Role of dephasing in modifying the evolution of the cavity radiation of a coherent beat laser

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Detailed derivation of the master equation and corresponding time evolution of the cavity radiation of coherent beat laser in terms of the c-number variables associated with the normal ordering is presented. Although the obtained expressions in general turn out to have the same form as when dephasing is not taken into consideration, there is clear evidence that the quantum features of the cavity radiation can be significantly affected by the rate of dephasing. It is shown based on intuitive argument that the degree of two-mode squeezing and entanglement decrease with the rate of dephasing. Moreover, a thorough study of the effects of the rate at which the atomic coherent superposition is decaying is expected to aid in determining the actual available various quantum properties of the generated radiation.

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I. INTRODUCTION

Nonclassical properties such as squeezing and entanglement of the cavity radiation of the three-level cascade laser of various forms have received a great deal of attention in recent years [1–10]. It has been established that the atomic coherence is accountable for observing the quantum features. The atomic coherence in the three-level cascade scheme, where a direct transition from the upper energy level $|a\rangle$ to the lower energy level $|c\rangle$ is dipole forbidden, can be induced via coupling these levels by external radiation [1–5] or by preparing the atoms initially in arbitrary coherent superposition of these levels [6–9] or by using the two mechanisms simultaneously [10]. In the nondegenerate configuration, when the atoms decay from energy level $|a\rangle$ to $|c\rangle$ via intermediate level $|b\rangle$, two photons with different frequencies are generated. For the sake of convenience, the amplification of light when spontaneously emitted photons in the cascade transitions are correlated by atomic coherence induced by the initial preparation of the superposition and external driving mechanism can be taken as coherent beat laser. In order to demonstrate such a laser, the initially prepared atoms are assumed to be injected into the cavity at constant rate and then removed after they spontaneously decay to levels that are not involved in the lasing process. Moreover, these atoms are pumped externally with resonant radiation while they are in the cavity.

In view of the assumption that the injected coherent superposition creates population transfer path way which is a basis for the correlated two-photon emission, nondegenerate three-level cascade laser in general has been shown to be a source of light characterized by strong correlation of radiation modes with two different frequencies. In connection to the strong correlation between the two modes, the prediction for substantial degree of two-mode squeezing [6], entanglement [1,2,11,12], violation of Cauchy-Schwarz inequality [3], and quantum nonlocality [13] in various forms of nondegenerate three-level cascade laser have been reported. In these works, the atomic decay (corresponding to every sort of decay process) from each energy level to any lower energy level including those involved in establishing the lasing process is taken to be the same. In the present paper, in addition to this assumption, the initially prepared atomic coherent superposition is believed to decay due to the arising physical processes such as vacuum fluctuations [14]. This entails that the quantum properties of the radiation are affected by this decay process since the quantum features of the generated radiation are by large attributed to the atomic coherence. In light of this argument, it seems imperative taking the rate at which the atomic coherent superposition decays due to the changes in the environment (dephasing) into consideration in order to know the actually realizable quantum features. In this regard despite the expectation that decoherence (the process in which quantum properties are degraded while the cavity is coupled to the environment via a coupler mirror) inhibits the manifestation of quantum features, dephasing is found to enhance the same by providing an indirect correlation between totally uncorrelated states as recently shown in some cases [15,16]. Moreover, the contribution of each decay channel toward establishing correlation responsible for the appearance of the quantum properties needs to be investigated by comparing the separate decay rates with the parameters that describe the initially prepared and externally induced coherent superposition.

The master equation and expressions that represent the evolution of the cavity radiation in terms of the c-number variables associated with the normal ordering are derived. In order to analyze the effects of dephasing on the quantum features of the cavity radiation it is presumed, contrary to a well established approach, that the atomic and coherent superposition decay rates are different. A similar assumption has been considered earlier, though the contribution of dephasing has not been thoroughly studied [17,18]. Basically, taking the two damping rates as different is anticipated to lead to a considerable deviation in statistical and quantum properties of the generated radiation. Generally, the atomic decay rate corresponds directly to the spontaneous emission of the radiation whereas the coherent superposition decay rate is related to the process by which the quantum phenomenon is destroyed due to the pertinent fluctuations in the environment. In other words, the spontaneous emission of

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the cavity radiation is responsible for establishing the correlation which, on the other hand, is unfortunately destroyed by dephasing. No doubt that the competition between these two processes is profoundly interesting in knowing how the quantum properties of the radiation are affected. It is worth noting that the effect of the fluctuations in environment modes in the form of decoherence on quantum features of the cavity radiation of various quantum optical systems has been considerably studied [8,15,19]. But in this paper the contribution of the environment via coupling the energy levels involved in the lasing process in the form of dephasing [20] is also taken into consideration. Though the scope of this paper is restricted to deriving the time evolution of the cavity variables, it is not difficult to observe that the statistical and quantum properties of the generated cavity radiation can be investigated employing the obtained results.

II. MASTER EQUATION

It has been well known for a long time that the description of any isolated system can be provided using density operator where the time evolution is governed by the Liouville equation. However, as thoroughly discussed recently by different authors, when the system of interest is allowed to exchange energy and fluctuations with the surrounding environment, the evolution of the system is describable applying the master equation [21–23]. Though the master equation generally looks simple and elegant in form, it is usually difficult to solve the differential equations following from it. In connection to this, Ficek and Drummond [17] developed a procedure in which the effect of the environment is systematically incorporated when the nondegenerate three-level cascade atom interacts with broadband squeezed vacuum reservoir. Although different approaches can be employed, the master equation is derived following the procedure outlined in [17,21,24–26].

In the process of finding the master equation, it is customary to begin with the interaction Hamiltonian of the system under consideration. To this effect, the interaction of resonantly pumped nondegenerate three-level cascade atom with two-mode cavity radiation is describable in the rotating-wave approximation and interaction picture by the Hamiltonian of the form

$$\hat{H} = ig[\hat{a}|a\rangle\langle b| - |b\rangle\langle a|\hat{a}^\dagger + \hat{b}|b\rangle\langle c| - |c\rangle\langle b|\hat{b}^\dagger] + i\frac{\Omega}{2}[|c\rangle\langle a| - |a\rangle\langle c|], \quad (1)$$

where Ω is a real-positive constant proportional to the amplitude of the driving radiation and g is a coupling constant chosen to be the same for both transitions. $\hat{a}$ and $\hat{b}$ are the annihilation operators that represent the two cavity modes. In the cascade configuration, the atomic transitions from upper energy level $|a\rangle$ to the intermediate energy level $|b\rangle$ and from level $|b\rangle$ to the lower energy level $|c\rangle$ are presumed to be resonant with the cavity radiation whereas the direct spontaneous transition from $|a\rangle$ to $|c\rangle$ is dipole forbidden.

In addition, it is assumed that the atoms are initially prepared to be in arbitrary coherent superposition of the upper and lower energy levels; that is, the initial state of the three-level atom can be taken as

$$|\Psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle, \quad (2)$$

where $C_a(0)=\langle a|\Psi_A(0)\rangle$ and $C_c(0)=\langle c|\Psi_A(0)\rangle$ are probability amplitudes for the atom to be initially in the upper and lower energy levels. In line with Eq. (2), the corresponding initial density operator would be

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \quad (3)$$

where $\rho_{aa}^{(0)}=|C_a(0)|^2$, $\rho_{ac}^{(0)}=C_a(0)C_c^*(0)$, $\rho_{ca}^{(0)}=C_c(0)C_a^*(0)$, and $\rho_{cc}^{(0)}=|C_c(0)|^2$. Hence it is straight forward to notice that $\rho_{aa}^{(0)}$ and $\rho_{cc}^{(0)}$ are the probability for the atom to be initially in the upper and lower energy levels, whereas $\rho_{ac}^{(0)}$ represents the initial atomic coherence. The review of the ways of preparing atoms in arbitrary coherent superposition is found, for instance, in [14].

It is found advantageous setting

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \quad (4)$$

with $-1\leq\eta\leq 1$. It is not difficult to verify when the three-level atom is initially prepared in arbitrary atomic superposition and the relative random phase between the upper and lower states is neglected that

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \quad (5)$$

$$\rho_{ac}^{(0)} = \frac{\sqrt{1-\eta^2}}{2}. \quad (6)$$

Therefore, one can readily infer that when $\eta=-1$, $\rho_{aa}^{(0)}=1$ and $\rho_{cc}^{(0)}=\rho_{ac}^{(0)}=0$. This indicates that the case $\eta=-1$ corresponds to when the atoms are initially prepared to be in the upper energy level. On the other hand, when $\eta=1$, $\rho_{aa}^{(0)}=\rho_{ac}^{(0)}=0$ and $\rho_{cc}^{(0)}=1$ which show that the atoms are initially prepared to be in the lower energy level, and when $\eta=0$, $\rho_{aa}^{(0)}=\rho_{cc}^{(0)}=\rho_{ac}^{(0)}=1/2$. Moreover, it is an obvious matter to see from Eq. (6) that $0\leq\rho_{ac}^{(0)}\leq 1/2$. Hence $\eta=0$ corresponds to a maximum, whereas $\eta=-1$ and $\eta=1$ to a minimum injected atomic coherence.

Here the situation in which three-level atoms in a cascade configuration and initially prepared in coherent superposition of the upper and lower energy levels are injected into resonant cavity at constant rate and removed after sometime T, that is, long enough for the atoms to spontaneously decay to levels other than the middle or the lower energy level, is considered. In this case, the density operator for the cavity radiation plus a single atom injected into the cavity at time t_i is represented by $\hat{\rho}_{AR}(t,t_i)$ in which $t-T\leq t_i\leq t$. Thus the density operator that describes all the atoms plus radiation in the cavity when the atoms are continuously injected into it at constant rate r_a can be expressed as

$$\hat{\rho}_{AR}(t) = r_a \int_{t-T}^{t} \hat{\rho}_{AR}(t,t')dt'. \tag{7}$$

It is good to note that replacing the summation over randomly injected atoms to integration in a similar manner is known for quite a long time [27,28]. It is also a well established fact that the density operator $[\hat{\rho}_{AR}(t,t')]$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i[\hat{H},\hat{\rho}_{AR}(t,t')]. \tag{8}$$

With the assumption that the atom-radiation density operator can be decorrelated into the atom and radiation parts at a time when the atoms are injected into the cavity and when they just left the cavity, it is possible to write

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{9}$$

$$\hat{\rho}_{AR}(t,t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{10}$$

Now in view of Eqs. (8)–(10), integration of Eq. (7) results in

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) - i[\hat{H},\hat{\rho}_{AR}(t)], \tag{11}$$

where $\hat{\rho}_A(t)=\hat{\rho}_A(0)$. Moreover, taking the trace over the atomic variables and using the fact that

$$\mathrm{Tr}_A[\hat{\rho}_A(0)] = \mathrm{Tr}_A[\hat{\rho}_A(t-\tau)] = 1 \tag{12}$$

lead to

$$\frac{d\hat{\rho}(t)}{dt} = -i\,\mathrm{Tr}_A[\hat{H},\hat{\rho}_{AR}(t)]. \tag{13}$$

Furthermore, with the aid of Eq. (1), one can readily find

$$\frac{d\hat{\rho}(t)}{dt} = g[\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} - \hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{bc}\hat{b}^\dagger + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b}], \tag{14}$$

in which $\hat{\rho}_{\alpha\beta}=\langle\alpha|\hat{\rho}_{AR}|\beta\rangle$, with $\alpha,\beta=a,b,c$.

On the other hand, it is not difficult to see on the basis of Eq. (11) that

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho} - r_a\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle\hat{\rho} - i\langle\alpha|[\hat{H},\hat{\rho}_{AR}(t)]|\beta\rangle - \gamma_{\alpha\beta}\hat{\rho}_{\alpha\beta}, \tag{15}$$

where the last term is added to account for various atomic decay processes including the spontaneous emission. Assuming the atoms to be removed from the cavity after they successfully decay to energy levels other than the intermediate or the lower implies that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0. \tag{16}$$

Consequently, making use of Eqs. (1), (3), (15), and (16), one easily gets

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_a[\rho_{aa}^{(0)}\delta_{\alpha a}\delta_{a\beta} + \rho_{ac}^{(0)}\delta_{\alpha a}\delta_{c\beta} + \rho_{ca}^{(0)}\delta_{\alpha c}\delta_{a\beta} + \rho_{cc}^{(0)}\delta_{\alpha c}\delta_{c\beta}]\hat{\rho}(t) - g[\hat{a}^\dagger\hat{\rho}_{a\beta}\delta_{\alpha b} + \hat{b}^\dagger\hat{\rho}_{b\beta}\delta_{\alpha c} - \hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} - \hat{b}\hat{\rho}_{c\beta}\delta_{\alpha b} + \hat{\rho}_{\alpha a}\hat{a}\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta} - \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta} - \hat{\rho}_{\alpha c}\hat{b}^\dagger\delta_{b\beta}] - \frac{\Omega}{2}[\hat{\rho}_{c\beta}\delta_{a\alpha} - \hat{\rho}_{a\beta}\delta_{c\alpha} - \hat{\rho}_{\alpha a}\delta_{c\beta} + \hat{\rho}_{\alpha c}\delta_{a\beta}] - \gamma_{\alpha\beta}\hat{\rho}_{\alpha\beta} \tag{17}$$

from which follows

$$\frac{d}{dt}\hat{\rho}_{aa}(t) = r_a\rho_{aa}^{(0)}\hat{\rho}(t) + g(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger) - \frac{\Omega}{2}(\hat{\rho}_{ac} + \hat{\rho}_{ca}) - \Gamma_a\hat{\rho}_{aa}, \tag{18}$$

$$\frac{d}{dt}\hat{\rho}_{bb}(t) = -g(\hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{ba}\hat{a} - \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{bc}\hat{b}^\dagger) - \Gamma_b\hat{\rho}_{bb}, \tag{19}$$

$$\frac{d}{dt}\hat{\rho}_{cc}(t) = r_a\rho_{cc}^{(0)}\hat{\rho}(t) - g(\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) + \frac{\Omega}{2}(\hat{\rho}_{ac} + \hat{\rho}_{ca}) - \Gamma_c\hat{\rho}_{cc}, \tag{20}$$

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = g(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^\dagger) - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma_{ab}\hat{\rho}_{ab}, \tag{21}$$

$$\frac{d}{dt}\hat{\rho}_{ac}(t) = r_a\hat{\rho}_{ac}^{(0)}\hat{\rho} + g(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}) - \frac{\Omega}{2}(\hat{\rho}_{cc} - \hat{\rho}_{aa}) - \gamma_{ac}\hat{\rho}_{ac}, \tag{22}$$

$$\frac{d}{dt}\hat{\rho}_{cb}(t) = -g(\hat{\rho}_{ca}\hat{a} - \hat{\rho}_{cc}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}_{bb}) + \frac{\Omega}{2}\hat{\rho}_{ab} - \gamma_{cb}\hat{\rho}_{cb}, \tag{23}$$

where $\Gamma_i=\gamma_{ii}$ and $\gamma_{ij(i\neq j)}$ with $i,j=a,b,c$ stand for the atomic decay rate and the rate of dephasing (the rate at which the atomic coherent superposition decays), respectively. These coefficients have been defined as radiative spontaneous-emission rate and generalized decay constant that arises from the coupling between these transitions in [17].

In the good cavity limit, where the cavity damping rate κ is much smaller than atomic decay rates (Γ_i and γ_{ij}), the cavity mode variables change slowly when compared to the atomic variables. In this case, the atomic variables will reach steady state in relatively short time. The time derivatives of such variables can be then set to zero while keeping the remaining atomic and cavity mode variables at time t. This procedure is usually known as the adiabatic approximation scheme. Confining to linear analysis, which amounts to dropping the terms containing g in Eqs. (18)–(20) and (22), and then applying the adiabatic approximation scheme result in

$$r_a\rho_{aa}^{(0)}\hat{\rho}(t) - \Omega\hat{\rho}_{ac} - \Gamma_a\hat{\rho}_{aa} = 0, \quad (24)$$

$$\hat{\rho}_{bb} = 0, \quad (25)$$

$$r_a\rho_{cc}^{(0)}\hat{\rho}(t) + \Omega\hat{\rho}_{ac} - \Gamma_c\hat{\rho}_{cc} = 0, \quad (26)$$

$$r_a\rho_{ac}^{(0)}\hat{\rho}(t) - \frac{\Omega}{2}(\hat{\rho}_{cc} - \hat{\rho}_{aa}) - \gamma_{ac}\hat{\rho}_{ac} = 0, \quad (27)$$

where we set, on the basis of the assumption that, $\rho_{ac}^{(0)} = \rho_{ca}^{(0)}$, in case the random phase between the upper and lower energy levels is neglected, that $\hat{\rho}_{ac} = \hat{\rho}_{ca}$. Essentially, the linear analysis is required so that the resulting differential equations can be analytically solvable. The linearization approach still holds when the nonclassical properties of the radiation are studied since the quantum features in this system are associated with the correlation induced in the cascading process rather than the nonlinear process as in the other quantum optical systems.

Moreover, upon setting $\Gamma_a = \Gamma_b = \Gamma_c = \Gamma$ and $\gamma_{ab} = \gamma_{ac} = \gamma_{cb} = \gamma$, it is possible to get using Eqs. (24), (26), and (27) that

$$\hat{\rho}_{aa} = \frac{r_a\hat{\rho}}{2\Gamma(\gamma\Gamma + \Omega^2)}[\gamma\Gamma(1-\eta) - \Gamma\Omega\sqrt{1-\eta^2} + \Omega^2], \quad (28)$$

$$\hat{\rho}_{cc} = \frac{r_a\hat{\rho}}{2\Gamma(\gamma\Gamma + \Omega^2)}[\gamma\Gamma(1+\eta) + \Gamma\Omega\sqrt{1-\eta^2} + \Omega^2], \quad (29)$$

$$\hat{\rho}_{ac} = \frac{r_a\hat{\rho}}{2(\gamma\Gamma + \Omega^2)}[\Gamma\sqrt{1-\eta^2} - \Omega^2], \quad (30)$$

with $\hat{\rho} = \hat{\rho}(t)$. Making use of Eqs. (21), (23), (25), and (28)–(30) and applying the adiabatic approximation scheme once again,

$$\hat{\rho}_{ab} = -\frac{gr_a\hat{\rho}}{\gamma^2(4+\varepsilon^2)(1+\varepsilon\varepsilon')}\{\hat{a}[2(\varepsilon'^2 + \varphi) + \eta(\varepsilon'\varepsilon - 2\varphi) - (2\varepsilon' + \varepsilon)\sqrt{1-\eta^2}] + \hat{b}^\dagger[\varepsilon'(1+\varepsilon'\varepsilon) + 3\eta\varepsilon - (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2}]\}, \quad (31)$$

$$\hat{\rho}_{cb} = -\frac{gr_a\hat{\rho}}{\gamma^2(4+\varepsilon^2)(1+\varepsilon'\varepsilon)}\{\hat{a}[\varepsilon'(1+\varepsilon\varepsilon') - 3\eta\varepsilon' + (2-\varepsilon\varepsilon')\sqrt{1-\eta^2}] - \hat{b}^\dagger[2(\varepsilon'^2 + \varphi) - \eta(\varepsilon'\varepsilon + 2\varphi) + (2\varepsilon' + \varepsilon)\sqrt{1-\eta^2}]\}, \quad (32)$$

where $\varepsilon = \Omega/\gamma$, $\varepsilon' = \Omega/\Gamma$, and $\varphi = \frac{\gamma}{\Gamma}$.

Now employing Eqs. (31) and (32), Eq. (14) can be put in the form

$$\frac{d\hat{\rho}}{dt} = \frac{AC_+}{2B}[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}] + \frac{AC_-}{2B}[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}] - \frac{AD_+}{2B}[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger + \hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho}] - \frac{AD_-}{2B}[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} + \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}], \quad (33)$$

where

$$A = \frac{2r_a g^2}{\gamma^2}, \quad (34)$$

$$B = (4+\varepsilon^2)(1+\varepsilon'\varepsilon), \quad (35)$$

$$C_\pm = 2\varepsilon'^2 + 2\varphi(1-\eta) \pm [(\eta\varepsilon'\varepsilon - (2\varepsilon' + \varepsilon)\sqrt{1-\eta^2})], \quad (36)$$

$$D_\pm = (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2} - 3\eta\varepsilon \mp \varepsilon'(1+\varepsilon'\varepsilon). \quad (37)$$

Besides, the contribution of the cavity damping which corresponds to the coupling of the cavity modes to environment modes via the coupler mirror can be incorporated following the usual standard approach [21,24]. Hence the master equation that describes the cavity radiation of the quantum optical system under consideration when it is coupled to two-mode vacuum reservoir can be expressed in the form

$$\frac{d\hat{\rho}(t)}{dt} = \frac{\kappa}{2}[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}] + \frac{AC_+}{2B}[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger] + \frac{1}{2}\left(\frac{AC_-}{B} + \kappa\right)[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}] - \frac{AD_+}{2B}[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a}] - \frac{AD_-}{2B}[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{b}\hat{\rho}\hat{a}]. \quad (38)$$

This master equation is found to have the same form as when dephasing is not taken into consideration [3] as it should be. In view of the form of this master equation, it is possible to infer that C_+ stands for the gain of mode a whereas C_- for the lose of mode b. It is also good to note that $D_\pm$ are associated with the correlation between the two modes that accounts for the manifestation of nonclassical features. In earlier communications, $A = \frac{2r_a g^2}{\gamma^2}$ is defined as the linear gain coefficient where γ is taken as the atomic decay rate [3,7,29]. However, it so happens that γ corresponds to the rate of dephasing. In this respect, it is worth noting that this study unequivocally asserts that the linear gain coefficient should have been directly related to the rate at which the atomic superposition decays rather than the spontaneous atomic decay rate in the explanation of the effects of the linear gain coefficient on the quantum properties of the cavity radiation.

On the basis of recent discussion that the degree of two-mode squeezing and entanglement increase with the linear gain coefficient [7], it can be argued that the nature of the quantum features of the cavity radiation significantly depends on the rate at which the coherent superposition is decaying. Taking the rate at which the atoms are injected into the cavity (r_a) and the coupling constant (g) as constant parameters, one can readily see that $A \propto \frac{1}{\gamma^2}$. In view of the discussion presented elsewhere, it is not difficult to infer from this study that the degree of two-mode squeezing and entanglement increase with decreasing rate of dephasing [7]. This actually means that the more the rate at which the atomic coherence is lost, the lesser would be the chance for demonstrating quantum features of the cavity radiation. This concurs with the earlier claim that the manifestation of the quantum property is attributed to the atomic coherent superposition. Since the actual quantum property of the generated radiation is determined by the interplay between the initially prepared and externally induced coherent superpositions (coherence beating), it is basically required to compare γ with Γ, Ω, and η to get the real picture of the situation.

III. EVOLUTION EQUATIONS

It is, nowadays, common practice deriving the stochastic differential equations from the Fokker-Planck equation of one of the quasiprobability distribution functions when the system interacts with the environment. In this approach, the effect of the external agent is fundamentally accounted for via the correlation between the noise forces characterizing the environment. Particularly, as recent discussion indicates, the stochastic differential equations corresponding to the Langevin equations can be applied in studying the quantum properties of the radiation inside and outside the cavity [30]. It is good to note that mapping of quantum equations onto the corresponding c-number stochastic differential equations relies on a similarity between the partial differential equations derived from the master equation and Fokker-Planck equation associated with the Brownian motion. In many instances, the c-number Langevin equations are found to be easier mathematically to handle than the corresponding operator equations. This is one of the advantages of the stochastic differential equations when compared to the operator equations that can be derived from the master equation directly. Moreover, recent works make it clear that the stochastic differential equations associated with the normal ordering of the cavity mode variables are an important tool in studying the quantum features of the radiation [3,11].

Therefore, in this section, the stochastic differential equations associated with the normal ordering for the cavity mode variables would be determined applying the pertinent master equation. To this end, employing Eq. (38) and the fact that

$$\frac{d}{dt}\langle \hat{O}(t)\rangle = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{O}\right), \tag{39}$$

where $\hat{O}$ is any operator, it can be verified that

$$\frac{d}{dt}\langle \hat{a}(t)\rangle = -\frac{\mu_+}{2}\langle \hat{a}(t)\rangle - \frac{AD_+}{2B}\langle \hat{b}^\dagger(t)\rangle, \tag{40}$$

$$\frac{d}{dt}\langle \hat{b}(t)\rangle = -\frac{\mu_-}{2}\langle \hat{b}(t)\rangle + \frac{AD_-}{2B}\langle \hat{a}^\dagger(t)\rangle, \tag{41}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu_+\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle - \frac{AD_+}{2B}[\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \langle \hat{a}(t)\hat{b}(t)\rangle] + \frac{AC_+}{B} + \kappa N, \tag{42}$$

$$\frac{d}{dt}\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle = -\mu_-\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{AD_-}{2B}[\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \langle \hat{a}(t)\hat{b}(t)\rangle] + \kappa N, \tag{43}$$

$$\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t)\rangle = -\frac{\mu_- + \mu_+}{2}\langle \hat{a}(t)\hat{b}(t)\rangle + \frac{AD_-}{2B}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle - \frac{AD_+}{2B}\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{AD_-}{2B} + \kappa M, \tag{44}$$

in which $\mu_\pm = \frac{\kappa B \mp AC_\pm}{B}$. It is not difficult to notice that the operators in the above equations are already put in the normal order. Hence the corresponding expressions in terms of c-number variables associated with normal ordering are

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\frac{\mu_+}{2}\langle \alpha(t)\rangle - \frac{AD_+}{2B}\langle \beta^*(t)\rangle, \tag{45}$$

$$\frac{d}{dt}\langle \beta^*(t)\rangle = -\frac{\mu_-}{2}\langle \beta^*(t)\rangle + \frac{AD_-}{2B}\langle \alpha(t)\rangle, \tag{46}$$

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = -\mu_+\langle \alpha^*(t)\alpha(t)\rangle - \frac{AD_+}{2B}[(\langle \alpha^*(t)\beta^*(t)\rangle + \langle \alpha(t)\beta(t)\rangle] + \frac{AC_+}{B} + \kappa N, \tag{47}$$

$$\frac{d}{dt}\langle \beta^*(t)\beta(t)\rangle = -\mu_-\langle \beta^*(t)\beta(t)\rangle + \frac{AD_-}{2B}[\langle \alpha^*(t)\beta^*(t)\rangle + \langle \alpha(t)\beta(t)\rangle] + \kappa N, \tag{48}$$

$$\frac{d}{dt}\langle \alpha(t)\beta(t)\rangle = -\frac{\mu_- + \mu_+}{2}\langle \alpha(t)\beta(t)\rangle + \frac{AD_-}{2B}\langle \alpha^*(t)\alpha(t)\rangle - \frac{AD_+}{2B}\langle \beta^*(t)\beta(t)\rangle + \frac{AD_-}{2B} + \kappa M. \tag{49}$$

On the basis of Eqs. (45) and (46), one can write

$$\frac{d}{dt}\alpha(t) = -a_+\alpha(t) - b_+\beta^*(t) + f_a(t), \tag{50}$$

$$\frac{d}{dt}\beta(t) = -a_-\beta(t) - b_-\alpha^*(t) + f_b(t), \tag{51}$$

where

$$a_{\pm} = \frac{\kappa}{2} + \frac{A}{2B}[(2\varepsilon' + \varepsilon)\sqrt{1-\eta^2} - \eta(\varepsilon'\varepsilon \mp 2\varphi) \mp 2(\varepsilon'^2 + \varphi)], \quad (52)$$

$$b_{\pm} = -\frac{A}{2B}\{\varepsilon'(1+\varepsilon\varepsilon') \pm [3\eta\varepsilon - (2-\varepsilon\varepsilon')\sqrt{1-\eta^2}]\}. \quad (53)$$

Moreover, $f_a(t)$ and $f_b(t)$ are the noise forces the properties of which remain to be determined. For instance, the expectation values of Eqs. (50) and (51) would be identical to Eqs. (45) and (46) provided that $\langle f_a(t)\rangle=0$ and $\langle f_b(t)\rangle=0$ which indicate that the noise forces have stochastic nature.

Furthermore, the correlations of these noise forces are found by comparing the expressions following from Eqs. (50) and (51) with the corresponding equations obtained from the master equation to be

$$\langle f_a(t')f_a^*(t)\rangle = \frac{AC_+}{B}\delta(t-t'), \quad (54)$$

$$\langle f_b(t')f_b^*(t)\rangle = 0, \quad (55)$$

$$\langle f_b(t')f_a(t)\rangle = \frac{AD_-}{2B}\delta(t-t'), \quad (56)$$

$$\langle f_b^*(t')f_a(t)\rangle = \langle f_a(t')f_a(t)\rangle = \langle f_b(t')f_b(t)\rangle = 0. \quad (57)$$

It is evident that the effect of the noise source appears in these c-number equations just as it does in the corresponding quantum description. The properties of the correlations of these noise forces are related to the operator ordering which directly associated with the vacuum fluctuations in the environment and cavity modes. Comparing Eqs. (54) and (55) reveals that the correlation properties of the noise associated with mode a and mode b are different unlike the other quantum optical systems. This disparity is basically due to the difference in the number of photons in mode a and mode b [11,12]. It can also be deduced from Eq. (56) that the nonclassical properties of the radiation depend on $\frac{AD_-}{B}$. As clearly indicated in previous discussion A, D_-, and B are expressed in terms of the rate of dephasing (γ). Hence it is possible to conclude that the quantum properties of the radiation depend on dephasing one way or the other.

It is straight forward to see that Eqs. (50) and (51) form coupled differential equations. In order to solve these differential equations, one can construct a matrix equation of the form

$$\frac{d}{dt}\mathcal{U}(t) = -\mathcal{M}\mathcal{U}(t) + \mathcal{W}(t), \quad (58)$$

where

$$\mathcal{U}(t) = \begin{pmatrix}\alpha(t)\\ \beta^*(t)\end{pmatrix}, \quad (59)$$

$$\mathcal{M} = \begin{pmatrix}a_+ & b_+\\ b_- & a_-\end{pmatrix}, \quad (60)$$

$$\mathcal{W}(t) = \begin{pmatrix}f_a(t)\\ f_b^*(t)\end{pmatrix}. \quad (61)$$

The formal solution of Eq. (58) can be written as

$$\mathcal{U}(t) = \mathcal{V}e^{-\mathcal{R}t}\mathcal{V}^{-1}\mathcal{U}(0) + \int_0^t \mathcal{V}e^{-\mathcal{R}(t-t')}\mathcal{V}^{-1}\mathcal{W}(t')dt', \quad (62)$$

where $\mathcal{V}$ is the eigenvector of matrix $\mathcal{M}$ which is found to be

$$\mathcal{V} = \begin{pmatrix}\frac{b_+}{\sqrt{b_+^2 + (\lambda_+ - a_+)^2}} & \frac{b_+}{\sqrt{b_+^2 + (\lambda_- - a_+)^2}}\\ \frac{\lambda_+ - a_+}{\sqrt{b_+^2 + (\lambda_+ - a_+)^2}} & \frac{\lambda_- - a_+}{\sqrt{b_+^2 + (\lambda_- - a_+)^2}}\end{pmatrix} \quad (63)$$

and

$$e^{-\mathcal{R}\tau} = \begin{pmatrix}e^{-\lambda_+\tau} & 0\\ 0 & e^{-\lambda_-\tau}\end{pmatrix}, \quad (64)$$

in which $\lambda_{\pm}$ are the corresponding eigenvalues,

$$\lambda_{\pm} = \frac{\kappa}{2} + \frac{A}{2B}([(2\varepsilon' + \varepsilon)\sqrt{1-\eta^2} - \eta\varepsilon'\varepsilon \pm \{\varepsilon'^2(1+\varepsilon\varepsilon')^2 + 4(\varepsilon'^2 + \varphi)^2 - [3\eta\varepsilon - (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2}]^2\}^{1/2}). \quad (65)$$

Finally, carrying out the required algebra leads to

$$\alpha(t) = E_+(t)\alpha(0) + F_+(t)\beta^*(0) + G_+(t) + H_+(t), \quad (66)$$

$$\beta(t) = E_-(t)\beta(0) + F_-(t)\alpha^*(0) + G_-(t) + H_-(t), \quad (67)$$

where

$$E_{\pm}(t) = \tfrac{1}{2}[(1\pm p)e^{-\lambda_- t} + (1\mp p)e^{-\lambda_+ t}], \quad (68)$$

$$F_{\pm}(t) = \frac{q_{\pm}}{2}[e^{-\lambda_+ t} - e^{-\lambda_- t}], \quad (69)$$

$$G_+(t) = \frac{1}{2}\int_0^t [(1+p)e^{-\lambda_-(t-t')} + (1-p)e^{-\lambda_+(t-t')}]f_a(t')dt', \quad (70)$$

$$G_-(t) = \frac{1}{2}\int_0^t [(1-p)e^{-\lambda_-(t-t')} + (1+p)e^{-\lambda_+(t-t')}]f_b(t')dt', \quad (71)$$

$$H_+(t) = \frac{q_+}{2}\int_0^t [e^{-\lambda_+(t-t')} - e^{-\lambda_-(t-t')}]f_b^*(t')dt', \quad (72)$$

$$H_-(t) = \frac{q_-}{2}\int_0^t [e^{-\lambda_+(t-t')} - e^{-\lambda_-(t-t')}]f_a^*(t')dt', \quad (73)$$

with

$$p = \frac{2(\varepsilon'^2 + \varphi)}{\{\varepsilon'^2(1+\varepsilon\varepsilon')^2 + 4(\varepsilon'^2+\varphi)^2 - [3\eta\varepsilon - (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2}]^2\}^{1/2}}, \tag{74}$$

$$q_{\pm} = \frac{-\varepsilon'(1+\varepsilon'\varepsilon) \mp [3\eta\varepsilon - (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2}]}{\{\varepsilon'^2(1+\varepsilon\varepsilon')^2 + 4(\varepsilon'^2+\varphi)^2 - [3\eta\varepsilon - (2-\varepsilon'\varepsilon)\sqrt{1-\eta^2}]^2\}^{1/2}}. \tag{75}$$

It is perhaps worth mentioning that Eqs. (66) and (67) are used to calculate various quantities of interest. It is noticeable that these solutions are well behaved functions at steady state provided that $\lambda_{\pm} \geq 0$. As a result, the case for which $\lambda_{\pm}=0$ is designated as threshold condition. As critical scrutiny of the expressions following from the master equation reveals that this mathematical condition is directly related to the uncertainty condition [3].

It has been argued that the form of the solution of Eqs. (50) and (51) is independent of the reservoir to which the cavity is coupled, but the corresponding correlations of the stochastic noise forces considerably affected by it [11]. As clearly discussed elsewhere the parameters in Eqs. (68)–(75) strongly depend on the amplitude of the driving radiation and the way in which the atoms are initially prepared, but the form of the solution in general remains the same under various conditions [3]. It is also shown in this study that though the form of the solution still remains the same, the involved parameters $(\lambda_{\pm}, p, q_{\pm})$ are significantly reliant on the rate of dephasing. As a result, it would not be wrong if one expects that the degree of the resulting quantum features of the radiation is affected by the rate of dephasing. It may not be difficult to observe that this result complements what has been discussed in relation to $A \propto \frac{1}{\gamma^2}$. Ironically, the investigation of the actual dependence of the quantum features of the cavity radiation on dephasing is quite an involving task. Since the property of the radiation is associated with the rate at which the atoms are injected into the cavity, the way in which they are initially prepared, the atomic damping rate, the cavity damping rate, the amplitude of the driving radiation, and of course the rate of dephasing, it appears that even considering some cases of interest is not an easy task. In light of this, for the sake of clarity, the detailed study of the quantum properties of the cavity radiation is differed to subsequent communications.

IV. CONCLUSION

In this contribution, detailed derivation of the evolution of the cavity radiation in terms of c-number variables associated with the normal ordering for coherent beat laser is presented. The master equation and the solution of the corresponding stochastic differential equations are found to have the same form as when dephasing is not taken into consideration. However, the parameters closely related to the quantum properties of the radiation turn out to be strongly dependent on the rate of dephasing. Though the detailed analysis in this respect is not provided due to the formidable complications associated with the number of involved parameters, setting some of these parameters as constant indicates that the degree of two-mode squeezing and entanglement decrease with increasing rate of dephasing. This result agrees with earlier claim that the coherent superposition is responsible for the appearance of two-mode squeezing and entanglement. Consequently, it is possible to observe that the detailed study of the effects of dephasing on quantum features of the radiation under various conditions is required.

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