Comment on "General relation between the transformation operator and an invariant under stochastic local operations and classical communication in quantum teleportation"

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In this paper, we make a few corrections to a paper by Zha and Ren [Phys. Rev. A **77**, 014306 (2008)]. The Zha-Ren protocol for teleportation in principle is equivalent to Rigolin's protocol [G. RigolinPhys. Rev. A **71**, 032303 (2005)] and the associated Comment [F. G. DengPhys. Rev. A **72**, 036301 (2005)]. We feel that the transformation operator is not well suited as a criterion for the faithful teleportation, but can be used as a means to transform an arbitrary four-qubit entangled state into a tensor product state of two Bell states. We give all the transformation operators.

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Recently, Zha and Ren [1] (ZR) introduced a transformation operator for teleportation (hereafter the ZR paper refers to Ref. [1]). First, let us briefly review the ZR protocol for teleportation and the transformation operator.

The sender Alice wants to transmit an unknown arbitrary two-qubit state:

$$|\chi\rangle_{12}=(x_0|00\rangle+x_1|01\rangle+x_2|10\rangle+x_3|11\rangle)_{12}, \quad (1)$$

where x_0, x_1, x_2, and x_3 are arbitrary complex numbers satisfying $\Sigma_{i=0}^{3}|x_i|^2=1$.

Alice and the receiver Bob share an arbitrary four-qubit entangled channel:

$$\begin{aligned}|\varphi\rangle_{3456}=(&a_0|0000\rangle+a_1|0001\rangle+a_2|0010\rangle+a_3|0011\rangle\\&+a_4|0100\rangle+a_5|0101\rangle+a_6|0110\rangle+a_7|0111\rangle\\&+a_8|1000\rangle+a_9|1001\rangle+a_{10}|1010\rangle+a_{11}|1011\rangle\\&+a_{12}|1100\rangle+a_{13}|1101\rangle+a_{14}|1110\rangle\\&+a_{15}|1111\rangle)_{3456}.\end{aligned} \quad (2)$$

The particles 1234 are in Alice's possession, and the particles 56 belong to Bob. The protocol for teleportation in Ref. [1] was expressed as

$$|\psi\rangle_{123456}=|\chi\rangle_{12}\otimes|\varphi\rangle_{3456}=\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\varphi_{13}^{i}\varphi_{24}^{j}\sigma_{56}^{ij}|\chi\rangle_{56}. \quad (3)$$

Here, $\sigma_{56}^{ij}=\sigma_{56}^{11}(\sigma_5^i\otimes\sigma_6^j)$, and $\sigma_m^k=I_m,\sigma_{mz},\sigma_{mx},-i\sigma_{my}$, where $m=5,6$. I_m is the two-dimensional identity; σ_{mz}, σ_{mx}, and σ_{my} are the Pauli matrices. The transformation operator σ_{56}^{11} is

$$\sigma_{56}^{11}=2\begin{pmatrix}a_0 & a_8 & a_4 & a_{12}\\ a_1 & a_9 & a_5 & a_{13}\\ a_2 & a_{10} & a_6 & a_{14}\\ a_3 & a_{11} & a_7 & a_{15}\end{pmatrix} \quad (4)$$

and φ_{mn}^{i} and φ_{mn}^{j} ($mn=13,24$) are Bell states,

$$\varphi_{mn}^{1}=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{mn},\quad \varphi_{mn}^{2}=\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)_{mn}, \quad (5)$$

$$\varphi_{mn}^{3}=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)_{mn},\quad \varphi_{mn}^{4}=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)_{mn}.$$

It can be seen that in the ZR protocol for teleportation, Alice performs two Bell-state measurements on particles 13 and 24, and Bob needs to perform $(\sigma_{56}^{ij})^{-1}=(\sigma_5^i\otimes\sigma_6^j)^{-1}(\sigma_{56}^{11})^{-1}$ on particles 56 to recover the original state.

In addition, ZR gave a criterion for faithfully teleporting an arbitrary two-qubit state via a four-qubit entangled state in terms of the operator σ_{56}^{ij}. On one hand, σ_{56}^{ij} is invertible. If σ_{56}^{ij} is not only invertible but also unitary (σ_{56}^{11} is also unitary), an unknown two-particle entangled state can be teleported perfectly. If σ_{56}^{ij} is not unitary but invertible (σ_{56}^{11} is also invertible), Alice and Bob can realize the teleportation with certain probability. On the other hand, if the transformation operator is not invertible (σ_{56}^{11} is also not invertible), the unknown two-particle arbitrary entangled state cannot be teleported.

In this Comment, we first show that there are some mistakes in Ref. [1]. We present the corrections here. From Eqs. (2)–(5), we calculate and find that the relation between the σ_{56}^{ij} and the Bell base measurement is not $\sigma_{56}^{ij}=\sigma_{56}^{11}(\sigma_5^i\otimes\sigma_6^j)$ [Eq. (13) in Ref. [1]], but $\sigma_{56}^{ij}=\sigma_{56}^{11}(\sigma_6^i\otimes\sigma_5^j)$.

The protocol for teleportation is also not Eq. (3), but

$$|\psi\rangle_{123456}=\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\varphi_{13}^{i}\varphi_{24}^{j}\sigma_{56}^{11}(\sigma_6^i\otimes\sigma_5^j)|\chi\rangle_{65}. \quad (6)$$

In order to realize the ZR protocol for teleportation in Eq. (3) via an arbitrary four-particle entangled channel, we propose a transformation operator B_1,

$$B_1 = 2\begin{pmatrix} a_0 & a_4 & a_8 & a_{12} \\ a_1 & a_5 & a_9 & a_{13} \\ a_2 & a_6 & a_{10} & a_{14} \\ a_3 & a_7 & a_{11} & a_{15} \end{pmatrix}. \tag{7}$$

When B_1 is unitary, the ZR protocol for teleportation,

$$|\psi\rangle_{123456} = \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4} \varphi_{13}^{i}\varphi_{24}^{j} B_1(\sigma_5^i \otimes \sigma_6^j)|\chi\rangle_{56}, \tag{8}$$

can be completed perfectly.

Comparing with previous protocols, we find that the ZR protocol for teleportation in principle is equivalent to Rigolin's protocol [2] and the associated Comment [3]. In Ref. [1], the receiver must perform additionally an operation $(B_1)^{-1}$ which is not relevant to Alice's operations. Thus, Bob can first perform the operation $(B_1)^{-1}$ before teleportation happens.

Theorem [4]. Let the matrix $A=(a_{ij}) \in P^{n\times n}$; then

$$\sum_{i=1}^{n} a_{ij}A_{ik} = \delta_{jk}|A|, \quad j,k = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} a_{ji}A_{ki} = \delta_{jk}|A|, \quad j,k = 1,2,\ldots,n,$$

where

$$\delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

A_{ij} denotes the cofactor of a_{ij}, and $|A|$ is the determinant of A.

Based on the above theorem, perform the operation $(B_1)^{-1}$ on the particles 56 of the state in Eq. (2),

$$(B_1)^{-1}|\varphi\rangle_{3456} = \frac{1}{|B_1|}(B_1)^{*}|\varphi\rangle_{3456} = \frac{1}{2}\frac{1}{|B_1|}[|00\rangle_{56}(B_1)^{*}2(a_0|00\rangle + a_4|01\rangle + a_8|10\rangle + a_{12}|11\rangle)_{34} + |01\rangle_{56}(B_1)^{*}2(a_1|00\rangle + a_5|01\rangle + a_9|10\rangle + a_{13}|11\rangle)_{34} + |10\rangle_{56}(B_1)^{*}2(a_2|00\rangle + a_6|01\rangle + a_{10}|10\rangle + a_{14}|11\rangle)_{34} + |11\rangle_{56}(B_1)^{*}2(a_3|00\rangle + a_7|01\rangle + a_{11}|10\rangle + a_{15}|11\rangle)_{34}] = \frac{1}{2}(|0011\rangle + |0101\rangle + |1010\rangle + |1111\rangle)_{3456} = \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{35}\right] \otimes \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{46}\right], \tag{9}$$

where $(B_1)^{*}$ is the adjoint of B_1.

It is clearly seen that the quantum channel is just a tensor product state of two Bell states. Alice performs two Bell-state measurements and Bob performs the appropriate Pauli operation $(\sigma_5^i \otimes \sigma_6^j)^{-1}$ to recover the original state $|\chi\rangle_{56}$. Obviously, this protocol for teleportation is equivalent to Rigolin's protocol [2] and the associated Comment [3]. The main reason is that Bob is required to perform and can first perform operation $(B_1)^{-1}$ which does not depend on Alice's measurement outcomes. Thus, the quantum channel in essence has been transformed into a tensor product state of two Bell states in the process of teleportation. Moreover, the other operations performed by Alice and Bob are the same as Rigolin's protocol [2] and the associated Comment [3].

As far as Eqs. (2) and (4) are concerned, we have

$$(\sigma_{56}^{11})^{-1}|\varphi\rangle_{3456} = \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{36}\right] \otimes \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{45}\right]. \tag{10}$$

The quantum channel is just a tensor product state of two Bell states about particles 36 and 45.

Note that the evolution of a closed quantum system is described by a unitary transformation [5]. When the transformation operator in Eq. (7) is invertible and unitary, a tensor product state of two Bell states can be perfectly obtained. When the transformation operator is invertible but not unitary, Alice and Bob can take some strategies, for instance, introduction of the auxiliary particle, to probabilistically extract a certain number of entangled Bell pairs.

It is well known that an arbitrary two-qubit state can be perfectly teleported via a tensor product state of two Bell states [2,3,6]. The invertible transformation operator can perfectly or probabilistically transform the four-qubit entangled state as a quantum channel into a tensor product state of two Bell states. When the transformation operator has no inverse, Alice and Bob cannot share a tensor product state of two Bell states. However, it is possible that there exist other protocols for teleportation. So, we feel that the transformation operator is not well suited as a criterion for faithful teleportation of an arbitrary two-qubit state via a four-qubit entangled state. In essence, the transformation operator only determines whether the four-qubit entangled state can be transformed into a tensor product state of two Bell states or not.

The tensor product state of two Bell states has 16 kinds in all and can be divided into four groups [2,3]. As a means to transform an arbitrary four-qubit entangled state into a tensor product state of two Bell states, we find that there are many transformation operators. The transformation operator σ_{56}^{11} given in Ref. [1] is only one of them. Each of Alice and Bob has 16 kinds of transformation operators which can transform an arbitrary four-particle entangled state into a tensor product state of two Bell states about particles 35 and 46 or about particles 36 and 45.

If columns 1 and i of the B_1 are swapped and the other two columns of the B_1 are also swapped, the transformation operator $B_{4(i-1)+1}$ $(i=2,3,4)$ can be obtained. If to the elements of columns 1 and j (or the other two columns) in $B_{4(i-1)+1}$ we add the minus, the transformation operator $B_{4(i-1)+j}$ $(i=1,2,3,4;\ j=2,3,4)$ can be obtained.

If the transformation operator $B_{4(i-1)+j}$ $(i,j=1,2,3,4)$ is invertible and we perform $(B_{4(i-1)+j})^{-1}$ on the state of the particles 56 in Eq. (2), the jth state of group i $|g_{4(i-1)+j}\rangle$ can be obtained.

If columns 2 and 3 of the transformation operator $B_{4(i-1)+j}$ $(i,j=1,2,3,4)$ are swapped and the inverse of it is performed on the state in Eq. (2), the tensor product state of two Bell states about particles 36 and 45 can be obtained. The transformation operator σ_{56}^{11} proposed in the ZR paper [1] is to swap the columns 2 and 3 of the B_1.

Certainly, Alice can also perform transformation operators on particles 34 in her hand to obtain a tensor product state of two Bell states. Let $A_1=(B_1)^T$. $(B_1)^T$ is the transpose of the B_1. Similarly, we can obtain the other transformation operators.

For example, if we swap columns 1 and 4 of the A_1 (the other two columns 2 and 3 of the A_1 must be swapped, too), and then add minus on elements of columns 1 and 3, we can get the transformation operator A_{15},

$$A_{15}=2\begin{pmatrix} -a_3 & a_2 & -a_1 & a_0 \\ -a_7 & a_6 & -a_5 & a_4 \\ -a_{11} & a_{10} & -a_9 & a_8 \\ -a_{15} & a_{14} & -a_{13} & a_{12} \end{pmatrix}. \tag{11}$$

When the A_{15} is invertible, the operation $(A_{15})^{-1}$ on the particles 34 can transform the arbitrary four-qubit entangled state in Eq. (2) into a tensor product state of two Bell states,

$$(A_{15})^{-1}|\varphi\rangle_{3456}=\left[\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)_{35}\right]\otimes\left[\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)_{46}\right]. \tag{12}$$

To summarize, we have made a few corrections to Ref. [1]. Because the transformation operator is not relevant to Alice's measurement outcomes and can transform the quantum channel into a tensor product state of two Bell states in the process of teleportation, the ZR protocol in principle is equivalent to Rigolin's protocol [2] and the associated Comment [3]. All protocols for teleportation utilized the transformation operator in principle are equivalent. In addition, we feel that the transformation operator expression is not well suited as a criterion for faithful teleportation, but can be used as a means to transform an arbitrary four-qubit entangled state into a tensor product state of two Bell states. All transformation operators have been given.

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