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Optimal estimation of losses at the ultimate quantum limit with non-Gaussian states

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We address the estimation of the loss parameter of a bosonic channel probed by arbitrary signals. Unlike the optimal Gaussian probes, which can attain the ultimate bound on precision asymptotically either for very small or very large losses, we prove that Fock states at any fixed photon number saturate the bound unconditionally for any value of the loss. In the relevant regime of low-energy probes, we demonstrate that superpositions of the first low-lying Fock states yield an absolute improvement over any Gaussian probe. Such few-photon states can be recast quite generally as truncations of de-Gaussified photon-subtracted states.

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I. INTRODUCTION

Suppose an experimenter is given different stations connected by channels manufactured before he or she joined the laboratory. Assume that he or she is allowed to routinely transmit light beams prepared, e.g., in coherent, squeezed, or number states through these channels with the purpose of implementing some communication network. Given the state at one station, one finds out that in general the state has been altered at the next node. The problem is then to determine what sort of noise is affecting the transmissions. This is a typical issue of *quantum parameter estimation*, whose solution is clearly of direct interest to practical situations such as the one described. The experimenter wishes to determine the optimal probe state that has to be sent through the channel and the optimal measurement that needs to be performed at the output in order to estimate (after repeating the process N times) the loss parameter with the maximum possible precision. In the case of amplitude damping bosonic channels, Monras and Paris [1] provided a solution to the problem in the particular case of Gaussian input probe states (displaced and squeezed vacuum states). Gaussian states are easier to engineer by quantum optical means than more sophisticated non-Gaussian states. On the other hand, the optimal measurement needed for loss estimation involves manipulations such as displacement and squeezing of the output signal and photon counting, which is a non-Gaussian measurement not belonging to the standard toolbox of linear optics. At a fundamental level, using Gaussian probe inputs allows to saturate the ultimate bound on precision only asymptotically in the unphysical limits of infinitesimal or infinite losses, while in the realistic regime of intermediate loss the Gaussian-based estimation is clearly suboptimal [1]. Therefore considering Gaussian inputs does not solve the important problem of *optimal* estimation of loss in bosonic channels, both on theoretical and practical grounds.

In this work, we study the estimation of loss in bosonic channels probed by arbitrary non-Gaussian states. For any energy of the probes, we show that there exist non-Gaussian states improving the estimation compared to Gaussian states in all regimes of loss. Specifically, we prove that Fock states $|n\rangle$ (which can be produced deterministically in the laboratory [2]) are the truly optimal probes that attain the ultimate quantum limit exactly, for any n and any value of the loss. The optimal estimation then requires only photon counting, resulting in a technological simplification compared to the Gaussian case. For low-energy probes (mean photon number smaller than 1), we construct optimal superpositions of the first k low-lying Fock states which improve the estimation over the Gaussian case already for $k=2,3$ and approach the ultimate limit in a much broader range of losses. Interestingly, we find that the optimal superpositions for $k=2$ correspond to qutritlike two-photon truncations of photon-subtracted states, showing that de-Gaussification procedures generally allow enhanced performance in the task of loss estimation in quantum channels. This result adds to the diverse existing instances of non-Gaussianity as a "powerup" for quantum information encountered in the optimal cloning of coherent states [3], continuous variable teleportation [4], nonlocality tests [5], and entanglement distillation [6].

II. BOSONIC CHANNELS AND QUANTUM ESTIMATION

We consider a bosonic channel described by the master equation $d\rho/dt=(\gamma/2)\mathcal{L}[a]\rho$ for quantum states ρ, where $\mathcal{L}[a]\rho=2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a$, a being the annihilation operator on the Fock space of a single bosonic mode. The aim of our study is the optimal estimation of the loss parameter γ or equivalently of $\phi\in(0,\pi/2)$ defined by $\tan^2\phi=\exp(\gamma t)-1$. In terms of ϕ, the master equation reads as $d\rho/d\phi=\tan\phi\mathcal{L}[a]\rho$, whose general solution is of the form

$$\rho_\phi=\sum_{n=0}^{\infty}\frac{(\sin^2\phi)^n}{n!}(\cos\phi)^{a^\dagger a}a^n\rho_0(a^\dagger)^n(\cos\phi)^{a^\dagger a}. \quad (1)$$

Let us recall the basic elements of the quantum estimation theory [7,8] and the relevant tools of interest for the present

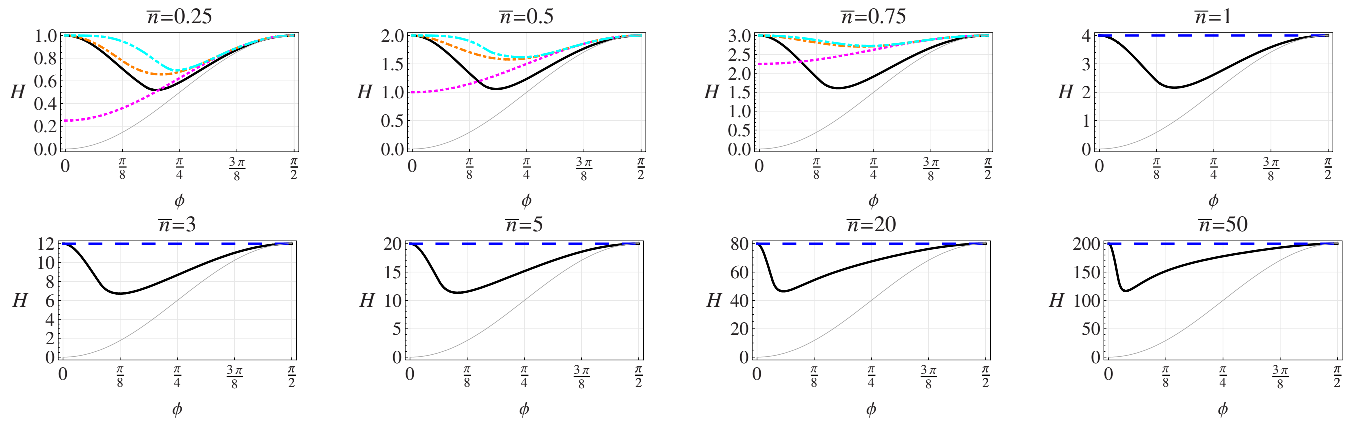

FIG. 1. (Color online) Quantum Fisher information $H(\phi)$ versus the loss parameter ϕ for different values of the input energy $\bar{n}$. Solid black line: optimal Gaussian probes [1]; thin gray line: coherent states; dashed (blue) line: Fock states $|\bar{n}\rangle$; dotted (magenta) line: qubitlike states $|\psi_0^{(1)}\rangle$; dotted-dashed (orange) line: optimal qutritlike states $|\psi_0^{(2)}\rangle$; and dot-dotted-dashed (cyan) line: optimal quartetlike states $|\psi_0^{(3)}\rangle$.

case [1,9]. The optimal estimation of ϕ is achieved asymptotically by sending N independent and identically distributed optimal probe states ρ_0 into the channel and performing at each run the optimal measurement on the output signals ρ_ϕ, in order to construct an estimator $\hat{\phi}$ to infer the true value of ϕ with minimal variance. For any given ρ_0, the optimal output measurement can be exactly determined in terms of the symmetric logarithmic derivative (SLD) $\Lambda(\phi)$ defined implicitly as the Hermitian operator that satisfies $d\rho_\phi/d\phi=(1/2)[\rho_\phi\Lambda(\phi)+\Lambda(\phi)\rho_\phi]$. Using the spectral decomposition $\rho_\phi=\Sigma_k\rho_k|\psi_k\rangle\langle\psi_k|$, one finds for the SLD

$$\Lambda(\phi)=2\tan\phi\sum_{pq}\frac{\langle\psi_q|\mathcal{L}[a]\rho|\psi_p\rangle}{\rho_p+\rho_q}|\psi_q\rangle\langle\psi_p|. \qquad (2)$$

The resulting minimum variance saturates the quantum Cramér-Rao bound $\mathrm{Var}_\phi[\hat{\phi}]\geq 1/[NH(\phi)]$, where the quantum Fisher information (QFI) $H(\phi)$ reads as $H(\phi)=\mathrm{Tr}[\rho_\phi\Lambda(\phi)^2]$. The problem is thus recast in the determination of the optimal single-mode pure input states with a given finite mean energy (or mean photon number $\bar{n}$), such that the QFI of the corresponding output states is maximal. The ultimate quantum limit on the precision that is computable in the ideal assumption that the experimenter may have access also to the degrees of freedom of the environment (i.e., to the oscillators internal to the channel) is achieved for estimators with [1] $\mathrm{Var}_\phi[\hat{\phi}]\geq 1/(4\bar{n}N)$. This means that a truly optimal estimation requires input probes which yield at the output a QFI exactly equal to $4\bar{n}$ (for any single run). If the ensemble of input signals is limited to Gaussian states [1], the estimation is never optimal: the ultimate limit is attained only asymptotically for ϕ approaching 0 or $\pi/2$, while $H(\phi)$ for the best Gaussian probes can get as low as $\sim 2\bar{n}$ for intermediate losses (see Fig. 1). Here we show that non-Gaussian probes, Fock states, and low-lying superpositions thereof are indeed optimal for the estimation of loss in bosonic channels with the maximum precision allowed by laws of quantum mechanics.

III. FOCK STATES

Let us consider, as input probes, Fock states $\rho_0=|n\rangle\langle n|$ $(\bar{n}=n)$. The evolved state, according to Eq. (1), reads as $\rho_\phi=\Sigma_{k=0}^n(\sin^2\phi)^k\binom{n}{k}(\cos^2\phi)^{(n-k)}|n-k\rangle\langle n-k|$. The SLD for this case is $\Lambda(\phi)=\tan\phi\Sigma_{k=0}^n(g_k/f_{n-k})|k\rangle\langle k|$, where $g_k=2[f_{n-k-1}(k+1)(1-\delta_{k,n})-f_{n-k}k]$ and $f_k=\binom{n}{k}(\sin^2\phi)^k(\cos^2\phi)^{n-k}$. The QFI reads as $H(\phi)=\tan^2\phi\Sigma_{k=0}^n(g_k^2/f_{n-k})=4n$. Fock states thus enable the *optimal unconditional* estimation of loss regardless of the actual value of the parameter to be estimated (see Fig. 1). This makes an adaptive estimation scheme unnecessary (unlike the Gaussian case [1]). Moreover, the measurement that has to be performed obtained by projecting onto one-dimensional eigenspaces of $\Lambda(\phi)$ [9] can be implemented only by the simple photon counting. Given the recently achieved degree of control in this measuring technique [10] and in the high-fidelity engineering of Fock states with a small number $n\lesssim 10$ of photons (conditionally for running optical fields and even deterministically in microwave cavity or circuit QED) [2], with $n=2$ standing as an ideal work point, our results might pave the way for an experimental verification of the quantum theory of optimal estimation and a measurement of the SLD to infer the value of such a relevant parameter as the loss factor in dissipative channels. While one may argue that in practice it is easier to produce and manipulate "classical" fields, i.e., coherent states obtained from attenuated laser beams, than nonclassical resources such as squeezed (Gaussian) and Fock states, we remark that—as shown in Fig. 1—the performance of coherent probes for the loss estimation is quite far from optimality. In particular in the regime of small and intermediate losses ($\phi<\pi/4$), corresponding to routinely available good quality fiber channels, the precision achieved by—say—a two-photon Fock state would be matched by that of a coherent field with much higher mean photon number (e.g., $\bar{n}=26$ for $\phi=\pi/16$ and $\bar{n}=105$ for $\phi=\pi/32$), an increase in energy which may be not worth paying in terms of efficiency of the estimation. In fact, in actual implementations it is very desirable to have probes of low energy in order not to alter the channel significantly [11] and to enable repeatability of the input-and-

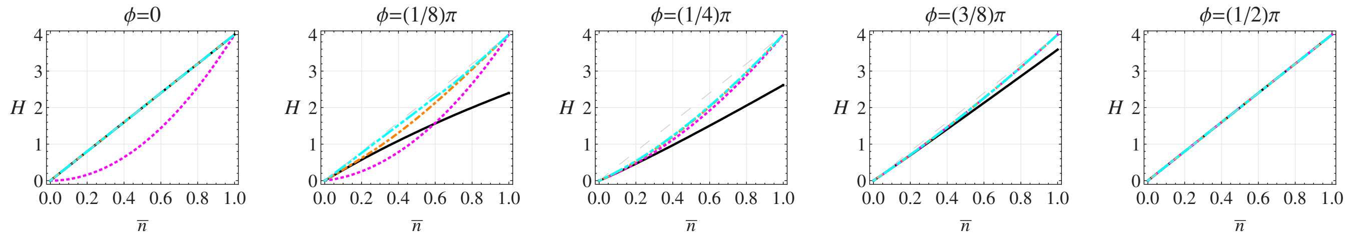

FIG. 2. (Color online) Quantum Fisher information $H(\phi)$ versus the input energy $\bar{n}$ for different values of the loss parameter ϕ. Solid line: optimal Gaussian probes [1]; dotted (magenta) line: qubitlike states $|\psi_0^{(1)}\rangle$; dotted-dashed (orange) line: optimal qutritlike states $|\psi_0^{(2)}\rangle$; and dot-dotted-dashed (cyan) line: optimal ququartetlike states $|\psi_0^{(3)}\rangle$. The thin dashed line depicts the ultimate quantum limit $H(\phi)=4\bar{n}$.

measure scheme. In this respect, it appears crucial to identify classes of non-Gaussian states which may attain the ultimate precision for any value of the input energy, especially in the relevant regime of $0\leq\bar{n}\leq 1$.

IV. PHOTONIC QUBIT STATES

The simplest candidate probe state in the low-energy regime is the superposition of the vacuum and the one-photon Fock state, $\rho_0=|\psi_0^{(1)}\rangle\langle\psi_0^{(1)}|$, $|\psi_0^{(1)}\rangle=\cos\theta|0\rangle+e^{i\varphi}\sin\theta|1\rangle$ characterized by a mean photon number $\bar{n}=\sin^2\theta$. It is rather straightforward to obtain the evolved state ρ_ϕ and to diagonalize it in order to compute the SLD. The resulting expression for the QFI is found to be independent of the phase φ and given by $H^{(1)}(\phi)=4\bar{n}[1-(1-\bar{n})\cos^2\phi]$. We notice (see Figs. 1 and 2) that the considered simple example of the non-Gaussian superposition state (with no free parameter left for optimization) yields a significant improvement over the best Gaussian estimation for intermediate-high losses, although in the regime of small losses and small energies Gaussian states (which in this limit are simply squeezed states [1]) remain better probes.

V. PHOTONIC QUTRIT STATES

Next, we consider superpositions of the vacuum and the first two Fock states $\rho_0=|\psi_0^{(2)}\rangle\langle\psi_0^{(2)}|$, with $|\psi_0^{(2)}\rangle=\cos\alpha|0\rangle+e^{i\mu}\sin\alpha\sin\beta|1\rangle+e^{i\nu}\sin\alpha\cos\beta|2\rangle$. Here α can be fixed as a function of β and $\bar{n}$, $\alpha=\arcsin(\sqrt{\frac{2\bar{n}}{\cos(2\beta)+3}})$. The evaluation of the SLD involves the diagonalization of the 3×3 matrix corresponding to the output state. The QFI has to be optimized, for a given $\bar{n}$, over the phases μ and ν and over the weight β (the latter ranges from $\beta=0$, corresponding to a superposition of $|0\rangle$ and $|2\rangle$, to $\beta=\pi/2$, corresponding to the previously considered qubitlike state superposition of $|0\rangle$ and $|1\rangle$). Maximization over the phases yields $\mu=\nu=\pi$. The optimal β can instead be found numerically for each $\bar{n}$, ϕ, and is reported in Fig. 3. The resulting optimal QFI is shown in Figs. 1 and 2. The photonic qutrit states improve over the qubitlike state and, more remarkably, over the optimal Gaussian probes with the same mean energy in the whole range of parameters (i.e., for any value of the loss). In the limit of vanishing probe energy $\bar{n}\to 0$, the optimal Gaussian probe is a purely squeezed vacuum [1] with QFI $H^{(G)}(\phi)=4\bar{n}[1+z^2]/[1+2z(1+\bar{n})+z^2]$ (where $z=\tan^2\phi$), while the optimal qutritlike state is a pure superposition of $|0\rangle$ and $|2\rangle$ ($\beta=0$) with QFI $H^{(2)}(\phi)=4\bar{n}[1+z^2]/[1+z(2-\bar{n}+z]\geq H^{(G)}(\phi)$. In the limit $\bar{n}\to 1$, the weight β increases up to $\pi/2$ and the qutritlike state converges to the optimal Fock state $|1\rangle$.

VI. RELATION TO DE-GAUSSIFIED STATES

A very important and natural question concerns the nature of such an optimal qutritlike state, in particular, whether such state can be interpreted as a finite-dimensional truncation of an infinite-dimensional non-Gaussian state and how to determine the latter. To this aim, we consider de-Gaussified photon-subtracted, displaced, and squeezed states $\rho_0^{(nG)}=\mathcal{N}^{-1}aD(\eta)S(r)|0\rangle\langle 0|S^\dagger(r)D^\dagger(\eta)a^\dagger$ and study their projections on the subspace spanned by the vacuum and the first two Fock states. The choice for comparison is inspired by the fact that the strategies of photon addition and subtraction are the current royal avenues to the experimental production of optical non-Gaussian resources [12]. Truncation of the photon-subtracted displaced squeezed state $\rho_0^{(nG)}$ yields the state $|\psi^{(nGtr)}\rangle=c_0|0\rangle+c_1|1\rangle+c_2|2\rangle$, where the coefficients $c_j=k_j/(k_0^2+k_1^2+k_2^2)^{1/2}$, with $k_0=\eta(\tanh r+1)$, $k_1=k_0^2-\tanh r$, and $k_2=2^{-1/2}k_0(k_0^2-3\tanh r)$. The coefficients k_j are functions of the modulus of the displacement η and the real squeezing amplitude r (the relative phase can be set to zero in order to maximize the QFI). Remarkably, we find that for any η and r such that $0\leq\bar{n}\leq 1$, the pure states $\rho_0^{(nG)}$ and $\rho_0^{(nGtr)}=|\psi^{(nGtr)}\rangle\langle\psi^{(nGtr)}|$ always possess a high-fidelity overlap $\mathcal{F}=\mathrm{Tr}[\rho_0^{(nG)}\rho_0^{(nGtr)}]>92\%$. Adding one or few further

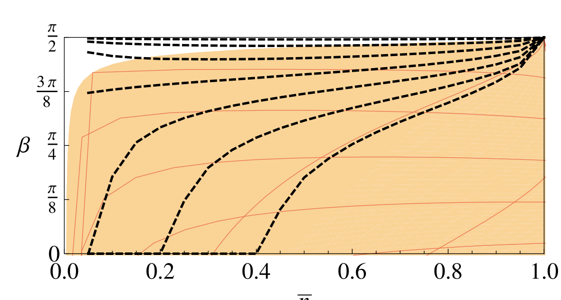

FIG. 3. (Color online) Shaded surface: attainable values in the space of the coefficients β and $\bar{n}$ of qutritlike states, as functions of the parameters r and η associated to the truncation of photon-subtracted states. Dashed lines: optimal β as a function of $\bar{n}$, corresponding to the maximum quantum Fisher information among all qutrit states, for different values of the loss parameter ϕ (ranging from $\pi/16$ to $\pi/2$ from bottom to top).

terms in the superpositions quickly rises the fidelity well above 99%. The mean photon number $\bar{n}$ is a highly nonlinear function of η and r. Thus, in order to visualize the maximization of H over these two resource parameters, at fixed $\bar{n}$, we let η and r vary in the real space and depict as a shaded region in Fig. 3 the achievable range of β (characterizing the originally introduced parametrization of fixed-energy qutrit states) versus the values of $\bar{n}$ that are spanned by the variation in the parameters. We find that there exist combinations of values of η and r for the truncated photon-subtracted states such that the angle β can take almost any value in the range $[0,\pi/2]$ for each $\bar{n}\in[0,1]$. Superimposing the parametric region with the curves of the optimal β as a function of $\bar{n}$ yields the maximum QFI among all qutrit states, for different values of the loss parameter ϕ. Apart from a small range of extremely high losses and low energies (in which practically all states, such as nontruncated Gaussian states, qubit states, qutrit states, etc., yield the same optimal performance close to the ultimate limit), there always exist values of the displacement η and of the squeezing r such that the truncated photon-subtracted state reproduces exactly the optimal qutrit state (pictorially, the dashed lines fall in the attainable shaded region in Fig. 3). On the other hand, this conclusion does not apply to finite truncations of Gaussian states, which are never optimal among all pure qutrit states in many ranges of values of energies and losses, although they can still perform better than the original Gaussian states. This clearly shows that de Gaussification be it implemented by means of truncation, of photon subtraction, or both generally enhances the task of estimating the loss in bosonic Gaussian channels. It is reasonable to conjecture that there exist particular families of non-Gaussian states (e.g., non truncated photon-subtracted states) that represent optimal resources for the considered task in the regime of low energy, attaining the ultimate quantum limit also for intermediate losses, where superpositions of the first low-lying Fock states do not saturate the $4\bar{n}$ scaling of $H(\phi)$.

VII. HIGHER-ORDER SUPERPOSITIONS

The previous conclusions can be confronted by investigating the effect of adding terms of higher order in the superpositions. Consider states of the form $\rho_0=|\psi_0^{(3)}\rangle\langle\psi_0^{(3)}|$ with $|\psi_0^{(3)}\rangle=\Sigma_{n=0}^{3}c_n|n\rangle$, i.e., superpositions of Fock states up to $n=3$. The optimal QFI can be obtained by optimizing numerically the complex weights c_n for each $\bar{n}$ and ϕ (see Figs. 1 and 2). This yields a further improvement over the optimal qubit and qutrit states as well as over the Gaussian states. The succession of curves of $H^{(k)}(\phi)$ thus appears to converge to $4\bar{n}$ for $k\to\infty$. It is certainly true that the best possible performance requires non-Gaussian states in every energy range and for any value of the loss. Obviously, not all non-Gaussian states improve over Gaussian ones. For instance, the QFI for catlike superpositions of coherent states is almost always smaller than that of the optimal Gaussian probes, but for $\bar{n}\lesssim 2$ and $\phi\lesssim\pi/8$.

VIII. DISCUSSION

Estimating the loss factor of a channel is an important theoretical issue of direct practical relevance. In the case of purely dissipative bosonic channels, we have shown that proper nonclassical non-Gaussian states, such as Fock states and superpositions thereof, need to be employed as probes in order to achieve the most precise estimation. The optimality of Fock states can be understood on intuitive grounds by realizing that the parameter subject to estimation is essentially the decay rate of the field energy, and Fock states are eigenstates of the energy observable thus having zero energy uncertainty in their preparation.

The present work provides strong support for the need of going beyond the Gaussian scenario in applied quantum technology and quantum metrology and motivates further research to realize advanced tools of non-Gaussian quantum state engineering, manipulation, and detection.

ACKNOWLEDGMENT

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