$

$$\dot{B}^r(\rho,t) = \frac{\sqrt{\pi}Lg^2}{n\delta_0}\mathfrak{N}A\int_0^{L/2n} dX\delta_D(nX-ct-\rho) \times[RB^r(\rho,t)+\sqrt{RT}B^l(\rho,t)]. \quad (49b)$$

It is now possible to carry out the integral over X. In the limit that $L\sim\infty$, one finds

$$\dot{B}^l(\rho,t) = -\frac{\alpha c}{2n}[TB^l(\rho,t)+\sqrt{RT}B^r(\rho,t)]\Theta(\rho+ct), \quad (50a)$$

$$\dot{B}^r(\rho,t) = -\frac{\alpha c}{2n}[RB^r(\rho,t)+\sqrt{RT}B^l(\rho,t)]\Theta(\rho+ct), \quad (50b)$$

where α is given by Eq. (32) and Θ is a Heaviside function.

It is interesting to note that there is a dark state

$$B_d(\rho,t) = \sqrt{R}B^l(\rho,t)-\sqrt{T}B^r(\rho,t) \quad (51)$$

that is decoupled from the fields. As a consequence one has

$$\sqrt{R}B^l(\rho,t)-\sqrt{T}B^r(\rho,t) = \sqrt{R}B^l(\rho,0). \quad (52)$$

It also follows from Eqs. (50) that

$$d[\sqrt{T}B^l(\rho,t)+\sqrt{R}B^r(\rho,t)]/dt = -\frac{\alpha c}{2n}[\sqrt{T}B^l(\rho,t)+\sqrt{R}B^r(\rho,t)]\Theta(\rho+ct), \quad (53)$$

leading to the solution

$$\sqrt{T}B^l(\rho,t)+\sqrt{R}B^r(\rho,t) = \begin{cases} \sqrt{T}B^l(\rho,0) & t<-\rho/c \\ \sqrt{T}B^l(\rho,0)\exp\left[-\frac{\alpha c}{2n}\left(t+\frac{\rho}{c}\right)\right] & t>-\rho/c. \end{cases} \quad (54)$$

By combining Eqs. (52) and (54), one finds that the final solution for the Fourier transform of the field amplitudes is

$$B^l(\rho,t) = \begin{cases} B^l(\rho,0) & t<-\rho/c \\ (R+Te^{-\alpha c/2n(t+\rho/c)})B^l(\rho,0) & t>-\rho/c, \end{cases} \quad (55a)$$

$$B^r(\rho,t) = \begin{cases} 0 & t<-\rho/c \\ -\sqrt{RT}\left(1-\exp\left[-\frac{\alpha c}{2n}\left(t+\frac{\rho}{c}\right)\right]\right)B^l(\rho,0) & t>-\rho/c. \end{cases} \quad (55b)$$

Finally, by taking the inverse transform, one obtains field state amplitudes

$$b^l(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-ct} d\rho e^{-ik\rho}B^l(\rho,0)+\frac{1}{\sqrt{2\pi}}\int_{-ct}^{\infty} d\rho e^{-ik\rho} \times(R+Te^{-(\alpha c/2n)(t+\rho/c)})B^l(\rho,0), \quad (56a)$$

$$b^r(k,t) = -\frac{\sqrt{RT}}{\sqrt{2\pi}}\int_{-ct}^{\infty} d\rho e^{-ik\rho}\left\{1-\exp\left[-\frac{\alpha c}{2n}\left(t+\frac{\rho}{c}\right)\right]\right\} \times B^l(\rho,0). \quad (56b)$$

If one substitutes the inverse transform of $B^l(\rho,0)$ into Eqs. (56), one can obtain a solution for $b^{\beta=l,r}(k,t)$ in terms of $b^{\beta=l,r}(k,0)$. The fact that the resulting equation is an integral equation implies that different field modes are coupled by the atoms. To a good approximation, however, the field corresponds to the single-photon pulse partially reflected at the dielectric surface and partially transmitted, with an amplitude that diminishes exponentially in the medium. Not surprisingly, this result is consistent with the Maxwell-Bloch approach, since the approximations (e.g., neglect of atom position and transition frequency fluctuations) used in both approaches are equivalent.

As $t\sim\infty$,

$$b^l(k,t) \sim Rb^l(k,0), \quad (57a)$$

$$b^r(k,t) \sim -\sqrt{RT}b^l(k,0). \quad (57b)$$

From the quantization condition (5), it follows that the energy in the field is $(R^2+RT)=R$ times the initial energy in the field, as expected. Moreover, the energy density $2\epsilon_0\langle\mathbf{E}^-(X)\cdot\mathbf{E}^+(X)\rangle$ is that associated with the initial one-photon pulse that has been reflected at the dielectric surface.

By combining Eqs. (44c) and (56) and going over to continuum states, one finds that the atomic state amplitudes evolve according to

$$\dot{c}_m = -ig\frac{L}{2\pi}\sqrt{\frac{2\pi}{L}}\int_{-\infty}^{\infty} dk e^{inkX_m}e^{-i(\delta_k-\delta_m)t}\left[\sqrt{\frac{T}{n}}b^l(k,t)+\sqrt{\frac{R}{n}}b^r(k,t)\right]$$

$$= -ig\frac{1}{2\pi}\sqrt{\frac{TL}{n}}\int_{-\infty}^{\infty} d\rho e^{-ik\rho}\int_{-\infty}^{\infty} dk e^{inkX_m}e^{-i(\delta_k-\delta_m)t}\left\{\Theta(-\rho-ct)+\exp\left[-\frac{\alpha c}{2n}\left(t+\frac{\rho}{c}\right)\right]\Theta(\rho+ct)\right\}B^l(\rho,0)$$

$$= -ig\sqrt{\frac{TL}{n}}e^{ink_0X_m}\int_{-\infty}^{\infty} d\rho e^{-ik_0\rho}\delta_D(nX_m-ct-\rho)\left\{\Theta(-\rho-ct)+\exp\left[-\frac{\alpha c}{2n}\left(t+\frac{\rho}{c}\right)\right]\Theta(\rho+ct)\right\}B^l(\rho,0)$$

$$= -ig\sqrt{\frac{TL}{n}}\exp(-\alpha X_m/2)e^{i\delta_m t}e^{ik_0ct}B^l(nX_m-ct,0), \tag{58}$$

where the fact that $X_m>0$ has been used. The result takes on a remarkably simple form, with the atomic excitation driven by the transmitted pulse. The pulse amplitude decays exponentially as the pulse propagates in the medium.

If we choose

$$b^l(k;0)=\left(\frac{1}{\pi(\Delta k)^2}\right)^{1/4}\exp[-(k-k_0)^2/2(\Delta k)^2]e^{ikX_0}, \tag{59}$$

with $X_0\ll -1/\Delta k=-n\Delta X$, (i.e., a pulse incident from the left at $t=0$), then

$$B^l(\rho,0)=\pi^{1/4}\sqrt{\Delta k/\pi}\exp[-(\rho-X_0)^2(\Delta k)^2/2]e^{ik_0(\rho-X_0)} \tag{60}$$

and

$$\dot{c}_m = -\sqrt{\frac{T\Delta k\omega_0\mu^2}{2\pi\epsilon_0A\hbar n}}\exp(-\alpha X_m/2)e^{i\delta_m t}e^{ink_0X_m}e^{-ik_0X_0}\pi^{1/4}$$
$$\times\exp[-(nX_m-ct-X_0)^2(\Delta k)^2/2]. \tag{61}$$

Using Eqs. (61) and (20), it is easy to verify that

$$\mathfrak{N}A\int_0^{\infty} dX\int d\delta_m W(\delta_m)|c_m(X_m,t\gg n\alpha/c)|^2=T,$$

all the field energy transmitted into the medium is transferred to the atoms when the pulse is fully absorbed.

Explicitly, one finds the atomic state amplitude at time t is given by

$$c_m(X_m,t) = -\pi^{1/4}\sqrt{\frac{T\omega_0\mu^2\tau}{4\pi\epsilon_0A\hbar n}}\exp(-\alpha X_m/2)$$
$$\times\exp[in(X_m-X_0)(k_0+\delta_m/c)]\exp[-(\delta_m\tau)^2/2]$$
$$\times\left[\Phi\left(\frac{t}{\sqrt{2}\tau}-v\right)+\Phi(v)\right], \tag{62}$$

where

$$v=\frac{1}{\sqrt{2}}\left(\frac{nX_m-X_0}{n\Delta X}+i\delta_m\tau\right),$$

$\Delta X=1/n\Delta k$ and $\tau=1/c\Delta k$. For sufficiently long times $\frac{\alpha c}{n}(t+X_0/c)\gg 1$, all the energy initially in the single-photon pulse has been transferred to the atoms and the atoms are in an entangled state with a spatial phase factor $e^{ink_0X_m}$ that has been imprinted by the field, along with an exponentially decaying factor $e^{-\alpha X_m/2}$, resulting from absorption in the medium. In this limit, the result (62) reduces to Eq. (37) multiplied by $\sqrt{T}$.

IV. ELECTROMAGNETIC-INDUCED TRANSPARENCY PROTOCOL—SEMI-INFINITE MEDIUM

The astute reader will realize that the presence of the dielectric host did not play a critical role in the calculations, except insofar as it provided inhomogeneous broadening for the two-level atoms. In the limit that the index of refraction of the dielectric host goes to unity, all the previous calculations remain valid. In the case of a semi-infinite medium of two-level atoms with no dielectric host, it is not necessary to include any reflected wave whatsoever. As long as there is negligible scattering for the time scale of the experiment, the reflected wave is unimportant as the incident pulse is absorbed in the medium.

These considerations suggest that it might be possible to extend our methods to the EIT protocol. A qualitative discussion of the EIT protocol has been given in Sec. I. The physical system includes the incoming single-photon-pulsed field and a semi-infinite medium consisting solely of three-level atoms (no background dielectric) (Fig. 3). Each atom has a Λ configuration in which levels 1 and 2 are ground-state sublevels having the same parity (e.g., different ground-state hyperfine levels) and level 3 is an excited-state electronic level coupled to levels 1 and 2 via electric-dipole transitions. A classical control field drives the 2–3 transition, the signal field drives the 1–2 transition, and the atoms are all initially in state 1. The atoms are located in the halfspace $X>0$. We calculate the modification of the signal field by the atoms, assuming the frequency bandwidth of the pulse is narrower than the EIT transparency window. As the one-photon pulse propagates in the medium, quantum information can be

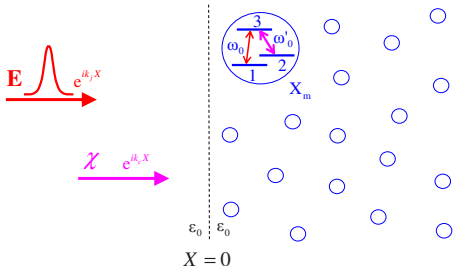

FIG. 3. (Color online). A single-photon pulse propagates to the right toward a semi-infinite medium consisting of three-level atoms only (no dielectric host). The incident pulsed field and a cw control field drive the atoms under conditions where EIT can occur. The one-photon pulse is compressed and travels with reduced group velocity in the medium accompanied by an entangled state of the atoms.

stored in a collective excitation involving a single excitation of state 2. This excitation adiabatically follows the field. In EIT protocols, one often turns off the control field when the pulse is in the medium to store the information in the medium, but we limit our discussion to a constant control field.

We assume that there is no inhomogeneous broadening. Hence the transition frequencies for *each* atom can be specified as

$$\omega_0 = \omega_{31}, \quad \omega_0' = \omega_{32}. \tag{63}$$

The monochromatic control field is resonant with the 2–3 transition and has a propagation vector $\mathbf{k}_c = (\omega_0'/c)\hat{\mathbf{x}}$, while the incident one-photon pulse has a frequency bandwidth $\Delta\omega$ centered about ω_0. The one-photon pulse propagates in the x direction.

In dipole and rotating-wave approximation, the Hamiltonian appropriate to our atom-field system is given by

$$\begin{aligned} H = & \sum_m [\hbar\omega_0 \sigma_{33}^m + \hbar(\omega_0 - \omega_0')\sigma_{22}^m] + \sum_j \hbar\omega_j a_j^\dagger a_j \\ & + \sum_{j,m} \hbar g_j [e^{ik_j X} \sigma_+^m(31) a_j - a_j^\dagger e^{-ik_j X_m} \sigma_-^m(31)] \\ & + \hbar\chi \sum_m [e^{ik_c X} e^{-i\omega_0' t} \sigma_+^m(32) + e^{-ik_j X_m} e^{i\omega_0' t} \sigma_-^m(32)], \end{aligned} \tag{64}$$

$\sigma_+^m(\alpha\alpha') = |\alpha\rangle\langle\alpha'|$ is a raising operator and $\sigma_-^m(\alpha\alpha') = |\alpha'\rangle\langle\alpha|$ is a lowering operators for atom m; $\sigma_{\alpha\alpha}^m = |\alpha\rangle\langle\alpha|$ is the population operator for atom m, and χ is one half the Rabi frequency associated with the classical field driving the 2–3 transition. The energy of state 1 has been set equal to zero. The vacuum field is quantized in free space using periodic boundary conditions; Eqs. (4), (5), (8), and (24) remain valid.

The state vector for the system can be written as

$$\begin{aligned} |\psi(t)\rangle = & \sum_j b_j(t) e^{-i\omega_0 t} |G;j\rangle + \sum_m c_m(t) e^{-i\omega_0 t} |3,m;0\rangle \\ & + \sum_m d_m(t) e^{-i(\omega_0 - \omega_0')t} e^{-ik_c X_m} |2,m;0\rangle, \end{aligned} \tag{65}$$

where $b_j(t)$ is the probability amplitude for the field to be in mode j and all the atoms to be in state 1, $c_m(t)$ is the probability amplitude for the field to be in the vacuum mode and the atoms to be in a state where atom m is in state 3, and all the other atoms are in state 1, while $d_m(t)$ is the probability amplitude for the field to be in the vacuum mode and the atoms to be in a state where atom m is in state 2 and all the other atoms are in state 1. The initial state for the system is given by Eq. (12).

From Eqs. (64) and (65) and the Schrödinger equation, one can obtain the following equations for the state amplitudes:

$$\dot{b}_j = -i\delta_j b_j + i g_j \sum_m e^{-ik_j X_m} c_m, \tag{66a}$$

$$\dot{c}_m = -i \sum_{j'} g_{j'} e^{ik_{j'} X_m} b_{j'} - i\chi d_m, \tag{66b}$$

$$\dot{d}_m = -i\chi c_m, \tag{66c}$$

where δ_j is given by Eq. (14). It will prove advantageous to use semiclassical dressed state amplitudes defined by

$$A_m = \frac{1}{\sqrt{2}}(c_m - d_m), \tag{67a}$$

$$B_m = \frac{1}{\sqrt{2}}(c_m + d_m), \tag{67b}$$

allowing us to transform Eq. (66) into

$$\dot{b}_j = -i\delta_j b_j + i\frac{g_j}{\sqrt{2}} \sum_m e^{-ik_j X_m}(B_m - A_m), \tag{68a}$$

$$\dot{A}_m = i\chi A_m + i \sum_{j'} \frac{g_{j'}}{\sqrt{2}} e^{ik_{j'} X_m} b_{j'}, \tag{68b}$$

$$\dot{B}_m = -i\chi B_m - i \sum_{j'} \frac{g_{j'}}{\sqrt{2}} e^{ik_{j'} X_m} b_{j'}, \tag{68c}$$

or, in terms of slightly modified amplitudes defined by

$$\tilde{A}_m = e^{-i\chi t} A_m, \quad \tilde{B}_m = e^{i\chi t} B_m, \quad \tilde{b}_j = e^{i\delta_j t} b_j, \tag{69}$$

$$d\tilde{b}_j/dt = i\frac{g_j}{\sqrt{2}} e^{i\delta_j t} \sum_m e^{-ik_j X_m}(e^{-i\chi t}\tilde{B}_m - e^{i\chi t}\tilde{A}_m), \tag{70a}$$

$$d\tilde{A}_m/dt = i e^{-i\chi t} \sum_{j'} \frac{g_{j'}}{\sqrt{2}} e^{ik_{j'} X_m} e^{-i\delta_{j'} t} \tilde{b}_{j'}, \tag{70b}$$

$$d\tilde{B}_m/dt = -ie^{i\chi t}\sum_{j'}\frac{g_{j'}}{\sqrt{2}}e^{ik_{j'}X_m}e^{-i\delta_{j'}t}\tilde{b}_{j'}. \tag{70c}$$

The calculation now closely mirrors that of Secs. II and III. By formally integrating Eqs. (70b) and (70c) and substituting the result into Eq. (70a), one finds that the field amplitude evolves as

$$d\tilde{b}_j/dt = g_j\sum_{m,j'} g_{j'}e^{-i(k_j-k_{j'})X_m}e^{i\delta_j t}\int_0^t dt' e^{-i\delta_{j'}t'} \times\cos[\chi(t-t')]\tilde{b}_{j'}(t'). \tag{71}$$

To convert this to a "local" equation, we assume that the control field Rabi frequency is much greater than the bandwidth of the one-photon pulse. In that limit, one can use integration by parts (twice) in Eq. (71) to obtain

$$d\tilde{b}_j/dt = \frac{g_j}{\chi^2}\sum_{m,j'} g_{j'}e^{-i(k_j-k_{j'})X_m}e^{i(\delta_j-\delta_{j'})t}[d\tilde{b}_{j'}/dt - i\delta_{j'}\tilde{b}_{j'}(t)], \tag{72}$$

or, going over to continuum states,

$$d\tilde{b}(k,t)/dt = \frac{g^2}{\chi^2}\frac{L}{2\pi}\sum_m\int_{-\infty}^{\infty}dk' e^{-i(k-k')(X_m-ct)} \times[d\tilde{b}(k',t)/dt - ic(k'-k_0)\tilde{b}(k',t)], \tag{73}$$

where g is the value of g_k evaluated at $\omega_k=\omega_0$.

Following a procedure identical to that in Sec. III, we can then transform this equation into

$$\frac{\partial B(\rho,t)}{\partial t} = -\frac{\eta c}{\chi^2}\left[\frac{\partial B(\rho,t)}{\partial t} + c\frac{\partial B(\rho,t)}{\partial \rho} - i\omega_0 B(\rho,t)\right]\Theta(\rho+ct), \tag{74}$$

where

$$B(\rho,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk e^{ik\rho}\tilde{b}(k,t), \tag{75}$$

$$\eta = \frac{\mathfrak{N}\omega_0\mu^2}{2\hbar\epsilon_0 c} = \frac{3}{8\pi}\mathfrak{N}\lambda_0^2\gamma_3, \tag{76}$$

$\lambda_0=2\pi c/\omega_0$, and γ_3 is the excited-state decay rate. The second and third terms in Eq. (74) arise from the second term in Eq. (73). For $\rho>-ct$, Eq. (74) reduces to

$$\frac{\partial B(\rho,t)}{\partial \rho} - \frac{1}{v_0}\frac{\partial B(\rho,t)}{\partial t} = ik_0 B(\rho,t), \tag{77}$$

where $k_0=\omega_0/c$ and

$$v_0 = \frac{\eta c}{\eta c+\chi^2}c. \tag{78}$$

The solution of Eq. (74) for $\rho<-ct$ is

$$B(\rho,t) = B(\rho,0), \quad t<-\rho/c. \tag{79}$$

For $\rho>-ct$, Eq. (74) reduces to Eq. (77); the solution of Eq. (77) subject to the boundary condition $B(\rho,-\rho/c)=B(\rho,0)$ imposed by Eq. (79) is

$$B(\rho,t) = \exp[i(k_0-\omega_0/v_g)\rho - i\omega_0 v_0 t/v_g]B[(\rho+v_0 t)c/v_g,0]; \quad t>-\rho/c, \tag{80}$$

where

$$v_g = \frac{\chi^2}{\eta c+\chi^2}c. \tag{81}$$

As we shall see, this solution corresponds to a pulse that propagates in the medium with group velocity v_g and is compressed spatially by a factor v_g/c.

We are now in a position to calculate the state amplitudes b_j, c_m, and d_m appearing in Eq. (65). The amplitude $b(k,t)$ is just the Fourier transform of $B(\rho,t)$ multiplied by e^{-ikct}. Rather than calculate $b(k,t)$ directly, it is more instructive to see how the field propagates into the medium. Of course, the expectation value of the field for any single-photon state of the field vanishes. Instead, we must calculate the expectation value of the electric field intensity,

$$\begin{aligned}\langle \mathbf{E}^-(X)\mathbf{E}^+(X)\rangle &= \langle\psi(t)|\mathbf{E}^-(X)\mathbf{E}^+(X)|\psi(t)\rangle \\ &= \left|\sum_{j=-\infty}^{\infty} g_j e^{ik_j X}b_j(t)\right|^2 \\ &= g^2 L\left|\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}dk e^{ik(X-ct)}\tilde{b}(k,t)\right|^2 \\ &= g^2 L|B(X-ct,t)|^2,\end{aligned} \tag{82}$$

where Eqs. (2), (8), (24), (65), (69), and (75) have been used. Thus the field intensity is determined by $|B(X-ct,t)|^2$.

For $X<0$ $[X-ct<-ct]$, it follows from Eq. (79) that

$$B(X-ct,t) = B(X-ct,0), \tag{83}$$

this corresponds to the original pulse displaced by an amount ct. In other words, before entering the medium, the pulse simply propagates at speed c. On the other hand, for $X>0$ $[X-ct>-ct]$, the pulse is in the medium. Using Eq. (80), one finds

$$|B(X-ct,t)| = |B[(\rho+v_0 t)c/v_g - ct,0]|. \tag{84}$$

As a consequence, the pulse intensity (82) propagates with reduced group velocity and is compressed in the medium. This is seen most easily if we use a specific form for the original pulse.

For the pulse (60), with

$$|B(\rho,0)| = \pi^{1/4}\sqrt{\Delta k/\pi}\exp[-(\rho-X_0)^2(\Delta k)^2/2], \tag{85}$$

corresponding to an initial pulse centered at $X=X_0<0$ at $t=0$ having spatial width of order $1/\Delta k$, one finds that

$$|B(X-ct,t)| = |B[(\rho+v_0t)c/v_g - ct,0]| = \pi^{1/4}\sqrt{\Delta k/\pi} \times \exp\{-[X-v_g(t+X_0/c)]^2(c\Delta k/v_g)^2/2\}. \quad (86)$$

As a result, once the pulse has fully entered the medium, the average pulse intensity (82) at time t is centered at

$$X = v_g(t+X_0/c), \quad (87)$$

having a spatial width of order $(v_g/c)(1/\Delta k)$. Thus, the pulse propagates with the group velocity v_g in the medium and is compressed spatially by a factor (v_g/c). These results are well known in the theory of EIT, but have been derived here using a state vector approach.

It remains to calculate the amplitudes c_m and d_m. Let us first calculate d_m. Using Eqs. (67) and (69), one sees that

$$d_m = \frac{1}{\sqrt{2}}(e^{-i\chi t}\tilde{B}_m + e^{i\chi t}\tilde{A}_m). \quad (88)$$

By formally integrating Eqs. (70b) and (70c) and going over to continuum variables, one then finds

$$\begin{aligned} d_m(t) &= -g\sqrt{L/2\pi}\int_{-\infty}^{\infty}dk\int_0^t dt'\, e^{ik(X_m-ct')}e^{ik_0ct'} \times \sin[\chi(t-t')]\tilde{b}(k,t') \\ &= -g\sqrt{L}\int_0^t dt'\, e^{ik_0ct'}\sin[\chi(t-t')]B(X_m-ct',t') \\ &= -g\sqrt{L}e^{ik_0X_m}\int_0^t dt'\,\sin[\chi(t-t')] \times \exp[-i\omega_0(X_m-v_gt')/v_g]B[(X_m-v_gt')c/v_g,0], \end{aligned} \quad (89)$$

where Eqs. (75) and (80) have been used.

Consistent with the approximations used to obtain $\tilde{b}(k,t)$, one can use an integration by parts to approximate $d_m(t)$ as

$$d_m(t) \sim (g/\chi)\sqrt{L}e^{ik_0X_m}\exp[-i\omega_0(X_m-v_gt)/v_g] \times B[(X_m-v_gt)c/v_g,0], \quad (90)$$

valid when the pulse bandwidth $\Delta\omega \ll \chi$. For the pulse (60), this reduces to

$$d_m(t) = (g/\chi)\pi^{1/4}e^{ik_0(X_m-X_0)}\sqrt{\Delta kL/\pi} \times \exp[[-X_m-v_g(t+X_0/c)]^2(c\Delta k/v_g)^2/2]. \quad (91)$$

The atomic excitation adiabatically follows the pulse envelope (86) in the medium.

Moreover, by substituting Eq. (91) into the third term in the state vector (65), one can rewrite this term as

$$\sum_m D_m(t)e^{i(k_0-k_c)X_m}e^{-i(\omega_0-\omega_0')t}|2,m;0\rangle, \quad (92)$$

where

$$D_m(t) = d_m(t)e^{-ik_0X_m} \quad (93)$$

no longer contains any atomic position-dependent phase factor. The entangled state (92) has spatial phases that have been imprinted on the atoms by the control and signal fields. Pulse compression and reduced group velocity are evident in Eq. (91). In contrast to the absorption protocol where the signal field leaves atomic excitation in its wake, the collective excitation in EIT adiabatically follows the signal field.

One can repeat the calculation for the excited-state amplitude c_m. In the same manner, one finds that the second term in the state vector (65) can be written as

$$\sum_m C_m(t)e^{ik_0X_m}e^{-i\omega_0t}|3,m;0\rangle, \quad (94)$$

where

$$C_m(t) = -ig\sqrt{L}\int_0^t dt'\,\cos[\chi(t-t')]\exp[-i\omega_0(X_m - v_gt')/v_g]B[(X_m-v_gt')c/v_g,0]. \quad (95)$$

In this case, however, the lead term vanishes when one carries out an integration by parts. Integrating by parts twice, one obtains

$$C_m(t) \sim -i\left(\frac{g\sqrt{L}}{\chi^2}\right)\frac{d}{dt}\{e^{-i\omega_0(X_m-v_gt)/v_g}B[(X_m-v_gt)c/v_g,0]\}, \quad (96)$$

For the pulse (60), this reduces to

$$C_m(t) = (ig/\chi)\pi^{1/4}e^{-ik_0X_0}\sqrt{\Delta kL/\pi}(c\Delta k/\chi)[X_m-v_g(t+X_0/c)] \times (c\Delta k/v_g)\exp[-[X_m-v_g(t+X_0/c)]^2(c\Delta k/v_g)^2/2]. \quad (97)$$

This term is down from Eq. (91) by a factor $c\Delta k/\chi = \Delta\omega/\chi \ll 1$. Thus, any excited-state population is suppressed by this factor. In this calculation, any diminution of the one-photon pulse's amplitude resulting from the absorption to the excited state or scattering into vacuum field modes has been neglected.

V. DISCUSSION

We have formulated a theory of single-photon pulse propagation using a Schrödinger-picture approach. The Schrödinger-picture approach allows us to follow the entanglement of the atoms with the field (and with each other) as the system evolves. Specifically we derived equations that could be used to calculate the state vector of the atom-field system for both the "pulse absorption" and EIT protocols.

In general, the atom-field dynamics is quite complicated, owing to the fact that different frequency components in the initial field pulse are absorbed differently in the medium. Moreover, when one considers an arbitrary spatial array of the atoms and an inhomogeneous distribution of transition frequencies, the complexity of the problem becomes even more severe. It is possible to solve the evolution equations analytically only if certain simplifying assumptions are

made. Consistent with a Maxwell-Bloch approach to the problem, one can neglect the fluctuations in atomic position and the fluctuations in atomic transition frequencies. In effect, one solves *amplitude* equations in which a discrete distribution of positions and frequencies (in the case of inhomogeneous broadening) is replaced by a continuum. The atomic density is assumed to be uniform.

These assumptions alone are not sufficient to obtain an analytical result. In the case of probe absorption, it is necessary to assume that the inhomogeneous width associated with the transition frequencies is much larger than the bandwidth of the excitation pulse. Only in this manner are all frequency components of the pulse absorbed in the same manner by the atoms, leading to pulse propagation without distortion. In the case of EIT, the control field provides a transparency window, obviating the need for inhomogeneous broadening.

With these assumptions, the results are consistent with what one would expect intuitively from a Maxwell-Bloch approach. In the case of absorption, the pulsed field enters the medium and is absorbed at each atomic site. As a consequence, the field intensity diminishes as it propagates in the medium. For an optically thick sample, the pulse is absorbed within a few absorption lengths and the atoms are left in an entangled state involving a single excitation. The phase of the incident field is imprinted on the atoms. As such, it is possible to restore the excitation pulse at some later time by reversing the excitation process. For EIT, the pulse is compressed as it enters the medium and propagates with the reduced group velocity. A collective coherence is created in the medium that adiabatically follows the field pulse envelope.

It might be possible to extend our method to the DLCZ protocol. In the DLCZ protocol, one can send a one-photon pulse into an ensemble of three-level atoms in the same Λ configuration as that used for EIT. One would then calculate the manner in which the signal field is converted into a Stokes field that is entangled with a collective excitation of ground-state coherence. Either a background dielectric that provides inhomogeneous broadening of the ground-state transition or a modified DLCZ protocol making use of EIT [3,22,23] is needed to allow the field pulses to propagate without distortion.

We have neglected any cooperative effects between the atoms in the absorption protocol. Clearly, if we had looked at pulsed excitation in an *optically thin* sample, there would be minimal absorption of the pulse. As a result, the atoms would be excited in a type of symmetric-phased array that could emit coherently; however, *cooperative* decay processes would be unimportant owing to the fact that the sample is optically thin. In our case, the medium is assumed to be optically thick, $\mathcal{N}\lambda^2 L \gg 1$, suggesting that cooperative decay process could play a role. However, owing to the exponential decay in the medium, the atoms are not excited to a *symmetric*-phased array and it is unlikely that cooperative decay plays an important role. In other words, if N atoms share a single excitation, there are N decay modes, but very few of these (e.g., the fully symmetric, phased excitation) decay with an enhanced decay rate. The exponentially decaying collective state in our examples corresponds to a superposition of a large number of different decay modes that is expected to decay with the free-atom decay rate.

Finally we would like to stress once again that in the absorption protocol, we are dealing with absorption rather than scattering. All the energy of the incident pulse is transferred to the atoms, with a minimal amount going into radiation. Of course, this result is valid only if the pulse is absorbed in a time that is much less than the excited-state lifetime if one is considering dipole-allowed optical transitions. The extension of this approach to multiphoton fields is not so obvious, as a large number of excitation states of the medium would have to be included. On the other hand, such an extension could provide a useful model of a photodetector, where the excited-state population in the medium is used as a measure of the "number of photons" absorbed by the detector.

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[1] M. Fleischhauer, A. Imamoglu, and J. P. Marangos, Rev. Mod. Phys. **77**, 633 (2005).

[2] M. Fleischhauer and M. D. Lukin, Phys. Rev. Lett. **84**, 5094 (2000); Phys. Rev. A **65**, 022314 (2002).

[3] T. Chanelière, D. N. Matsukevich, S. D. Jenkins, S.-Y. Lan, T. A. B. Kennedy, and A. Kuzmich, Nature (London) **438**, 833 (2005).

[4] M. D. Eisaman, A. Andre, F. Massou, M. Fleischhauer, A. S. Zibrov, and M. D. Lukin, Nature (London) **438**, 837 (2005).

[5] K. Honda, D. Akamatsu, M. Arikawa, Y. Yokoi, K. Akiba, S. Nagatsuka, T. Tanimura, A. Furusawa, and M. Kozuma, Phys. Rev. Lett. **100**, 093601 (2008).

[6] J. Appel, E. Figueroa, D. Korystov, M. Lobino, and A. I. Lvovsky, Phys. Rev. Lett. **100**, 093602 (2008).

[7] A. E. Kozhekin, K. Molmer, and E. S. Polzik, Phys. Rev. A **62**, 033809 (2000).

[8] S. A. Moiseev and S. Kröll, Phys. Rev. Lett. **87**, 173601 (2001).

[9] B. Kraus, W. Tittel, N. Gisin, M. Nilsson, S. Kröll, and J. I. Cirac, Phys. Rev. A **73**, 020302(R) (2006).

[10] M. Afzelius, C. Simon, H. de Riedmatten, and N. Gisin, e-print arXiv:0805.4164.

[11] G. Hétet, J. J. Longdell, A. L. Alexander, P. K. Lam, and M. J. Sellars, Phys. Rev. Lett. **100**, 023601 (2008); J. J. Longdell, G.

Hétet, P. K. Lam, and M. J. Sellars, Phys. Rev. A **78**, 032337 (2008).
[12] G. Hétet, M. Hosseini, B. M. Sparkes, D. Oblak, P. K. Lam, and B. C. Buchler, Opt. Lett. **33**, 2323 (2008).
[13] A. L. Alexander, J. J. Longdell, M. J. Sellars, and N. B. Manson, Phys. Rev. Lett. **96**, 043602 (2006).
[14] H. de Riedmatten, M. Afzelius, M. Staudt, C. Simon, and N. Gisin, e-print arXiv:0810.0630.
[15] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Nature (London) **414**, 413 (2001).
[16] A. Kuzmich, W. P. Bowen, A. D. Boozer, A. Boca, C. W. Chou, L.-M. Duan, and H. J. Kimble, Nature (London) **423**, 731 (2003).
[17] C. W. Chou, S. V. Polyakov, A. Kuzmich, and H. J. Kimble, Phys. Rev. Lett. **92**, 213601 (2004).
[18] B. Dubetsky and P. R. Berman, Phys. Rev. A **47**, 1294 (1993).
[19] J. R. Jeffers, N. Imoto, and R. Loudon, Phys. Rev. A **47**, 3346 (1993).
[20] In such absorption protocols (e.g., [8]), one usually applies a second classical pulse to transfer the ground-excited-state coherence to a Raman coherence before the excited state has had time to decay.
[21] See, for example, K. J. Blow, R. Loudon, S. J. D. Phoenix, and T. J. Shepherd, Phys. Rev. A **42**, 4102 (1990); H. Khosravi and R. Loudon, Proc. R. Soc. London, Ser. A **433**, 337 (1991); B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992); P. W. Milonni, Int. J. Mod. Phys. A **42**, 1991 (1995).
[22] D. A. Braje, V. Balić, S. Goda, G. Y. Yin, and S. E. Harris, Phys. Rev. Lett. **93**, 183601 (2004).
[23] D. N. Matsukevich, T. Chanelière, M. Bhattacharya, S. Y. Lan, S. D. Jenkins, T. A. B. Kennedy, and A. Kuzmich, Phys. Rev. Lett. **95**, 040405 (2005).