Bounding the dimension of bipartite quantum systems

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Let us consider the set of joint quantum correlations arising from two-outcome local measurements on a bipartite quantum system. We prove that no finite dimension is sufficient to generate all these sets. We approach the problem in two different ways by constructing explicit examples for every dimension d, which demonstrates that there exist bipartite correlations that necessitate d-dimensional local quantum systems in order to generate them. We also show that at least ten two-outcome measurements must be carried out by the two parties altogether so as to generate bipartite joint correlations not achievable by two-dimensional local systems. The smallest explicit example we found involves 11 settings.

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I. INTRODUCTION

We consider the following standard bipartite measurement scenario [1]. Two distant separated parties, conventionally called Alice and Bob, share a joint quantum state ρ. The two participants carry out measurements on their respective states and may obtain two possible classical outcomes of ±1. Let us specify the situation, that Alice (Bob) has m_A (m_B) number of ±1-valued observables $A_1,\ldots,A_{m_A}$ ($B_1,\ldots,B_{m_B}$). In a particular run of the experiment Alice and Bob each measures one observable, A_i and B_j, getting the respective outputs $a_i\in\pm1$ and $b_j\in\pm1$, and then makes the product a_ib_j. By repeating this process many times they can form the joint correlation

$$E(a_i,b_j)=\langle A_iB_j\rangle=\mathrm{Tr}(\rho A_i\otimes B_j), \tag{1}$$

which is the expected value of the product a_ib_j. These joint correlations for all $\{i,j\}$ form an m_Am_B-dimensional vector, designated by $\mathbf{c}$.

We are interested in the following problem. We are given a vector $\mathbf{c}$ of joint correlations $E(a_i,b_j)$, with $i=1,\ldots,m_A$ and $j=1,\ldots,m_B$, for a particular number of settings m_A and m_B. What is the minimum Hilbert-space dimension which is needed to reproduce this vector $\mathbf{c}$? Is it the case that all sets of these vectors for any m_A and m_B are possibly reproduced by two-dimensional local Hilbert spaces? In a recent work we provided counterexamples for small values of m_A and m_B (numerically for $m_B=4$ and $m_A=8$; an exact treatment was carried out for $m_B=6$ and $m_A=15$), which shows that indeed higher than two-dimensional local Hilbert spaces are required in order to produce all the set of such vectors [2]. The same conclusion, based on the gap between a lower bound on the Grothendieck constant in infinite dimensions [3] and an upper bound on the three-dimensional Grothendieck constant [4], was obtained by Brunner *et al.* [5]. In this paper the idea of dimension witnesses was put forward as well. Brunner *et al.* [5] defined a d-dimensional witness as a certain kind of generalization of Bell inequalities [1], such that all quantum correlations arising from observables on d-dimensional component Hilbert spaces satisfy the inequality. Hence, correlations which violate the inequality could be established only by measuring systems of dimension larger than d. Thus dimension witness is a useful tool for measuring the dimension of multipartite quantum systems relying only on experimental data.

The results in [2,5] prove the existence of two-dimensional witnesses in bipartite systems based on only joint correlations. On the other hand, Pérez-García *et al.* [6] gave a nonconstructive proof of the existence of d-dimensional witnesses for tripartite systems for any dimension d. In particular, they proved in Ref. [6] in their Theorem 1 that for every dimension d, there exists a dimension D, a pure state $|\psi\rangle$ on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^D$, and a Bell inequality with ±1-valued observables such that the violation by $|\psi\rangle$ is greater than $\sqrt{d}$ up to some universal constant. We also mention a recent paper by Wehner *et al.* [7], in which a method was given to bound from below the dimension of a Hilbert space by relating this problem to the construction of quantum random access codes. Their method works for quantum systems involving any number of parties (can be applied even for a single system) and is most useful for small number of settings.

In this paper we tackle the problem of whether there exist dimension witnesses for bipartite systems built from two-outcome measurements for any dimension d. Note that the tripartite case was affirmatively answered in [6]. On the other hand, the fact that this may also hold for the bipartite case was conjectured in Ref. [5] based on the plausible assumption that the Grothendieck constant [8] for infinite dimension is strictly larger than the Grothendieck constant for every finite order. In the numerical works in [9,10] we found various dimension witnesses up to dimensions of three and four, respectively, using bipartite Bell inequalities up to five binary measurement settings, $m_A=m_B=5$. But there, besides joint correlations, local marginal terms were also involved.

In particular, in Sec. II of the present paper, we discuss the link between joint correlations of two-party outcomes arising from two-outcome measurements and dot products of unit vectors in the Euclidean space. Armed with this connection two families of Bell inequalities are presented (in Secs. III and IV), proving the existence of bipartite correlations for any dimension d which require at least d-dimensional sys-

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tems in order to generate them. The first family (Sec. III) involves a finite number of settings for any finite d, while in case of the other family the number of settings is infinite. In Sec. III, as a by-product, it is shown analytically that joint correlations of two parties resulting from two-outcome measurements, which require more than two-dimensional component spaces, cannot be generated by $m_B=4$ and $m_A=6$ or by less settings. That is, these numbers of settings are not enough to define a two-dimensional witness. On the contrary, examples are given for a two-dimensional witness for settings $m_B=4$ and $m_A=7$ and for settings $m_B=5$ and $m_A=6$. The paper concludes with Sec. V.

II. REPRESENTATION OF JOINT CORRELATIONS WITH DOT PRODUCTS

This section contains two lemmas. The first one is borrowed from [11] (their Lemma 2), establishing a link between dot product of unit vectors in the Euclidean space and joint correlations originating from projective measurements in the Hilbert space, whereas the second one is a strengthening of this lemma regarding dimension witnesses.

Here we state the first lemma without proof.

Lemma 1. (Acín *et al.* [11]) Suppose Alice and Bob measure observables A and B on a pure quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Then we can associate a real unit vector $\vec{a} \in \mathbb{R}^{2d^2}$ with A (independent of B) and a real unit vector $\vec{b} \in \mathbb{R}^{2d^2}$ with B (independent of A) such that $E(a,b)=\vec{a}\cdot\vec{b}$. Moreover, if $|\psi\rangle$ is maximally entangled, then we can assume that the vectors $\vec{a}$ and $\vec{b}$ lie in $\mathbb{R}^{d^2-1}$.

Let us fix some notations regarding dimension witnesses [5] based on joint correlations. A d-dimensional witness [5], in terms of joint correlations (1), is a linear function of these correlators, described by a vector $\mathbf{m}$ of real coefficients M_{ij}, such that

$$\mathbf{m}\cdot\mathbf{c} = \sum_{i=1}^{m_A}\sum_{j=1}^{m_B} M_{ij}E(a_i,b_j) \le W^d \tag{2}$$

for all correlations of form (1) with ρ in $\mathbb{C}^d\otimes\mathbb{C}^d$, and such that there are joint correlations for higher than d-dimensional systems for which $\mathbf{m}\cdot\mathbf{c}>W^d$. Note that due to convexity arguments it is sufficient to consider only pure states $\rho=|\psi\rangle\langle\psi|$ in order to achieve the maximum on the left-hand side of Eq. (2).

Lemma 1 implies the following. Let us take an arbitrary vector $\mathbf{m}$ in Eq. (2) defined by some coefficients M_{ij}, and denote the maximum value

$$Q^d = \max_{\mathbf{c}}\{\mathbf{m}\cdot\mathbf{c}\} \tag{3}$$

achievable by correlations (1) from $\mathbb{C}^d\otimes\mathbb{C}^d$. Because of Lemma 1 we can associate with the elements of $\mathbf{c}$, $E(a_i,b_j)$, two normalized vectors $\vec{a}_i,\vec{b}_j \in \mathbb{R}^n$ such that $E(a_i,b_j)=\vec{a}_i\cdot\vec{b}_j$. Defining

$$\mathcal{B} = \sum_{i=1}^{m_A}\sum_{j=1}^{m_B} M_{ij}\vec{a}_i\cdot\vec{b}_j, \tag{4}$$

and denoting the extremal value in the n-dimensional Euclidean space by

$$T^n = \max_{\vec{a}_i,\vec{b}_j\in\mathbb{R}^n} \mathcal{B}, \tag{5}$$

Lemma 1 implies that $Q^d \le T^n$ for $n=2d^2$.

This way we can bound the quantum value Q^d achievable in d-dimensional component Hilbert spaces from above by dot product of unit vectors in $\mathbb{R}^n$. On the other hand, due to the construction by Cirel'son [12], all dot products of unit vectors $\vec{a},\vec{b}\in\mathbb{R}^m$ can be realized as ±1-valued observables on a maximally entangled state on $\mathbb{C}^D\otimes\mathbb{C}^D$, where $D=2^{\lfloor m/2\rfloor}$. Then in light of this and the fact that $Q^d\le T^n$ for $n=2d^2$, constructing a Bell expression (4) for which $T^n<T^m$, with $n<m$, implies a d-dimensional witness. The next lemma sharpens this statement.

Lemma 2. In the notation defined above by Eqs. (3) and (6), we have $Q^d\le T^n$ for $n=2d-1$. Furthermore, if the Hilbert space is restricted to be real, then the above relation holds true with $n=d$. Thus, by constructing a Bell expression $\mathcal{B}$ for which $T^n<T^{n+1}$ implies a d-dimensional witness with $d=\lfloor\frac{n+1}{2}\rfloor$.

In Sec. III we present a family of Bell expressions, characterized by Bob's number of settings $n=m_B$, for which one can prove that the above condition $T^n<T^{n+1}$ holds for the $(n+1)$th member of the family. In case of this family, however, we can give the gap $(T^{n+1}-T^n)$ explicitly only for small n values. In Sec. IV another Bell expression is provided, where this gap can be calculated analytically for every n.

Proof. Here we prove Lemma 2 and start by proving the relation $Q^d\le T^d$ in the case of Hilbert space $\mathbb{C}^d\otimes\mathbb{C}^d$ restricted to be real. Since A^i and B^j are ±1-valued observables, the normalization conditions $\sum_{l=1}^d (a_{kl}^i)^2=1$ and $\sum_{l=1}^d (b_{kl}^j)^2=1$ are satisfied for all $k=1,\ldots,d$, where $A^i=(a_{kl}^i)$ and $B^j=(b_{kl}^j)$ are $d\times d$ symmetric matrices with real coefficients (designations i and j are moved to upper indices for convenience). On the other hand, for given $\mathbf{m}$, $Q^d=\max_{\mathbf{c}}\{\mathbf{m}\cdot\mathbf{c}\}=\max\sum_{ij}M_{ij}\langle\psi|A^i\otimes B^j|\psi\rangle$, where maximization is over all $|\psi\rangle\langle\psi|$ in $\mathbb{C}^d\otimes\mathbb{C}^d$ and for all observables A^i and B^j in $\mathbb{C}^d$. The state $|\psi\rangle$ can be written in Schmidt form, $|\psi\rangle=\sum_{i=1}^d\alpha_i|ii\rangle$, where α_i can be chosen positive and their squares add up to unity. Then we have

$$\mathbf{m}\cdot\mathbf{c} = \sum_{ij} M_{ij}\langle\psi|A^i\otimes B^j|\psi\rangle = \sum_{kl}\alpha_k\alpha_l N_{kl}, \tag{6}$$

where $N_{kl}=\sum_{ij}M_{ij}(a_{kl}^ib_{kl}^j+a_{lk}^ib_{lk}^j)/2$. Note that in the case of real matrices A^i and B^j, the products $a_{kl}^ib_{kl}^j$ and $a_{lk}^ib_{lk}^j$ are the same. However, this form will turn out to be useful in the proof of the complex case. Now let us flip the signs of a_{kl}^i and a_{lk}^i for all $i=1,\ldots,d$ in each pair $\{k,l\}$ if N_{kl} is negative. By performing the necessary sign flips we obtain the matrices $\tilde{A}^i$ defined by elements $\tilde{a}_{kl}^i$. In this way the sum $\tilde{N}_{kl}=\sum_{ij}M_{ij}(\tilde{a}_{kl}^ib_{kl}^j+\tilde{a}_{lk}^ib_{lk}^j)/2$ becomes positive for any given $\{k,l\}$ pair. Thus, as a result the value of $\mathbf{m}\cdot\mathbf{c}$ cannot decrease. Furthermore the normalization condition still holds for the sign-flipped matrices, $\Sigma_{l=1}^{d}(\tilde{a}_{kl}^{i})^2=1$. In the following, through a chain of inequalities we obtain an upper bound for Q^d. First of all, we can write

$$\sum_{kl}\alpha_k\alpha_l N_{kl} \le \sum_{kl}(\alpha_k^2+\alpha_l^2)\tilde{N}_{kl}/2 \tag{7}$$

$$=\sum_{kl}\alpha_k^2\sum_{ij}M_{ij}(\tilde{a}_{kl}^{i}b_{kl}^{j}+\tilde{a}_{lk}^{i}b_{lk}^{j})/2 \tag{8}$$

$$=\sum_{k}\alpha_k^2\sum_{ij}M_{ij}\vec{a}_k^{i}\cdot\vec{b}_k^{j} \tag{9}$$

$$\le\sum_{k}\alpha_k^2\sum_{ij}M_{ij}\vec{a}^{i}\cdot\vec{b}^{j}=\sum_{ij}M_{ij}\vec{a}^{i}\cdot\vec{b}^{j}, \tag{10}$$

where the unit vectors $\vec{a}_k^i,\vec{b}_k^j$ and $\vec{a}^i,\vec{b}^j$ are in the d-dimensional Euclidean space. In inequality (7) we used $N_{kl}\le\tilde{N}_{kl}$ and the relation $2xy\le x^2+y^2$ holding for any real number x and y. Equality (8) follows from the fact that the matrices $\tilde{A}^i$ and B^j are symmetric. In equality (9) we exploited that $\tilde{a}_{kl}^i$ and b_{kl}^j can be treated as the d real components of unit vectors $\vec{a}_k^i=(\tilde{a}_{k1}^i,\tilde{a}_{k2}^i,\ldots,\tilde{a}_{kd}^i)$ and $\vec{b}_k^j=(b_{k1}^j,b_{k2}^j,\ldots,b_{kd}^j)$. In the last inequality we omitted index k, but keeping in mind the normalization conditions for $\vec{a}_k^i$ and $\vec{b}_k^j$ we have $\vec{a}^i\cdot\vec{a}^i=1$ and $\vec{b}^j\cdot\vec{b}^j=1$. Finally the last equality owes to $\Sigma_k\alpha_k^2=1$. According to Eqs. (3) and (6), Q^d is the maximum of $\Sigma_{kl}\alpha_k\alpha_l N_{kl}$ over all observables and states of dimension d. Thus, by use of the chain of inequalities (10), we obtain the upper bound

$$Q^d\le\max_{\vec{a}_i,\vec{b}_j\in R^d}\sum_{ij}M_{ij}\vec{a}^i\cdot\vec{b}^j=T^d, \tag{11}$$

where $\vec{a}_i$ and $\vec{b}_j$ are unit vectors.

The proof regarding the complex case goes along the same line as in the real case, but in this case $A^i=(a_{kl}^i)$ and $B^j=(b_{kl}^{*j})$ are by definition Hermitian matrices and their components are complex valued, where $*$ denotes complex conjugation. Similarly as in the real case, $\tilde{a}_{kl}^i$ is defined by flipping the sign of a_{kl}^i if N_{kl} is negative for a given pair (k,l). Then in the complex case, in place of the real number $(\tilde{a}_{kl}^ib_{kl}^j+\tilde{a}_{lk}^ib_{lk}^j)$ one can write $2[\mathrm{Re}\,\tilde{a}_{kl}^i\,\mathrm{Re}(b_{kl}^j)-\mathrm{Im}\,\tilde{a}_{kl}^i\,\mathrm{Im}(b_{kl}^j)]$, and then in Eq. (9) we have $\vec{a}_k^i=(\mathrm{Re}\,\tilde{a}_{k1}^i,\mathrm{Im}\,\tilde{a}_{k1}^i,\ldots,\tilde{a}_{kk}^i,\ldots,\mathrm{Re}\,\tilde{a}_{kd}^i,\mathrm{Im}\,\tilde{a}_{kd}^i)$ and $\vec{b}_k^j=(\mathrm{Re}\,b_{k1}^j,-\mathrm{Im}\,b_{k1}^j,\ldots,b_{kk}^j,\ldots,\mathrm{Re}\,\tilde{b}_{kd}^j,-\mathrm{Im}\,\tilde{b}_{kd}^j)$. Since components $\tilde{a}_{kk}^i$ and b_{kk}^j are real, these vectors lie in the $(2d-1)$-dimensional Euclidean space, and they can be checked to be of unit length. Therefore, in the complex case in Eq. (11) one has the unit vectors $\vec{a}^i,\vec{b}^j\in\mathbb{R}^{2d-1}$ and the right-hand side becomes T^{2d-1}, which completes the proof. □

Some remarks about Lemma 2 are in order. First, provided that the real Hilbert-space result is given, one can also obtain the result concerning the complex case by mapping the $d\times d$ Hermitian observables to $2d\times 2d$ real observables (as discussed in the Appendix of [9] and in the multipartite setting in [13]), which latter matrices have the property that at each column there is at least one zero component, entailing the $(2d-1)$-dimensional Euclidean vectors $\vec{a}^i$ and $\vec{b}^j$ in the upper bound expression on Q^d. Second, in light of the proofs in Refs. [2,5] regarding the qubit case, in Lemma 2, Q^2 is equal to T^3 and T^2 in the respective cases of complex and real qubits.

III. BOUNDS ON DIMENSIONS WITH FINITE NUMBER OF SETTINGS

In this section an example for dimension witness is provided for every dimension d. Let us consider the expression $\mathcal{B}$ defined earlier in Sec. II by Eq. (4). According to Lemma 2, a Bell expression with a specific matrix $\mathbf{m}$ for which $T^n<T^{n+1}$ [T^n denoting the maximum value of Eq. (4) in $\mathbb{R}^n$] implies the existence of a d-dimensional Hilbert-space witness in a bipartite system with $d=\lfloor\frac{n+1}{2}\rfloor$.

In Sec. III A it is shown that if $m_A\le m_B(m_B-1)/2$, then vectors from a smaller space than dimension m_B are enough to maximize $\mathcal{B}$. Then in Sec. III B it is shown via a particular family of Bell inequalities (labeled by Bob's measurements m_B) that when $m_A=m_B(m_B-1)/2+1$ for any given value of m_B, the whole m_B-dimensional Euclidean space is required to maximize $\mathcal{B}$. This proves for every $n=m_B-1$ that $T^n<T^{n+1}$, and implies that the Bell coefficients $\mathbf{m}=(M_{ij})$ of this particular family define a dimension witness in Hilbert-space dimension $d=\lfloor\frac{m_B}{2}\rfloor$, where m_B is the number of settings on Bob's side. In Sec. III C we determine the gap between the maximum value achievable in m_B and in (m_B-1)-dimensional Euclidean spaces for small m_B, and we also discuss the question of minimal number of settings in order for witnesses for two-dimensional component spaces to exist.

A. Limits on the number of settings

Let us start from expression (4). Without the loss of generality, we may suppose that $m_B\le m_A$. The dimensions of the real vector spaces are large enough to accommodate vectors that maximize the expression. From

$$\mathcal{B}=\sum_{i=1}^{m_A}\vec{a}_i\cdot\left(\sum_{j=1}^{m_B}M_{ij}\vec{b}_j\right) \tag{12}$$

it is clear that when $\mathcal{B}$ is maximal, each $\vec{a}_i$ points to the same direction as the linear combination of the $\vec{b}_j$ vectors it is multiplied with. Therefore, however large the number of Alice's measurement settings m_A is, her vector space need not have a larger dimensionality than that of Bob. Bob's m_B vectors can always be accommodated in an m_B-dimensional space. What we will show here is that if $m_A<m_B(m_B-1)/2+1$, then even vectors from a smaller space are enough to maximize $\mathcal{B}$.

Let $\overline{\mathcal{B}}$ be the same as $\mathcal{B}$ with Alice's vectors $\vec{a}_i$ chosen optimally. $\overline{\mathcal{B}}$ is only the function of Bob's vectors $\vec{b}_j$:

$$\overline{\mathcal{B}}=\sum_{i=1}^{m_A}\left|\sum_{j=1}^{m_B}M_{ij}\vec{b}_j\right|\equiv\sum_{i=1}^{m_A}l_i. \tag{13}$$

We can always find the maximum of $\mathcal{B}$ by maximizing $\overline{\mathcal{B}}$ in terms of the unit vectors $\vec{b}$ of an m_B-dimensional vector space. The terms of the above equation can further be written as

$$l_k = \sqrt{\sum_i M_{ki}^2 + 2\sum_{i>j} M_{ki}M_{kj}X_{ij}}, \tag{14}$$

where

$$X_{ij} \equiv \vec{b}_i \cdot \vec{b}_j. \tag{15}$$

The X_{ij} $(j<i\le m_B)$ values determine the relative directions of the unit vectors and, therefore, up to an irrelevant orthogonal transformation, the vectors themselves. We will regard $\overline{\mathcal{B}}$ as a function of these $N\equiv m_B(m_B-1)/2$ numbers. Besides $|X_{ij}|\le 1$, the variables must satisfy several other constraints. The domain of the function is where the Gramian matrix, the symmetric matrix containing X_{ij} and $X_{ii}=1$ is positive semidefinite. Its determinant, the Gramian, is zero at the boundary of the domain, which means that the vectors $\vec{b}_j$ are linearly dependent. If the X_{ij} variables maximizing $\overline{\mathcal{B}}$ lie there, then a space of less than m_B dimensions is enough to accommodate the corresponding vectors. The whole m_B-dimensional space is only required if the maximum is not at the boundary of the domain. The $\overline{\mathcal{B}}$ as a function of X_{ij} is nonanalytical only where $l_i=0$, but that occurs only at the boundary, as $l_i=0$ means linear dependence of the vectors.

From these it follows that if the vectors $\vec{b}_j$ maximizing $\overline{\mathcal{B}}$ span the m_B-dimensional space, then all partial derivatives of the function must vanish there. Let us introduce a single index denoted by a Greek letter instead of the pair $\{i,j\}$, say let there be $\nu=(i-1)(i-2)/2+j$. We also introduce the following notation: $x_\nu \equiv X_{ij}$, $Y_{k\nu}\equiv M_{ki}M_{kj}$, and $C_k\equiv \Sigma_i M_{ki}^2$. With this notation

$$l_k = \sqrt{C_k + 2\sum_{\nu=1}^{N} Y_{k\nu}x_\nu}. \tag{16}$$

At the maximum of $\overline{\mathcal{B}}$

$$\frac{\partial \overline{\mathcal{B}}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\sum_{k=1}^{m_A} l_k = \sum_{k=1}^{m_A} Y_{k\alpha}\frac{1}{l_k} = 0 \tag{17}$$

must hold. This is a set of N homogenous linear equations for $1/l_k$. If the number of variables m_A is less than $N+1$, the equations may only have nontrivial solution if no more than m_A-1 of them are independent. If such a solution exists, it may define l_k only up to a constant factor. Therefore, $l_k=y/\lambda_k$, where λ_k is a nontrivial solution. All λ_k must be nonzero and must have the same sign, because l_k must be positive. If these conditions hold, from Eq. (16) we get a set of m_A linear equations for y^2 and x_ν:

$$2\sum_{\nu=1}^{N} Y_{k\nu}x_\nu - y^2/\lambda_k^2 = -C_k. \tag{18}$$

The solution of these equations can only correspond to a solution of our problem if y^2 is positive and x_ν are within the boundary of the domain and if it defines a maximum. However, if all these are true, the solution is not unique if the number of equations m_A is less than the number of variables $N+1=m_B(m_B-1)/2+1$. In this case the equations are satisfied on a whole subspace of the space containing the domain of $\overline{\mathcal{B}}$. The function is constant in that subspace up to the boundary of the domain. Therefore, where the subspace crosses the boundary, the function will still have its maximum value. This way $\overline{\mathcal{B}}$ can be maximized in a less than m_B-dimensional space.

To summarize the argument, if m_A is not large enough, the equations requiring the partial derivatives to be zero usually have no solution corresponding to a maximum. In this case the function will take its maximum value with its variables at the boundary of their domain. With the Bell factors M_{ij} having some very special values the equations may be solvable, but then they cannot be independent. Therefore they cannot define the variables in a unique way. The function will have the same value on a subset of its domain, extending to the boundary. This means that either there is no solution or there is an infinity of them. Only vectors spanning a less than m_B-dimensional space will maximize $\overline{\mathcal{B}}$ in the first case, and there will exist such vectors maximizing $\overline{\mathcal{B}}$ in the second case.

B. Family of Bell expressions

When $m_A=m_B(m_B-1)/2+1=N+1$ then the whole m_B-dimensional space may be required to maximize $\overline{\mathcal{B}}$. An example of such a case is the following:

$$\mathcal{B} \equiv \sum_{m_B\ge i>j\ge 1} \vec{a}_{ij}(\vec{b}_i - \vec{b}_j) + \gamma\vec{a}_0\sum_{i=1}^{m_B}\vec{b}_i, \tag{19}$$

where the corresponding M_{ij} is defined through Eq. (4), parametrized by m_B, and $\gamma>0$. Note that setting $\gamma=0$, we obtain the Z_n family introduced in [2]. By choosing Alice's vectors optimally,

$$\overline{\mathcal{B}} = \sum_{i>j}|\vec{b}_i - \vec{b}_j| + \gamma\left|\sum_{i=1}^{m_B}\vec{b}_i\right| = \sum_{\mu=1}^{N}\sqrt{2-2x_\mu} + \gamma\sqrt{m_B + 2\sum_{\nu=1}^{N}x_\nu}. \tag{20}$$

Here we used the single Greek letter notation for the $\{i,j\}$ pair we introduced earlier. The partial derivatives of Eq. (20) are

$$\frac{\partial\overline{\mathcal{B}}}{\partial x_\alpha} = -\frac{1}{\sqrt{2-2x_\alpha}} + \frac{\gamma}{\sqrt{m_B + 2\sum_{\nu=1}^{N}x_\nu}}, \tag{21}$$

which are zero at

$$x_\alpha = x = \frac{2\gamma^2 - m_B}{2\gamma^2 + m_B(m_B-1)}. \tag{22}$$

Thus at the extremal point all nondiagonal entries of the Gramian will be equal, $x<1$, and the diagonal ones are all 1 (the $\vec{b}_i$ are normalized). Let us consider a determinant of size $k\times k$ with all diagonal elements having the value of p and all nondiagonal ones are equal to q. Let us subtract the (i

-1)th column from the ith one in turn for $i=k,k-1,\dots,3,2$, in this order. Then each column, except for the unchanged first one, will contain $p-q$ in the diagonal, $q-p$ just above it, and zero everywhere else. Then add the ith row to the $(i-1)$th one in turn for $i=k,k-1,\dots,3,2$, in this order. This way only the diagonal element will be nonzero in each column but the first one, with a value of $p-q$, while the first element of the first column will be the sum of all original entries in that column, that is, $p+(k-1)q$. Therefore, the value of the determinant, which is now the product of all diagonal entries, will simply be $[p+(k-1)q](p-q)^{k-1}$. With $p=1$ and $q=x>-1/(m_B-1)$, which is true if $\gamma>0$, the determinant is positive for any $k\le m_B$. Therefore, the Gramian is positive definite, which means that we are not at the boundary of the domain. This is the only extremal point, and at that point

$$\overline{\mathcal{B}}=T^{m_B}=m_B\sqrt{\gamma^2+m_B(m_B-1)/2}. \tag{23}$$

The second derivatives of Eq. (20) are

$$\frac{\partial\overline{\mathcal{B}}}{\partial x_\alpha\,\partial x_\beta}=-(2-2x_\alpha)^{-3/2}\delta_{\alpha\beta}-\gamma\left(m_B+2\sum_{\nu=1}^{N}x_\nu\right)^{-3/2}. \tag{24}$$

At the extremal point all nondiagonal elements $\alpha\neq\beta$ of -1 times the matrix are equal, $q'=\gamma[m_B+m_B(m_B-1)x]^{-3/2}>0$, and also all diagonal elements $\alpha=\beta$ are equal, $p'=q'+(2-2x)^{-3/2}>q'$. From the values of the determinants of matrices with such entries given above, it follows that minus one times the second derivative matrix is positive definite at the extremum; therefore the extremum is a true maximum. There is no other extremal point in the domain of the function, so this one has to be the absolute maximum of $\overline{\mathcal{B}}$. This proves $T^n<T^{n+1}$ for Bell expression (19) for $n=m_B-1$, entailing a dimension witness $d=\lfloor m_B/2\rfloor$ for every m_B according to Lemma 2.

The $\gamma=\sqrt{m_B/2}$ is an interesting special value, in which case $x=0$; that is, all $\vec{b}_i$ vectors are orthogonal to each other. Geometrically, we may consider $\vec{b}_i$ as edges, and $\vec{b}_i-\vec{b}_j$ as face diagonals of a hypercube, while $\sum_{i=1}^{m_B}\vec{b}_i$ is a vector pointing toward one of its space diagonals. If we decrease γ, that is, the weight of this space diagonal in the expression to be maximized, the geometrical object will flatten along this direction, and at $\gamma=0$ it will collapse to become an (m_B-1)-dimensional object. The corresponding Bell inequality has one less measurement settings for Alice, and it belongs to family Z in Ref. [2]. When $\gamma\neq0$, the one extra term is enough to prevent the collapse, and the whole m_B-dimensional space is required to accommodate the optimum object.

We can determine the classical limit as the maximum value of

$$\overline{\mathcal{B}}=\sum_{i>j}|z_i-z_j|+\gamma\left|\sum_{i=1}^{m_B}z_i\right|, \tag{25}$$

with $z_i=\pm1$ for all $i<m_B$. The expression is permutation invariant; therefore it only depends on the number of $z_i=+1$ values. Let it be k. Then $\Sigma_{i>j}|z_i-z_j|=2k(m_B-k)$, because $|z_i-z_j|=2$ when $z_i\neq z_j$, which occurs $k(m_B-k)$ times; otherwise it is zero. At the same time $|\Sigma_{i=1}^{m_B}z_i|=|k-(m_B-k)|=|m_B-2k|$; therefore, $\overline{\mathcal{B}}=2k(m_B-k)+\gamma|m_B-2k|$. This expression has the same value at k and at m_B-k; therefore it is enough to consider $k\le m_B/2$. It is easy to show that $\overline{\mathcal{B}}$ takes its maximum at $k_{\max}$, which is the non-negative integer nearest to $(m_B-\gamma)/2$, and its maximum value, that is, the classical limit, is $\overline{\mathcal{B}}_{\rm cl}=(m_B^2+\gamma^2-4\Delta^2)/2$, where $\Delta=|(m_B-\gamma)/2-k_{\max}|$. For $\gamma>m_B$ this value is γm_B.

As we have shown, to get the maximum value for this family of Bell inequalities we need m_B-dimensional vector spaces. With (m_B-1)-dimensional spaces we may only get a smaller value, but it is interesting to know how much smaller. In the following this gap will be determined for particular number of m_B.

C. Two-dimensional witnesses

We have determined numerically the maxima achievable in (m_B-1)-dimensional spaces for small m_B cases. The solutions are nontrivial; they have different structures for different ranges of γ. As three-dimensional spaces correspond to qubits [2,5], Bell expression (19) defined above with $m_B=4$ and $m_A=7$ are especially interesting. They have the smallest number of measurement settings for one of the parties among correlation-type Bell expressions (i.e., involving only joint correlation), whose maximum violation cannot be obtained with qubits, while the other participant has just as few measurement settings as absolutely necessary. The maximum value that can be achieved with three-dimensional vectors (i.e. with qubits when working with tensor products of Hilbert spaces) as a function of γ consists of three regions. For $\gamma=0$ the four vectors $\vec{b}_i$ point toward the vertices of a regular simplex. For finite γ this simplex becomes somewhat distorted; its shape will be a pyramid, whose base is a regular triangle and whose apex is above the center of the base. The maximum is given by such a solution up to $\gamma=1.4153$, a value very near to $\sqrt{2}$, where a less regular shape takes over. This is where the three-dimensional maximum value differs the most from the global, the four-dimensional, one; their ratio here is 1.010 716 1. In the third region, valid for larger γ, the four $\vec{b}_i$ vectors point toward the corners of the square.

Remembering (according to remarks at the end of Sec. II) that real qubits correspond to two-dimensional Euclidean spaces and recalling the proof of Sec. III A, it follows that correlation Bell inequalities with $m_A,m_B\le3$ do not require complex Hilbert spaces for their maximal violation. On the other hand, Bell inequality (19) with $m_B=3$ and $m_A=4$ and with $\gamma=1$ does require complex numbers to achieve maximum violation (in the Euclidean space Bob's optimal vectors form an orthogonal triad). Note that this inequality is just the same as the 3×4 setting elegant Bell inequality introduced in [14]. From Eq. (23) it follows that the maximum value is 6 in $\mathbb{R}^3$, corresponding to complex qubits. On the other hand, the maximum restricted to $\mathbb{R}^2$, i.e., to real qubits, can be obtained through semidefinite programming [15]. Expression (19) to be maximized can be brought to a form containing only quadratic variables subject to quadratic constraints (actually we have 14 variables with 7 constraints). This nonconvex problem can be solved via a series of convex relaxations of increasing size [16]. This technique provides in each step a better upper bound to the global optimum. We obtain the upper bounds of 6 in the first order and 5.8894 in the second order, with the latter value coinciding up to numerical precision with the value which can be attained numerically. We used the GLOPTIPOL 3 package [17] to solve this global optimization problem. This approach thus gives the gap between the maximum quantum value achievable in the complex and real qubit spaces through an exact treatment (the ratio being $6/5.8894=1.018\,78$). The gap takes its maximum at $\gamma=1.3946$ with a ratio of 1.020 804 7. On the other hand, using a heuristic method for the $m_B=5$ and $m_A=11$, and for the $m_B=6$ and $m_A=16$ cases, the maximum ratios are 1.006 231 7 (at $\gamma=1.6396$), and 1.004 196 4 (at $\gamma=1.7642$), respectively.

Now let us come back to Bell expression (19) with $m_B=4$ and $m_A=7$. From our proof it does not follow that this is actually the one with the minimum total number of measurement settings whose maximum violation requires higher-dimensional spaces than qubits. If we allow both participants to have more than the minimum number of measurements, the sum may be decreased; with $m_B=m_A=5$ it is just 10. We have generated and checked numerically all 44 685 nonequivalent nontrivial inequalities with M_{ij} values restricted to 0, 1, and -1, and we have found that qubits were enough to get the maximum value for each of them. We then allowed M_{ij} to be 0, 1, -1, 2, and -2 while confining ourselves to symmetric matrices. We found no case requiring more than qubits for maximum violation even among these 7.66×10^6 inequivalent cases. Although this still does not prove that there is no $m_B=m_A=5$ correlation-type Bell inequality with this property, but it makes it likely. With $m_A+m_B=11$ we found several examples even with allowing only 0, 1, and -1 values for M_{ij}. With $m_B=4$ and $m_A=7$ we found 11 inequivalent cases (including the one discussed above with $\gamma=1$), and with $m_B=5$ and $m_A=6$ our extensive search gave 79 examples. Note that in these cases all of our results are due to numerical search, since the complexity of this particular problem was too large to be handled by the semidefinite-programming approach discussed previously. We are quite confident, though, about the results obtained by heuristic numerical computations.

The necessity of four-dimensional Euclidean space in achieving the maximum value means that in the Hilbert-space picture a pair of qubits is not enough [2,5]. However, according to the construct of Cirel'son [12] (see paragraph above Lemma 2), what one can achieve in four-dimensional Euclidean spaces, one can also get with measurement operators in four-dimensional complex Hilbert spaces. We have calculated the maximum violation of the 79 $m_B=5$ and $m_A=6$ and the 11 $m_B=4$ and $m_A=7$ inequalities mentioned in the previous paragraph, the γ-dependent $m_B=4$ and $m_A=7$ case for a few γ values, and for the $m_B=4$ and $m_A=8$ example X_4 introduced in Ref. [2] with determining the appropriate measurement operators via numerical optimization according to Refs. [9,10]. In all cases complex four-dimensional Hilbert spaces were required; smaller spaces were never enough.

IV. BOUNDS ON DIMENSIONS WITH INFINITE NUMBER OF SETTINGS

In this section we consider a bipartite Bell inequality with a continuum infinite number of two-outcome measurement settings for each party, which will be proved to serve as dimension witnesses for arbitrary dimensions. Let the indices of the measurement settings be m-dimensional unit vectors, that is, elements of S^{m-1}, the surface of the unit sphere in R^m. Let the Bell coefficients be proportional to the inner product of their indices,

$$M(x,y)\equiv m\langle x,y\rangle, \tag{26}$$

where the factor of m has been introduced for the sake of convenience. By construction the coefficient matrix $M(x,y)$ is positive semidefinite. Equation (26) was worked out by Grothendieck [18] in context of functional analysis. We use it in the following in matrix version formalized in [19]. The Bell expression with n-dimensional Euclidean vectors will then be

$$\mathcal{B}\equiv\int_{S^{m-1}}d\sigma(x)\int_{S^{m-1}}d\sigma(y)m\langle x,y\rangle\vec{a}(x)\cdot\vec{b}(y), \tag{27}$$

where σ denotes the normalized surface of the m-dimensional unit sphere S^{m-1}, and $\vec{a}(x)$ $(x\in S^{m-1})$ and $\vec{b}(y)$ $(y\in S^{m-1})$ are the n-dimensional unit vectors corresponding to the measurement operators of Alice and Bob, respectively. As in the previous sections, we denote the scalar product of these vectors with a dot. Grothendieck [18] constructed this example to provide a lower bound for his constant [8]. Actually, he calculated expression (27) for a uniform distribution of $x,y\in S^{m-1}$ for $m\to\infty$, but to our knowledge the optimality of the solution for any m and n has not been proved yet. Below we give an optimal solution for any value of m and n, and show that by fixing $m\to\infty$ the maximum value in $\mathbb{R}^n$ is strictly increasing in n. As discussed in Lemma 2, this will provide us a dimension witness for any finite dimension. The maximum value of the Bell expression achievable with n-dimensional vectors will then be

$$T^n=\sup_{\vec{b}(y)\in S^{n-1}}m\int_{S^{m-1}}d\sigma(x)\left|\int_{S^{m-1}}d\sigma(y)\langle x,y\rangle\vec{b}(y)\right|, \tag{28}$$

where $\vec{a}(x)$ has been chosen optimally, that is, parallel to

$$\vec{h}(x)\equiv\int_{S^{m-1}}d\sigma(y)\langle x,y\rangle\vec{b}(y). \tag{29}$$

In the following an exact value will be given for the maximum value of this Bell expression for any dimension n in the Euclidean space with respect to the classical case $\vec{a},\vec{b}=\pm1$ (i.e., setting $n=1$). Due to the lemma of Rietz [20] this ratio can maximally be $\pi/2$ for a positive semidefinite matrix M. Due to Grothendieck [18] this ratio can be achieved for infinite m,n. This proves incidentally that the solution in infinite m,n is optimal. However, next we can calculate exactly the optimal values T^n for any n and m in contrast to the case in Sec. III B.

Due to the linearity of $\vec{h}(x)$, for each of its components $h_i(x)$ there exist a $z_i \in S^{m-1}$ and a scalar λ_i [21] such that

$$h_i(x) = \lambda_i \langle x, z_i \rangle. \tag{30}$$

The $h_i(x)$ is maximal at z_i, and its value there is λ_i:

$$\lambda_i = h_i(z_i) = \int_{S^{m-1}} d\sigma(y) \langle z_i, y \rangle b_i(y). \tag{31}$$

From these it follows that the maximum value of $\mathcal{B}$ in $\mathbb{R}^n$ may be written as

$$T^n = \sup_{\vec{b}(y) \in S^{n-1}} m \int_{S^{m-1}} d\sigma(x) \sqrt{\sum_{i=1}^{n} \lambda_i^2 \langle x, z_i \rangle^2}. \tag{32}$$

Let us introduce the following generalized spherical polar coordinates for the components of unit vector x:

$$x_1 = \cos \varphi_1,$$

$$x_i = \cos \varphi_i \prod_{\mu=1}^{i-1} \sin \varphi_\mu, \quad 1 < i < m,$$

$$x_m = \prod_{\mu=1}^{m-1} \sin \varphi_\mu. \tag{33}$$

The integral of a function $\mathcal{F}(x)$ on the normalized unit sphere is

$$\int_{S^{m-1}} d\sigma(x) \mathcal{F}(x) = \int_0^{2\pi} d\varphi_{m-1} \int_0^{\pi} d\varphi_{m-2} \sin \varphi_{m-2} \cdots,$$

$$\int_0^{\pi} d\varphi_i \sin^{m-i-1} \varphi_i \cdots \int_0^{\pi} d\varphi_1 \sin^{m-2} \varphi_1 \mathcal{F} \frac{1}{2 s_0 s_1 \cdots s_{m-2}}, \tag{34}$$

where $s_i \equiv \int_0^{\pi} \sin^i \varphi d\varphi = \sqrt{\pi}\Gamma((i+1)/2)/\Gamma((i+2)/2)$.

Let us consider the case of $n \le m$. Let us choose the basis such that the last n unit vectors $e_{m-n+1}, e_{m-n+2}, \ldots, e_m$ span an n-dimensional subspace which contains all z_i. The value of expression (32) does not change if we replace x in the integrand with its projection $P_n x$ onto this subspace. Let us define $\|P_n x\| = \sqrt{\langle P_n x, P_n x \rangle}$ and introduce

$$x' \equiv \frac{P_n x}{\|P_n x\|}, \tag{35}$$

a unit vector in the n-dimensional subspace. Then

$$T^n = \sup_{\vec{b}(y) \in S^{n-1}} m \int_{S^{m-1}} d\sigma(x) \sqrt{\sum_{i=1}^{n} \lambda_i^2 \langle x', z_i \rangle^2} \|P_n x\|. \tag{36}$$

With our choice of the basis one can easily see from definition (33) of the generalized polar coordinates that $\|P_n x\| = \sin \varphi_{m-n} \sin \varphi_{m-n-1} \cdots \sin \varphi_1$, depending only on the first $m-n$ angles. At the same time, x' is independent of these variables; therefore, the two factors in the integrand of Eq. (36) may be integrated separately. The integral of the first factor will be an integral on S^{n-1}, while the value of the integral of the second factor, as one can easily verify by substituting $\mathcal{F}$ with $\|P_n x\|$ in Eq. (34), is s_{m-1}/s_{n-1}.

We will determine T^n by constructing an upper limit, and then finding an explicit solution which saturates it. Let us consider

$$U_{kn} = \int_{S^{k-1}} d\sigma(x) \sqrt{\sum_{i=1}^{n} \lambda_i^2 \langle x, z_i \rangle^2}. \tag{37}$$

The function depends on $\vec{b}(y)$ (through λ_i and z_i), and $T^n = m s_{m-1}/s_{n-1} \sup_{\vec{b}(y) \in S^{n-1}} U_{nn}$ if $n \le m$, and $T^n = m \sup_{\vec{b}(y) \in S^{n-1}} U_{mn}$ otherwise. By using $\int f(x) dx \le \sqrt{\int dx \int f^2(x) dx}$ we get

$$U_{kn} \le \sqrt{\int_{S^{k-1}} d\sigma(x) \sum_{i=1}^{n} \lambda_i^2 \langle x, z_i \rangle^2} = \sqrt{\frac{1}{k} \sum_{i=1}^{n} \lambda_i^2}. \tag{38}$$

To get the last expression the integral of $\langle x, z_i \rangle^2$ on the k-dimensional sphere has been carried out, which can easily be done using appropriate polar coordinates. For $\Sigma_{i=1}^n \lambda_i^2$ we get

$$\sum_{i=1}^{n} \lambda_i^2 = \int_{S^{m-1}} d\sigma(y) \sum_{i=1}^{n} \lambda_i \langle z_i, y \rangle b_i(y), \tag{39}$$

by substituting one λ_i factor in each term according to Eq. (31) and by changing the order of the summation and the integral. We may increase the integrand by replacing the vector $\vec{b}(y)$ by the unit vector parallel to $\vec{h}(y)$, that is, the vector whose components are $\lambda_i \langle z_i, y \rangle$. We may not necessarily be allowed to choose $\vec{b}(y)$ this way, as λ_i and z_i are determined by $\vec{b}(y)$, and this choice may be inconsistent. However, we can overestimate the integral by making this replacement; therefore

$$\sum_{i=1}^{n} \lambda_i^2 \le \int_{S^{m-1}} d\sigma(y) \sqrt{\sum_{i=1}^{n} \lambda_i^2 \langle z_i, y \rangle^2}. \tag{40}$$

Comparing the right-hand side of this inequality with Eq. (32), we can see that the maximum value it may take is nothing else but T^n divided by m. Therefore, it follows that if $n \le m$,

$$T^n \le \frac{m s_{m-1}}{s_{n-1}} \sqrt{\frac{1}{n} \frac{1}{m} T^n}, \tag{41}$$

that is,

$$T^n \le \frac{m}{n} \frac{s_{m-1}^2}{s_{n-1}^2} = \frac{s_{m-1}}{s_m} \frac{s_n}{s_{n-1}}. \tag{42}$$

For the last form we used the identity $n s_n s_{n-1} = 2\pi$.

Let us choose $\vec{b}(x)$ to be the normalized projection of x onto the n-dimensional subspace of the m-dimensional space spanned by the last n members of the basis introduced earlier, that is,

$$b_i(x) = \frac{\langle e_{m-n+i},x\rangle}{\sqrt{\sum_{j=m-n+1}^{m}\langle e_j,x\rangle^2}} = \frac{\langle e_{m-n+i},x\rangle}{\|P_n x\|}$$

$$= \cos\,\varphi_{m-n+i}\prod_{\mu=m-n+1}^{m-n+i-1}\sin\,\varphi_\mu. \tag{43}$$

For the last form we used the form of generalized polar coordinates (33), and that $\|P_n x\| = \Pi_{\mu=1}^{m-n}\sin\,\varphi_\mu$. To get $h_i(e_j)$ we have to integrate the product of $\langle e_j,x\rangle = \cos\,\varphi_j \Pi_{\mu=1}^{j-1}\sin\,\varphi_\mu$ and $b_i(x)$ on S^{m-1} [see Eq. (29)]. From Eq. (34) one can see that for φ_j and for φ_{m-n+i}, we have to integrate the product of the cosine function and a power of the sine function if $j \neq m-n+i$. Therefore $h_i(e_j)=0$ for these values of j. From this it follows that $z_i = e_{m-n+i}$, being $h_i(x)$ maximal for that vector. Then we can explicitly calculate λ_i from Eq. (31) by integrating $\langle e_{m-n+i},x\rangle b_i(x) = \Pi_{\mu=1}^{m-n}\sin\,\varphi_\mu \Pi_{\nu=m-n+1}^{m-n+i-1}\sin^2\,\varphi_\nu\cos^2_{m-n+i}\varphi_\nu$ on S^{m-1}. Substituting this into Eq. (34), performing the integrations, and simplifying the expression by $\Pi_{\tau=m-n+i+1}^{m-1}s_{m-\tau-1}$, we get

$$\lambda_i = \frac{\prod_{\mu=1}^{m-n}s_{m-\mu}\prod_{\nu=m-n+1}^{m-n+i-1}s_{m-\nu+1}(s_{n-i-1}-s_{n-i+1})}{\prod_{\tau=1}^{m-n+i}s_{m-\tau-1}}$$

$$= \frac{s_{m-1}}{s_{n-1}}\frac{s_n s_{n-1}}{s_{n-i+1}s_{n-i}}\frac{s_{n-i-1}-s_{n-i+1}}{s_{n-i-1}} = \frac{s_{m-1}}{n s_{n-1}}. \tag{44}$$

For the last equality the identities $s_n/s_{n-2}=(n-1)/n$ and $n s_n s_{n-1} = 2\pi$ have been used. Thus λ_i is independent of i, as it has to be due to symmetries. Then by bringing $\lambda = \lambda_i$ in front of the integral in Eq. (32) and using $z_i = e_{m-n+i}$, we can see that the integrand remaining is nothing else but $\|P_n x\|$, whose integral is s_{m-1}/s_{n-1}, as we have seen earlier. The result we get then for T^n is equal to upper limit (42).

For $n \geq m$, it is easy to verify that the upper limit is 1 [instead of Eq. (41) we get $T^n \leq \sqrt{T^n}$], which is the well-known quantum limit for this inequality. It can be reached with m-dimensional vector space, with $\vec{b}(x) = x$. As the limit remains the same for $n > m$, there is no need for the extra dimensions. Although the Bell inequality involves an infinite number of measurement settings for each party, a finite-dimensional space is enough to reach the quantum limit.

Now we set $m \to \infty$ in Eq. (42) and calculate T^n for different n values. In this case $T^n = s_n/s_{n-1}$. By choosing $n=1$ we obtain the known classical limit $2/\pi$. Thus the maximum quantum violation (quantum limit per classical limit) is $(\pi/2)s_n/s_{n-1}$. For $n=2$ the ratio is $\pi^2/8 \approx 1.2337$, corresponding to measurement on pairs of real qubits, while with $n=3$, corresponding to complex qubits, it is 4/3. We have to go up to $n=5$ to get a maximum violation of $64/45 \approx 1.4222$, larger than the value of $\sqrt{2}$ one can achieve with the Clauser-Horne-Shimony-Holt (CHSH) inequality. For $n \to \infty$ the maximum violation is the well-known $\pi/2$, which is $3\pi/8 \approx 1.1781$ times larger than what we can achieve with qubits.

Most importantly the ratio s_n/s_{n-1} is a strictly increasing function of n. Notably for n even it is $(\pi/4)\Pi_{i=1}^{n/2-1}(2i+1)^2/[(2i+1)^2-1]$ and for n odd it is equal to $(2/\pi)\Pi_{i=1}^{(n-1)/2}(2i)^2/[(2i)^2-1]$. Clearly both are strongly monotone functions of n. This entails $T^n < T^{n+1}$ by $m \to \infty$ for all n and owing to Lemma 2 proves the existence of dimension witnesses for any finite dimension.

V. CONCLUSION

In the present work we focused our attention on joint quantum correlations arising from local measurements on bipartite systems. Cirel'son [12] established a connection between these joint correlations in Hilbert space and standard inner products of unit vectors in Euclidean space. Based on this result we give a proof that constructing a correlation Bell expression for which $T^n < T^{n+1}$ in the vectorial picture (where T^n denotes the maximum value achievable in the n-dimensional Euclidean space) implies joint correlations which cannot be reproduced in a d-dimensional Hilbert space, where $d = \lfloor\frac{n+1}{2}\rfloor$. This defines a d-dimensional witness. For this sake, we discuss two particular families of Bell inequalities. The one in Sec. III involves a finite number of measurement settings m_B and $m_A = m_B(m_B-1)/2+1$, whereas the other one in Sec. IV involves continuously many settings on both sides. Though in the former case we cannot give the difference $T^{n+1}-T^n$ explicitly, except for small values of m_B, in the latter case this difference can be analytically calculated for all n. This conclusively proves the existence of dimension witnesses for arbitrary dimension d in a bipartite quantum system. Recently, we learned that Briët *et al.* arrived in Ref. [22] at similar conclusions. Furthermore, we discuss the minimum number of measurement settings arising in correlation Bell expressions in order to generate dimension witnesses. In this respect we prove that the numbers of settings $m_B=4$ and $m_A=6$ (or less settings) are not sufficient to generate a two-dimensional witness. In contrast, examples are given for a two-dimensional witness for settings $m_B=4$ and $m_A=7$ and for settings $m_B=5$ and $m_A=6$. It remains an open question whether for the pair of settings $m_B=5$ and $m_A=5$ there exists a two-dimensional dimension witness or not.

Due to the work of Cirel'son [12] for a given finite number of binary measurement settings, it is always possible to generate all the joint bipartite quantum correlations of binary outcomes in a finite-dimensional Hilbert space. However, in generic Bell expressions involving local marginal terms as well, this may not be true. With respect to it, Navascués *et al.* [23] asked recently whether there exist scenarios with finite number of measurement settings for both parties, for which all quantum correlations can be attained by measuring an infinite-dimensional entangled system. We leave this interesting problem as a challenge for future investigations.

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