Scale-invariant thermodynamics of a toroidally trapped Bose gas

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We consider a system of bosonic atoms in an axially symmetric harmonic trap augmented with a two-dimensional repulsive Gaussian optical potential. We find an expression for the grand free energy of the system for configurations ranging from the harmonic trap to the toroidal regime. For large tori we identify an accessible regime where the ideal-gas thermodynamics of the system are found to be independent of toroidal radius. This property is a consequence of an invariant extensive volume of the system that we identify analytically in the regime where the toroidal potential is radially harmonic. In considering corrections to the scale-invariant transition temperature, we find that the first-order interaction shift is the dominant effect in the thermodynamic limit and is also scale invariant. We also consider adiabatic loading from the harmonic-to-toroidal trap configuration, which we show to have only a small effect on the condensate fraction of the ideal gas, indicating that loading into the scale-invariant regime may be experimentally practical.

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I. INTRODUCTION

While Bose-Einstein condensation of dilute gases is now routinely observed, the degree of control and interrogation now affords more detailed studies of the dynamics of the condensation process, such as the symmetry breaking associated with the Kibble-Zurek mechanism (KZM) [1–4]. Recent experiments loading toroidal traps with Bose-degenerate gases [5,6] have shown the need for a basic theoretical understanding of the properties of Bose-Einstein condensation in toroidal traps, which may be important for future studies of Bose-Einstein condensate (BEC) formation and tests of KZM.

Following the development of storage rings for neutral atoms [7], toroidal traps for BECs were proposed using various combinations of magnetic and optical techniques [8,9]. Sagnac interferometry is an important application for large toroidal traps, having the feature of resolution proportional to the area of the interferometer [10]. While traps of order $\sim$3 mm diameter have been created [11,12], BEC loading for such large traps was limited to launching the BEC into the toroid which then acts as a dispersive waveguide. Smaller toroids, which can be more easily loaded, have recently been produced [5,6,13,14] opening the way for studies of superfluidity and persistent currents in a nontrivial trapping topology.

Theoretical efforts have focused on BECs far below T_c. Many features of toroidal geometry have been studied, including topological phases [15], the stability of macroscopic persistent currents [16–21], excitation spectra [22], atomic phase interference devices [23], vortex-vortex interactions [24], generation of excitations via stirring [25], dynamics of sonic horizons [26], parametric amplification of phonons [27], rotational current generation [28], the interplay of interactions and rotation [29], giant vortices [30], and vortex signatures [31]. Ideal-gas theory has recently been used [32] to study the rapidly rotating Bose gas in a quartically stabilized harmonic trap realized at Laboratoire Kastler Brossel [33]. The BEC transition temperature in non-power-law traps is thus becoming more relevant, and a recent study of optical lattices [34] gives further indication that analytical expressions can be found for increasingly rich potentials. However, the ideal-gas thermodynamics of three-dimensional toroidal potentials—crucial for understanding the dynamics of Bose-Einstein condensation—have not been addressed.

The effect of trapping geometry on the BEC transition was emphasized by Bagnato *et al.* [35]. It was observed that for power-law traps, increasing the confinement of the system has the effect of increasing the peak phase-space density and thus raising the temperature of the BEC transition. An understanding of how phase-space density depends on the toroid size is crucial for making large ultracold persistent currents. There is also the role of topology to consider. In particular, in a toroidal trap the angular spatial coordinate becomes unavailable for thermalization, suggesting a potentially interesting interplay between topology and system size.

In this work we study the properties of a Bose gas trapped by a specific toroidal potential. The potential is created from a harmonic magnetic potential combined with a repulsive (blue detuned) optical potential with a Gaussian spatial profile [6]; we refer to this potential as harmonic Gaussian and show that it has uniquely interesting properties which advantage it for creating large toroidally trapped BECs. Using the semiclassical approach to the thermodynamics of the ideal gas we find an exact expression for the free energy.

Examining the properties of the system for increasing toroidal radius shows the existence of an analytically tractable regime of *scale invariance* with respect to the toroid radius. In this regime the toroidal trap is well approximated as radially harmonic. We use the theory of Romero-Rochin [36] to identify the invariant generalized extensive volume of the system which governs scale invariance. We further generalize this result to show that the system enters a scale-invariant regime even when this approximation is not valid. Focusing on the preservation of quantum degeneracy, we then treat the BEC transition temperature in detail and consider finite-size and mean-field corrections to the scale-invariant result. Finally, we discuss possible means to reach the scale-invariant regime.

Geometry of the harmonic-Gaussian potential

We consider a Bose gas confined in the trapping potential

$$V(\mathbf{x}) = \frac{m}{2}(\omega_r^2 r^2 + \omega_z^2 z^2) + V_{\rm OP}(r), \tag{1}$$

where axial and radial trapping frequencies are ω_z and ω_r, and the nonharmonic potential is given by

$$V_{\rm OP}(r) = V_0 \exp(-r^2/\sigma_0^2), \tag{2}$$

where r is the distance from the z axis. This potential can be created using a magnetic trap combined with a detuned laser field propagating along the z axis which forms an optical dipole potential [37].

To determine the geometry of the trap it is convenient to define the energy

$$V_\sigma \equiv \frac{1}{2} m\omega_r^2\sigma_0^2, \tag{3}$$

which is the potential energy of an atom at $r=\sigma_0$ in the harmonic potential; this energy will play a central role in determining the thermodynamics of the system. The minimum of the combined potential is located at $z=0$ and $r=r_m$, where

$$r_m \equiv \begin{cases} 0 & \text{for } V_0 < V_\sigma \\ \sqrt{\sigma_0^2 \ln(V_0/V_\sigma)} & \text{for } V_0 \geq V_\sigma, \end{cases} \tag{4}$$

from which we find that $V_m \equiv \min[V(\mathbf{x})] = V(r=r_m, z=0)$ is given by

$$V_m \equiv \begin{cases} V_0 & \text{for } V_0 < V_\sigma \\ V_\sigma[1+\ln(V_0/V_\sigma)] & \text{for } V_0 \geq V_\sigma. \end{cases} \tag{5}$$

There are three distinct regimes parametrized by the ratio V_0/V_σ which are (1) *dimple trap* ($V_0<0$)—the Gaussian forms a dimple in the center of the harmonic trap. (2) *Flat trap* ($0<V_0\leq V_\sigma$)—the trap is flattened but not toroidal, as shown in Fig. 1(a). (3) *Toroidal trap* ($V_\sigma<V_0$)—the trap becomes toroidal, as shown in Figs. 1(b)–1(d).

II. IDEAL BOSE GAS IN A HARMONIC-GAUSSIAN TRAP

In this section we develop the general grand-canonical theory of the ideal Bose gas in the harmonic-Gaussian trap.

A. Grand canonical free energy

In general, the grand potential function for the system is

$$\mathcal{F} = -k_B T \ln \mathcal{Z}, \tag{6}$$

where $\mathcal{Z}$ is the grand partition function. For the Bose gas distributed over levels with excitation energy ϵ_i, this becomes

$$\mathcal{F} = k_B T \sum_i \ln(1 - e^{\beta(\mu-\epsilon_i)}), \tag{7}$$

where $\beta = 1/k_B T$. For the situation of interest, the chemical potential is assumed to approach the ground-state energy of the trap ϵ_0, leading to macroscopic occupation of the ground state N_0. In this regime, using the semiclassical approximation for the excited states, $\mathcal{F}$ can be written as

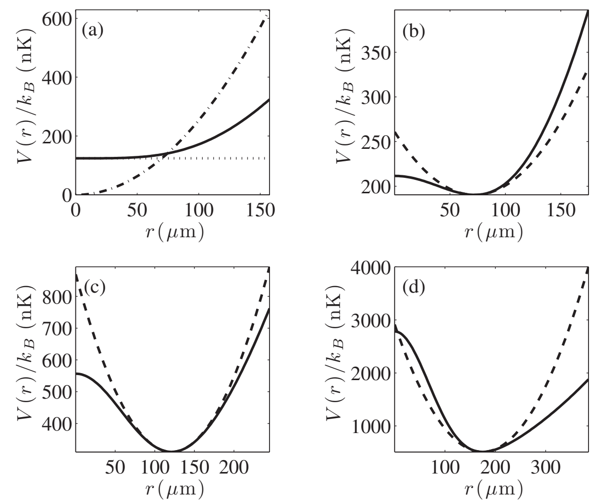

FIG. 1. The harmonic-Gaussian trapping potential is shown at $z=0$ (solid lines) for an optical potential of width $\sigma_0=70$ μm and (a) $V_0/k_B=124$ nK, (b) $V_0/k_B=211$ nK, (c) $V_0/k_B=557$ nK, and (d) $V_0/k_B=2784$ nK. In (a) the dashed-dotted curve is the bare harmonic potential ($V_0=0$) and the dotted line is $V_\sigma \equiv m\omega_r^2\sigma_0^2/2$, which in this case is also the critical case for bifurcation of the trap minimum: $V_0=V_\sigma$. The dashed lines in (b)–(d) give the harmonic approximation to the radial potential. The harmonic trap frequencies are $(\omega_z,\omega_r)=2\pi(15.3,7.8)$ Hz

$$\mathcal{F} = N_0(\epsilon_0 - \mu) - \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \sum_{k=1}^{\infty} \frac{e^{k\beta\mu}}{(k\beta)^{5/2}} \mathcal{G}(k\beta), \tag{8}$$

where

$$\mathcal{G}(\alpha) \equiv \int d^3\mathbf{x} \exp[-\alpha V(\mathbf{x})]. \tag{9}$$

All thermodynamic properties are determined by $\mathcal{G}(\alpha)$ and its derivatives. The BEC transition temperature is found from the total atom number at the point where the chemical potential reaches the ground-state energy,

$$N = -\left.\frac{\partial \mathcal{F}}{\partial \mu}\right|_{\mu=\epsilon_0}, \tag{10}$$

with $N_0=0$, giving the transition temperature T_c as the solution of

$$N\lambda_{dB}^3 = \sum_{k=1}^{\infty} \frac{e^{k\beta V_m}}{k^{3/2}} \mathcal{G}(k\beta), \tag{11}$$

where $\lambda_{dB} \equiv \sqrt{2\pi\hbar^2/mk_B T}$ is the thermal de Broglie wavelength.

B. Thermodynamics of the harmonic-Gaussian trap

For the general form of the harmonic-Gaussian trap (1) we can evaluate (9) to find

$$\mathcal{G}(\alpha) = 2\pi\sigma_0^2\sqrt{\frac{\pi}{\alpha 2m\omega_z^2}}\frac{\gamma(\alpha V_\sigma,\alpha V_0)}{(\alpha V_0)^{\alpha V_\sigma}}, \tag{12}$$

where

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt \tag{13}$$

is the incomplete Gamma function.

In the semiclassical approximation it is consistent to take the ground-state energy as the trap minimum $\epsilon_0 \to V_m$, and the free energy can now be written as

$$\mathcal{F} = N_0(V_m - \mu) - \frac{\zeta_4(e^{\beta\mu},\beta V_\sigma,\beta V_0)}{\beta^4\hbar^3\omega_r^2\omega_z}, \tag{14}$$

where we define the generalized ζ function,

$$\zeta_\nu(z,a,b) \equiv a\sum_{k=1}^{\infty}\frac{z^k}{k^{\nu-1}}\Gamma(ka)\gamma^*(ka,kb), \tag{15}$$

and $\gamma^*(a,x)=x^{-a}\gamma(a,x)/\Gamma(a)$ is a single valued analytic function of a and x with no singularities [38]. This function has the following asymptotics:

$$\lim_{V_0\to 0}\zeta_\nu(e^{\beta\mu},\beta V_\sigma,\beta V_0) = \zeta_\nu(e^{\beta\mu}), \tag{16}$$

$$\lim_{(V_0/V_\sigma)\to\infty;\ (V_\sigma/k_BT)\to\infty}\frac{\zeta_\nu(e^{\beta\mu},\beta V_\sigma,\beta V_0)}{\sqrt{2\pi\beta V_\sigma}} = \zeta_{\nu-1/2}(e^{\beta(\mu-V_m)}), \tag{17}$$

$$\lim_{V_0\to -\infty}\zeta_\nu(e^{\beta\mu},\beta V_\sigma,\beta V_0)\frac{|V_0|}{V_\sigma} = \zeta_\nu(e^{\beta(\mu+|V_0|)}), \tag{18}$$

where $\zeta_\nu(z)=\Sigma_{k=0}^{\infty}z^k/k^\nu$ is the polylogarithm function. In the regime of Bose-Einstein condensation, where $\mu - V_m \to 0$, this further reduces to the ordinary Riemann-zeta function $\lim_{V_0\to 0}\zeta_\nu(1,\beta V_\sigma,\beta V_0)=\zeta_\nu(1)=\zeta(\nu)$. We may check that familiar behavior is recovered by using (16) and (18) to recover the harmonic and dimple trap cases as the corresponding limits of (14).

The number of atoms in the system is given by

$$N = -\frac{\partial\mathcal{F}}{\partial\mu} = N_0 + \frac{\zeta_3(e^{\beta\mu},\beta V_\sigma,\beta V_0)}{\beta^3\hbar^3\omega_r^2\omega_z}, \tag{19}$$

from which the transition temperature T_c for N atoms is found as the solution of

$$N = \frac{\zeta_3(e^{\beta_c V_m},\beta_c V_\sigma,\beta_c V_0)}{\beta_c^3\hbar^3\omega_r^2\omega_z}, \tag{20}$$

where $\beta_c \equiv 1/k_BT_c$. From Eqs. (19) and (20) we then find the equation of state as

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3\frac{\zeta_3(e^{\beta V_m},\beta V_\sigma,\beta V_0)}{\zeta_3(e^{\beta_c V_m},\beta_c V_\sigma,\beta_c V_0)}, \tag{21}$$

which must be evaluated numerically and describes all the three regimes introduced above: dimple trap, flat trap, and the toroidal trap.

We remark that all ideal-gas properties of the system for arbitrary σ_0 and V_0 are not easily obtained from the current formulation. The difficulty arises when considering derivatives of Eq. (15) with respect to a and b, as required for obtaining the entropy and heat capacity. Function (15) is not closed with respect to differentiation, rather leading to a hierarchy of transcendental functions with each successive derivative operation [38]. This problem can be solved by introducing a generalization of Eq. (15), and a treatment that includes the double well, toroidal, and ellipsoidal cases will be provided elsewhere. For the remainder of this paper we will restrict our attention to the $0 \leq V_0$ case and focus primarily on the toroidal regime.

III. HARMONIC SCALE INVARIANCE

So far we have seen that known results are obtained in the appropriate limits. Our present result is found by using Eq. (17), where we find for the toroidal trap

$$\mathcal{F} = N_0(V_m - \mu) - \frac{\sqrt{2}\pi\sigma_0}{\lambda_{dB}}\frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\beta^3\hbar^2\omega_r\omega_z}. \tag{22}$$

What is immediately apparent here is that in this regime all of the ideal-gas thermodynamical properties become independent of V_0. In particular, the thermodynamics are independent of the toroidal size, and the system enters a *scale-invariant regime* defined by

$$k_BT \ll V_\sigma \ll V_0. \tag{23}$$

In this regime both of the energy scales V_0 and V_σ drop out of the problem. Specifically, for fixed harmonic frequencies (ω_r,ω_z) and a beam of fixed width (σ_0), the laser power can be increased to generate a toroidal potential with larger perimeter. Nevertheless, all ideal-gas properties of the system are invariant under this dilation. The only remnant of the Gaussian beam enters through the length scale σ_0 appearing in Eq. (22).

The invariance condition (23) can also be written in terms of appropriate length scales of the trap. We introduce a temperature associated with the optical potential height $V_0=k_BT_0$ and de Broglie wavelength $\lambda_0=\sqrt{2\pi\hbar^2/mk_BT_0}$. Then Eq. (23) can be written as

$$\lambda_0 \ll \sqrt{4\pi}\frac{a_r^2}{\sigma_0} \ll \lambda_{dB}. \tag{24}$$

The de Broglie wavelength must be long compared to the other length scales of the system, in the precise sense defined by Eq. (24).

We will adopt the more specific term *harmonic scale invariance* for this regime, as we will show below that condition (23) restricts the system to a regime where the radial trap is well approximated by a quadratic expansion of the potential about the toroidal minimum. In fact, as we shall show in Sec. IV, the scale-invariant property is more general, and the system enters a scale-invariant regime whenever $V_\sigma \ll V_0$.

A. Harmonic scale invariance

A more revealing description of the regime represented by Eqs. (22) and (23) is obtained by considering the radial curvature of the trap

$$\frac{\partial^2 V(\mathbf{x})}{\partial r^2} = m\omega_r^2 - \frac{2V_0}{\sigma_0^2}\left(1 - \frac{2r^2}{\sigma_0^2}\right)e^{-r^2/\sigma_0^2}, \tag{25}$$

so that at $r=r_m$, we have

$$\left.\frac{\partial^2 V(\mathbf{x})}{\partial r^2}\right|_{r=r_m} = \frac{2m\omega_r^2 r_m^2}{\sigma_0^2} \equiv m\omega_T^2, \tag{26}$$

which we have used to define the harmonic radial trapping frequency about the minimum of the toroid,

$$\omega_T = \sqrt{2}\frac{\omega_r r_m}{\sigma_0} = \omega_r\sqrt{2\ln(V_0/V_\sigma)}. \tag{27}$$

The toroidal trap potential can be expanded about its radial minimum as

$$V(\mathbf{x}) \approx V_T(\mathbf{x}) \equiv V_m + \frac{m\omega_z^2 z^2}{2} + \frac{m\omega_T^2 (r-r_m)^2}{2}, \tag{28}$$

with error $O((r-r_m)^3)$.

Using Eq. (27) for the toroidal frequency, we can write Eq. (22) in a more suggestive way,

$$\mathcal{F} = N_0(V_m - \mu) - \frac{2\pi r_m}{\lambda_{dB}}\frac{\zeta_{7/2}(e^{\beta(\mu - V_m)})}{\beta^3\hbar^2\omega_T\omega_z}, \tag{29}$$

where now we see explicitly the scaling with the toroidal perimeter $2\pi r_m$ and the appearance of ω_T as the physical parameter of the radial degrees of freedom.

Our treatment thus far has relied on the full semiclassical expression for the free energy for determining the scale-invariant regime. A description of the scale-invariant regime can also be found by applying the harmonic approximation [Eq. (28)] directly to the free energy [Eq. (8)]. In the scale-invariant regime given by Eq. (23), $\mathcal{G}(\alpha)$ can be approximated by the quadratic expansion of the potential about the minimum Eq. (28). Integral (9), given by

$$\mathcal{G}(\alpha) = \int_{-\infty}^{\infty} dz e^{-\alpha m\omega_z^2 z^2/2} 2\pi \int_0^{\infty} r dr e^{-\alpha m\omega_r^2 r^2/2 - \alpha V_0 e^{-r^2/2\sigma_0^2}}, \tag{30}$$

can then be approximated as

$$\begin{aligned}\mathcal{G}(\alpha) &\approx \sqrt{\frac{2\pi}{\alpha m\omega_z^2}}2\pi\int_{-r_m}^{\infty}(y+r_m)dy e^{-\alpha m\omega_T^2 y^2/2 - \alpha V_m} \\ &\approx \sqrt{\frac{2\pi}{\alpha m\omega_z^2}}2\pi r_m\int_{-\infty}^{\infty}dy e^{-\alpha m\omega_T^2 y^2/2-\alpha V_m} \\ &= 2\pi r_m\frac{2\pi}{\alpha m\omega_z\omega_T}e^{-\alpha V_m}.\end{aligned} \tag{31}$$

Using this with Eq. (8) gives Eq. (29) that we previously found using the exact semiclassical free energy in the harmonic scale-invariant regime.

Having found the scale-invariant grand potential through two different approaches, we now consider the transition temperature for the system, which is given by

$$T_c = \frac{1}{k_B}\left(\frac{\hbar^2\omega_z\omega_T N}{\zeta(5/2)}\right)^{2/5}\left(\frac{\hbar^2}{\pi m r_m^2}\right)^{1/5}. \tag{32}$$

Defining the toroidal kinetic energy

$$\hbar\omega_K = \frac{\hbar^2}{2\pi m r_m^2} \tag{33}$$

and modified geometric mean frequency

$$\bar{\omega}^5 = \omega_K\omega_T^2\omega_z^2 = \frac{\hbar}{\pi m\sigma_0^2}\omega_r^2\omega_z^2, \tag{34}$$

the transition temperature becomes

$$T_c = \frac{\hbar\bar{\omega}}{k_B}\left(\frac{N}{\zeta(5/2)}\right)^{2/5} \simeq 0.89\frac{\hbar\bar{\omega}}{k_B}N^{2/5}. \tag{35}$$

This expression closely resembles that for the three-dimensional harmonic trap $k_B T_c = \hbar[\omega_x\omega_y\omega_z N/\zeta(3)]^{1/3}$. The scaling with $N^{2/5}$ is caused by the reduction in the number of thermalized degrees of freedom in the system by 1, a consequence of toroidal trapping topology. The characteristic energy of the system $\hbar\bar{\omega}$, which also determines T_c, is seen from Eq. (34) to be scale invariant; invariance is caused by the precise dependence of ω_T on toroidal radius and the fact that this degree of freedom is harmonically bound so that it is doubly weighted in $\bar{\omega}$. In general harmonically bound degrees of freedom have double weight, whereas ω_K associated with the periodic coordinate is only singly weighted.

B. Density of states

The semiclassical density of states is given by

$$\rho(\epsilon) = \frac{2\pi(2m)^{3/2}}{h^3}\int_{V^*(\epsilon)} d^3\mathbf{x}\sqrt{\epsilon - V(\mathbf{x})}, \tag{36}$$

where $V^*(\epsilon)$ is the spatial volume available to a particle with energy ϵ. Making use of cylindrical symmetry, we obtain

$$\rho(\epsilon) = \frac{1}{\hbar^3}\frac{m}{\omega_z}\int_{R_-}^{R_+} dr r[\epsilon - V_{\text{eff}}(r)], \tag{37}$$

where $V_{\text{eff}}(r) = m\omega_r^2 r^2/2 + V_0 e^{-r^2/\sigma_0^2} - V_m \approx m\omega_T^2(r-r_m)^2/2$ is the shifted radial potential and energy is now expressed relative to V_m. The limits of integration are the two solutions of $\epsilon = V_{\text{eff}}(R_\pm)$, with $R_- < R_+$. Making the harmonic approximation to the potential, the semiclassical density of states becomes

$$g(\epsilon) \approx \frac{1}{\hbar^3}\frac{2mr_m}{\omega_z}\int_0^{\sqrt{2\epsilon/m\omega_T^2}} dy[\epsilon - m\omega_T^2 y^2/2] \tag{38}$$

$$= \frac{1}{\hbar^3}\frac{2mr_m}{\omega_z}\frac{2\epsilon}{3}\sqrt{\frac{2\epsilon}{m\omega_T^2}}. \tag{39}$$

Our analysis of the exact expression of the grand potential has shown that this regime is reached rigorously by taking

$V_0 \ll V_\sigma$ and $V_\sigma \ll k_B T$ simultaneously. Equation (39) can be written as

$$\lim_{(V_0/V_\sigma)\to\infty;\ (V_\sigma/k_BT)\to\infty} g(\epsilon) \equiv g_T(\epsilon) = \frac{4\epsilon^{3/2}}{3\sqrt{\pi}(\hbar\bar{\omega})^{5/2}}, \quad (40)$$

giving all thermodynamical properties of the system in the harmonic scale-invariant regime.

C. Thermodynamics

Using either Eq. (40) or Eqs. (29) and (34), the grand potential can now be written as

$$\mathcal{F} = N_0(V_m - \mu) - \frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\beta(\beta\hbar\bar{\omega})^{5/2}}. \quad (41)$$

For completeness we give the thermodynamic quantities in the harmonic scale-invariant regime, both above ($>$) and below ($<$) T_c. The total atom number in the system is

$$N = N_0 + \frac{\zeta_{5/2}(e^{\beta(\mu-V_m)})}{(\beta\hbar\bar{\omega})^{5/2}}, \quad (42)$$

where $N_0 > 0$ below T_c, and the condensate fraction is then given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{5/2}, \quad (43)$$

again showing the reduction to 5 degrees of freedom.

The entropy $S = -(\partial\mathcal{F}/\partial T)_{V,\mu}$ is

$$\frac{S_>}{Nk_B} = \frac{7}{2}\frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\zeta_{5/2}(e^{\beta(\mu-V_m)})} - \beta(\mu - V_m), \quad (44)$$

$$\frac{S_<}{Nk_B} = \frac{7}{2}\frac{\zeta(7/2)}{\zeta(5/2)}\left(\frac{T}{T_c}\right)^{5/2}. \quad (45)$$

The energy $U = \mathcal{F} + TS + \mu N$ is

$$\frac{U_>}{Nk_BT} = \frac{5}{2}\frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\zeta_{5/2}(e^{\beta(\mu-V_m)})} + \beta V_m, \quad (46)$$

$$\frac{U_<}{Nk_BT} = \frac{5}{2}\frac{\zeta(7/2)}{\zeta(5/2)}\left(\frac{T}{T_c}\right)^{5/2} + \beta V_m, \quad (47)$$

and the heat capacity $C = (\partial U/\partial T)_{N,V}$ takes the form

$$\frac{C_>}{Nk_B} = \frac{35}{4}\frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\zeta_{5/2}(e^{\beta(\mu-V_m)})} - \frac{25}{4}\frac{\zeta_{5/2}(e^{\beta(\mu-V_m)})}{\zeta_{3/2}(e^{\beta(\mu-V_m)})}, \quad (48)$$

$$\frac{C_<}{Nk_B} = \frac{35}{4}\frac{\zeta(7/2)}{\zeta(5/2)}\left(\frac{T}{T_c}\right)^{5/2}. \quad (49)$$

The discontinuity in the heat capacity across the transition $\Delta C(T_c) \equiv C_>(T_c) - C_<(T_c)$ is

$$\frac{\Delta C(T_c)}{Nk_B} = -\frac{25}{4}\frac{\zeta(5/2)}{\zeta(3/2)} \simeq -3.21. \quad (50)$$

D. Generalized volume and pressure and equation of state

A generalization of thermodynamical formalism to trapped systems has recently been developed and applied by Romero-Rochin *et al.* [36,39,40]. This formalism provides a definition of generalized volume and pressure variables for the system, allowing an equation of state to be usefully obtained. While in principle the formalism allows a generalized volume to be defined for any geometry, in general for this system it would have to be determined numerically. To gain some insight into the role of trapping topology in changing effective system volume, in this section we apply this approach to treat the harmonic scale-invariant regime of the ideal gas.

Proceeding in a similar manner to [36] we see that since N, U, S, and $\mathcal{F}$ are extensive variables, and T and μ are intensive, we may identify the generalized extensive volume in the harmonic scale-invariant regime as $\bar{\omega}^{-5/2}$,

$$\mathcal{V} \equiv (\omega_K\omega_T^2\omega_z^2)^{-1/2}. \quad (51)$$

The fact that it takes this form is not so surprising, since at a given temperature the gas will be mainly confined to a volume of order (toroidal circumference) $\times$(cross-sectional area)$\sim 2\pi r_m(k_BT/m\omega_z^2)^{1/2}(k_BT/m\omega_T^2)^{1/2} \propto \bar{\omega}^{-5/2}$. This should be compared to the volume for a harmonically trapped gas which is of order $(k_BT/m\omega^2)^{3/2} \propto \omega^{-3}$, where $\omega = (\omega_x\omega_y\omega_z)^{1/3}$ is the usual geometric mean [36]. We can now identify an important change in the system relative to the purely harmonic case; the system has effectively two thermally determined length scales parametrizing the cross section and one purely geometric length scale that is under direct experimental control, the toroidal circumference.

The conjugate generalized intensive pressure $\mathcal{P} = -(\partial\mathcal{F}/\partial\mathcal{V})_{N,T}$ is then given by

$$\mathcal{P} = -\frac{\zeta_{7/2}(e^{\beta(\mu-V_m)})}{\beta(\beta\hbar)^{5/2}}. \quad (52)$$

Above T_c this gives the equation of state $\mathcal{P}\mathcal{V} = -\mathcal{F}$ as required [36].

It is important to see that the thermodynamic limit is well defined in the scale-invariant regime. The appropriate limit is $N\to\infty$ and $\mathcal{V}\to\infty$, with $N/\mathcal{V} = N\bar{\omega}^{5/2} \to$ const. Since $\mathcal{V} \propto \sigma_0/\omega_r\omega_z$ we see that the thermodynamic limit requires

$$\frac{N\omega_r\omega_z}{\sigma_0} \to \text{const}, \quad (53)$$

corresponding to either relaxing the harmonic trapping frequencies or increasing the Gaussian beam width σ_0 appropriately. For the latter case, if the system is to remain in the scale-invariant regime ($V_\sigma \ll V_0$), we must also impose the further condition that $V_0 \to \infty$ at least as fast as σ_0^2 to obtain a consistent thermodynamic limit in this regime.

IV. GENERAL SCALE INVARIANCE

In this section we show that it is possible to further relax condition (23) for scale invariance. Our numerical investiga-

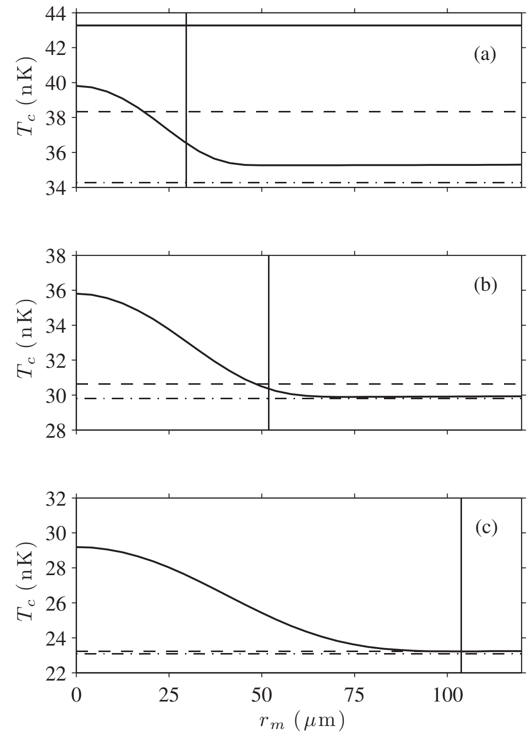

FIG. 2. Numerical T_c for ^{87}Rb atoms held in a cylindrically symmetric harmonic trap for a range of Gaussian potentials parametrized by the location of the trap minimum r_m. T_c (solid curves) is calculated for (a) $\sigma_0=14.1$ μm, (b) $\sigma_0=24.7$ μm, and (c) $\sigma_0=49.5$ μm by solving Eq. (20). The solid horizontal line in (a) shows $T_c=43.3$ nK for the purely harmonic trap ($V_0=0$). Dashed lines give the harmonic scale invariant T_c^0 [Eq. (35)], and dashed-dotted lines show the correction $T_c^0+\delta T_c^1$ given by Eq. (60). The ratio $k_BT_c(V_0\to\infty)/V_\sigma$ is 3.5, 0.95, and 0.19 for (a), (b), and (c), respectively, and the vertical lines show $V_0=3V_\sigma$. Other parameters are $N=10^6$ atoms and $(\omega_z,\omega_r)=2\pi(15.3,7.8)$ Hz.

tions show that a scale-invariant regime is always reached provided that $V_\sigma \ll V_0$ (see Fig. 2). Thus there are scale-invariant regimes parametrized by the ratio $V_\sigma/k_BT\sim 1$, corresponding to $\lambda_{dB}\sim a_r^2/\sigma_0$.

However, the properties of the system are not well described by the harmonic approximation to the potential (28) unless $k_BT\ll V_\sigma$ also holds. We now seek a general asymptotic expansion of the grand potential which does not require this condition to hold.

We make use of the representation of $\gamma(a,x)$ in terms of the confluent hypergeometric function $M(a,b,z)$ [38], $\gamma(a,x)=a^{-1}x^ae^{-x}M(1,1+a,x)$, and the asymptotic expansion of $M(a,b,z)$ for large real positive z and fixed a,b. To leading order in powers of z^{-1} we easily obtain

$$\gamma(a,x)=[a^{-1}\Gamma(a+1)-z^{a-1}e^{-z}][1+O(z^{-1})]. \tag{54}$$

At leading order in $(\beta V_0)^{-1}$ we then find

$$\zeta_\nu(e^{\beta\mu},\beta V_\sigma,\beta V_0)\simeq\sum_{k=1}^{\infty}\frac{e^{k\beta\mu}}{k^\nu}\frac{\Gamma(k\beta V_\sigma+1)}{(k\beta V_0)^{k\beta V_\sigma}}-\frac{V_\sigma}{V_0}\zeta_\nu(e^{\beta(\mu-V_0)}). \tag{55}$$

As $V_0/V_\sigma\to\infty$, $\zeta_\nu(e^{\beta(\mu-V_0)})\to e^{\beta(\mu-V_0)}$ and the second term vanishes. We note that for the vast majority of terms in the summation, the inequality $1\ll k\beta V_\sigma$ will hold even if $\beta V_\sigma\sim 1$. Introducing Stirling's expansion for $\Gamma(k\beta V_\sigma+1)$ and making use of the identity $e^{-\beta V_m}=e^{-\beta V_\sigma}(V_\sigma/V_0)^{\beta V_\sigma}$, we finally obtain the expansion

$$\lim_{V_0/V_\sigma\to\infty}\frac{\zeta_\nu(e^{\beta\mu},\beta V_\sigma,\beta V_0)}{\sqrt{2\pi\beta V_\sigma}}=\Bigg[\zeta_{\nu-1/2}(e^{\beta(\mu-V_m)})+\frac{\zeta_{\nu+1/2}(e^{\beta(\mu-V_m)})}{12\beta V_\sigma}+\frac{\zeta_{\nu+3/2}(e^{\beta(\mu-V_m)})}{288(\beta V_\sigma)^2}-\cdots\Bigg], \tag{56}$$

where the numerical coefficients of the asymptotic expansion are the coefficients of the Stirling expansion for $\Gamma(z)$. This can be used to give a more general expression for the free energy than the asymptotic form (41) which arises from the leading term in Eq. (56).

In the regime where $V_\sigma\ll V_0$ the grand free energy can now be written as

$$\mathcal{F}=N_0(V_m-\mu)-\frac{1}{\beta(\beta\hbar\bar{\omega})^{5/2}}\Bigg[\zeta_{7/2}(e^{\beta(\mu-V_m)})+\frac{\zeta_{9/2}(e^{\beta(\mu-V_m)})}{12\beta V_\sigma}+\frac{\zeta_{11/2}(e^{\beta(\mu-V_m)})}{288(\beta V_\sigma)^2}-\cdots\Bigg]. \tag{57}$$

This expression for the free energy is our main result and provides an asymptotically exact representation of all thermodynamics of the system in the scale-invariant regime. The energy scale V_0 only appears through the shift of the potential minimum V_m. Ignoring this trivial shift of energy, we now have

$$\lim_{V_0/V_\sigma\to\infty}\frac{\partial\mathcal{F}}{\partial r_m}\equiv 0, \tag{58}$$

and all ideal-gas thermodynamics of the system are scale invariant. In practice this regime is reached rather quickly. In the example shown in Fig. 2 the onset of scale invariance occurs around $V_0\sim 3V_\sigma$.

Nonharmonic corrections to T_c

As an application of the generalized expansion (57) we can now find a more accurate expression for T_c. We easily find the following asymptotic expansion for T_c:

$$\frac{T_c}{T_c^0}=\left[1+\frac{k_BT_c\zeta(7/2)}{12V_\sigma\zeta(5/2)}+\frac{(k_BT_c)^2\zeta(9/2)}{288V_\sigma^2\zeta(5/2)}-\cdots\right]^{-2/5}, \tag{59}$$

where the atom number has been eliminated in terms of the harmonic scale-invariant transition temperature (35) denoted by T_c^0. We now expand $T_c=T_c^0+\delta T_c^1+\delta T_c^2+\cdots$, which we treat formally as a perturbation expansion in powers of $k_BT_c^0/V_\sigma$. Solving for the first-order correction, we find

$$\frac{\delta T_c^1}{T_c^0}=-\frac{k_BT_c^0}{30V_\sigma}\frac{\zeta(7/2)}{\zeta(5/2)}\simeq -1.44\times 10^{-2}\frac{k_BT_c^0}{V_\sigma}. \tag{60}$$

In Fig. 2 we show the numerical solution of Eq. (20) and compare with the harmonic scale-invariant form (35) and the first correction for nonharmonic behavior (60). The full numerical solution reaches a scale-invariant regime for $V_\sigma \ll V_0$ and approaches the harmonic behavior for $k_B T_c \ll V_\sigma$. The rapid decay of the Stirling expansion coefficients renders the first correction given by Eq. (60) adequate even when $k_B T_c^0 \sim 3V_\sigma$, as can be seen from Fig. 2(a) where $T_c^0 + \delta T_c^1$ approaches T_c to within 3%.

V. NONIDEAL CORRECTIONS TO T_c

The ideal-gas behavior reveals a scale-invariant regime for the toroidal trap. While this gives a strong indication of what to expect in experiments, it is important to determine how additional effects will change this picture. We now consider the two most significant effects on the transition temperature: finite-size corrections and interactions. Our main focus here is on the modifications these effects will make to the scale invariance of quantum degeneracy.

A. Mean-field interaction shift in T_c

Recent experimental [41] and theoretical [42] studies of the harmonically trapped Bose gas have established that the value for T_c is well described by the combination of ideal-gas theory and the first-order mean-field interaction shift [43]. Within experimental error bars, the mean-field interaction shift is the dominant effect and critical fluctuations appear to be entirely negligible. Since the system we consider is in the semiclassical regime we evaluate the first-order interaction shift due to s-wave collisions from the expression derived by Giorgini *et al.* [43], which is

$$\frac{\delta T_c^{int}}{T_c^0} = -\frac{2U_0}{T_c^0} \frac{\int d^3\mathbf{x}\, \partial n_{th}^0/\partial\mu [n_{th}^0(\mathbf{x}=0) - n_{th}^0(\mathbf{x})]}{\int d^3\mathbf{x}\, \partial n_{th}^0/\partial T}, \quad (61)$$

where $n_{th}^0(\mathbf{x})$ is the noninteracting semiclassical thermal cloud density

$$n_{th}^0(\mathbf{x}) = \frac{1}{\lambda_{dB}^3} \sum_{k=1}^{\infty} \frac{e^{-\beta k[\mu - V(\mathbf{x})]}}{k^{3/2}}, \quad (62)$$

and $U_0 = 4\pi\hbar^2 a/m$ gives the interaction strength in terms of the s-wave scattering length a. As before, in the harmonic scale-invariant regime where Eq. (23) holds, we can carry out the harmonic approximation for the potential. A straightforward calculation similar to that of Ref. [43] gives

$$\frac{\delta T_c^{int}}{T_c^0} = -\frac{aN^{1/5}}{\bar{a}} \left(\frac{8}{5\sqrt{2}} \frac{\zeta(3/2)^2(1-G)}{\pi^{1/2}\zeta(5/2)^{6/5}} \right), \quad (63)$$

where $\bar{a} = \sqrt{\hbar/m\bar{\omega}}$ is the modified geometric mean of the toroid length scales defined as $\bar{a}^5 = a_K a_T^2 a_z^2$, with $a_j = \sqrt{\hbar/m\omega_j}$, and

$$G = \frac{1}{\zeta(3/2)^2} \sum_{j=1,\ k=1}^{\infty} \frac{1}{j^{3/2}k^{1/2}(j+k)} \equiv \frac{S}{\zeta(3/2)^2}. \quad (64)$$

We can obtain G by writing

$$S = \sum_{j=1,\ k=1}^{\infty} \left(1 - \frac{j}{j+k}\right) \frac{1}{j^{3/2}k^{3/2}} = \zeta(3/2)^2 - S \quad (65)$$

to give $G = 1/2$. Evaluating the numerical factors in Eq. (63) gives

$$\frac{\delta T_c^{int}}{T_c^0} \simeq -1.53 \frac{aN^{1/5}}{\bar{a}}. \quad (66)$$

This expression has the same structure as the well-known result for the shift in T_c in the harmonic trap [43]. A different characteristic length scale arises here and the dependence is now on $N^{1/5}$ rather than the $N^{1/6}$ behavior in the three-dimensional harmonic trap. As noted in Ref. [43], the first-order interaction shift depends on the geometric mean frequency of the trap. Here we see the interaction shift is invariant with the size of the toroid, depending on system size only through the modified geometric mean length scale $\bar{a}$.

B. Finite-size effect on T_c

The effect of finite particle number is calculated to first order by shifting the ground-state chemical potential up to the quantum-mechanical ground state of the potential (28),

$$\delta\mu = \frac{\hbar}{2}(\omega_T + \omega_z), \quad (67)$$

which shifts T_c according to

$$\frac{\delta T_c^{fs}}{T_c^0} = \frac{1}{T_c^0} \left. \frac{\partial T_c^0}{\partial \mu} \right|_N \delta\mu = -\frac{\zeta(3/2)}{5\zeta(5/2)} \left(\frac{\zeta(5/2)}{N} \right)^{2/5} \frac{\omega_T + \omega_z}{\bar{\omega}}. \quad (68)$$

Evaluating the numerical factors gives

$$\frac{\delta T_c^{fs}}{T_c^0} \simeq -0.44 \frac{\omega_T + \omega_z}{\bar{\omega}} N^{-2/5}. \quad (69)$$

As $\omega_T \propto r_m$ while $\bar{\omega}$ is invariant, finite-size effects can become a significant correction for large r_m. However, the scaling with $N^{-2/5}$ will strongly suppress this effect for large N.

For the parameters used in Fig. 2(c), the shifts have the values $\delta T_c^{int}/T_c^0 \sim -2 \times 10^{-5}$ (for 100 μm $< r_m$) and $-2 \times 10^{-2} < \delta T_c^{fs}/T_c^0 < -6 \times 10^{-2}$ (for 100 μm $< r_m$ < 500 μm). In contrast to the harmonically trapped gas, for these parameters the finite-size correction is the dominant shift in the toroidal trap. However, the finite-size shift vanishes in the thermodynamic limit, and for significant atom number (here $N = 10^6$) there is a wide range of toroidal radii where scale invariance of T_c holds to within a few percents.

VI. REACHING SCALE INVARIANCE

There are at least three means to reach the scale-invariant regime with degenerate Bose gases. First, a noncondensed gas may be evaporatively cooled below T_c into a toroidal trap as recently demonstrated experimentally [6]. A second method is to adiabatically load from the harmonic trap into the toroid by ramping up the optical potential which we further investigate below. Last, a gas in a harmonic trap with a small optical potential may be rotated which will push it into the scale-invariant regime.

The physics of the rotating case is rather simple; an ideal Bose gas in equilibrium in trap (1) in a frame rotating around the z axis with frequency $\Omega<\omega_r$ has thermodynamic properties of a system in the laboratory frame with effective radial trapping frequency $\omega_\perp=\sqrt{\omega_r^2-\Omega^2}$ [44]. Thus all the properties of the system follow from the laboratory-frame analysis presented above, after replacing V_σ with an effective rotating-frame energy $V_\sigma^\perp=m\omega_\perp^2\sigma_0^2/2=V_\sigma(1-\Omega^2/\omega_r^2)$. In the toroidal regime (where $V_\sigma^\perp<V_0$) the radius to the trap minimum $r_m^\perp$ is given by

$$r_m^\perp=\sigma_0\left[\ln(V_0/V_\sigma)+\ln\left(\frac{\omega_r^2}{\omega_r^2-\Omega^2}\right)\right]^{1/2}. \tag{70}$$

The essential condition for scale invariance becomes

$$V_\sigma(1-\Omega^2/\omega_r^2)\ll V_0 \tag{71}$$

for the rotating gas, and the harmonic approximation will be valid when $k_BT\ll V_\sigma(1-\Omega^2/\omega_r^2)$ also holds. Frequency (34) determining harmonic scale-invariant properties is modified to

$$\bar{\omega}_\perp^5=\frac{\hbar}{\pi m\sigma_0^2}\omega_r^2\omega_z^2(1-\Omega^2/\omega_r^2) \tag{72}$$

by the rotation. We conclude that for a gas in rotating equilibrium, scale invariance with respect V_0 is preserved but with appropriately modified characteristic energy $\hbar\bar{\omega}_\perp$ determining the density of states. However, since $\bar{\omega}_\perp$ depends on Ω, changes in the toroidal radius (70) caused by rotation do not have the scale-invariant property.

Adiabatic loading

We consider adiabatically loading a degenerate Bose gas from the harmonic trap into the toroidal trap. In order to maintain adiabaticity the timescale of loading T_L should greatly exceed the slowest timescale of the system, i.e., $\min(2\pi/\omega_j)\ll T_L$. Under these conditions entropy will be conserved during loading. Equating the entropy in the harmonic trap for the regime $T<T_c$,

$$\frac{S_<^H}{Nk_B}=\frac{4\zeta(4)}{\zeta(3)}\left(\frac{T}{T_c}\right)^3\equiv\frac{4\zeta(4)}{\zeta(3)}t_H^3, \tag{73}$$

with the scale-invariant toroidal entropy of Eq. (45),

$$\frac{S_<^T}{Nk_B}=\frac{7\zeta(7/2)}{\zeta(5/2)}t_T^{5/2}, \tag{74}$$

we find

$$t_T(t_H)=\left(\frac{8\zeta(4)\zeta(5/2)}{7\zeta(3)\zeta(7/2)}\right)^{2/5}t_H^{6/5} \tag{75}$$

for the reduced temperature in the scale-invariant toroid after adiabatic loading from the harmonic trap. From this expression we can find t_H^* defined by $t_T(t_H^*)\equiv t_H^*$, i.e., where adiabatic loading has no effect on the reduced temperature in the toroid. This is given by

$$t_H^*=\left(\frac{7\zeta(3)\zeta(7/2)}{8\zeta(4)\zeta(5/2)}\right)^2\simeq 0.67. \tag{76}$$

Evaluating the numerical factors gives

$$t_T=t_H^{6/5}/(t_H^*)^{1/5}\simeq 1.08t_H^{6/5}, \tag{77}$$

which is shown in Fig. 3(a). We see that to a good approximation $t_T\approx t_H$ for the region $t_H\lesssim 1$. Substituting Eq. (77) into Eq. (43), gives the expression

$$\left(\frac{N_0}{N}\right)=1-t_H^3/(t_H^*)^{1/2}\simeq 1-1.23t_H^3 \tag{78}$$

for the condensate fraction in the toroid after loading. This expression is shown in Fig. 3(b), where we see that the condensate fraction is well preserved during loading from the purely harmonic trap to the harmonic scale-invariant toroidal trap.

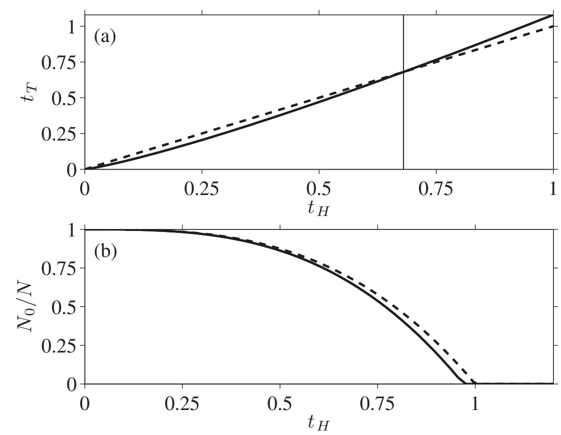

FIG. 3. Adiabatic loading from a harmonic trap into the scale-invariant regime. (a) Reduced temperature in the toroidal trap (solid line) after adiabatically ramping up the optical potential for a system with reduced temperature t_H in the harmonic trap. The line $t_T=t_H$ (dashed line) is shown for comparison. The point $t_H=t_H^*\simeq 0.67$, from Eq. (76), is shown by the vertical line. (b) Condensate fraction in the toroidal trap (solid line) after adiabatic loading from initial reduced temperature t_H. The condensate fraction for the harmonic trap is also shown (dashed line).

VII. CONCLUSIONS

We have provided a numerical and analytical treatment of the Bose gas trapped in a harmonic-Gaussian trap with toroidal topology. We have identified a regime where the harmonic-Gaussian toroidal trap has properties that are independent of the radius to the toroidal minimum. In particular, the transition temperature to Bose-Einstein condensation is scale invariant, and the toroid radius can then be increased without altering quantum degeneracy.

A. Harmonic scale invariance

In the regime where $k_B T_c \ll V_\sigma = m\omega_r^2\sigma_0^2/2$, the radial trap is well approximated by its quadratic expansion about the minimum. This regime affords an analytical description, and we have identified the invariant generalized extensive volume $\mathcal{V}=(\omega_K\omega_z^2\omega_T^2)^{-1/2}$ which determines the thermodynamics of the system. We have also shown that the first-order mean-field interaction shift to T_c is scale invariant in this regime. The main limitation on the invariance of T_c is the finite-size shift that vanishes in the thermodynamic limit.

B. General scale invariance

Relaxing the condition $k_B T_c \ll V_\sigma$, we find that the ideal gas always enters a scale-invariant regime when $V_\sigma \ll V_0$, for which the grand potential becomes independent of the toroidal radius: $\partial\mathcal{F}/\partial r_m=0$. In considering corrections to the harmonic approximation to the potential, we have evaluated the first-order perturbation of T_c in powers of $k_B T_c/V_\sigma$, and we find that it provides an accurate approximation even in the regime where $V_\sigma \sim k_B T_c$. In practice scale invariance is reached quite rapidly with increasing V_0, and the onset occurs at about $V_0 \sim 3V_\sigma$ for the specific system treated here.

C. Reaching scale invariance

We have considered adiabatic loading and applying rotation as ways to reach the scale-invariant regime. Adiabatic loading appears to preserve the condensate fraction of the ideal gas quite well, while rotation enhances the height of the Gaussian relative to the rotating-frame harmonic trap, thus deepening the toroidal potential. Another promising method is to evaporatively cool directly into the toroidal trap. The existence of a scale-invariant regime shows that rather than always decreasing with toroidal radius, T_c for such a system reaches a well-defined plateau. A system with these properties may be promising for creating and loading a large toroidal trap with a persistent current while maintaining quantum degeneracy. Future work will focus on more general thermodynamical quantities, the role of interactions, and extensions to trapped Fermi gases.

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