

# Quantum fluctuations in trapped time-dependent Bose-Einstein condensates

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Quantum fluctuations in time-dependent, harmonically trapped Bose-Einstein condensates are studied within Bogoliubov theory. An eigenmode expansion of the linear field operators permits the diagonalization of the Bogoliubov–de Gennes equation for a stationary condensate. When trap frequency or interaction strength is varied, the inhomogeneity of the background gives rise to off-diagonal coupling terms between different modes. This coupling is negligible for low energies, i.e., in the hydrodynamic regime, and an effective space-time metric can be introduced. The influence of the intermode coupling will be demonstrated in an example, where I calculate the quasiparticle number for a quasi-one-dimensional Bose-Einstein condensate subject to an exponential sweep of interaction strength and trap frequency.

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## I. INTRODUCTION

Ultracold atomic gases offer various opportunities for the study of interacting many-body quantum systems in a well-controlled environment [1,2]. For instance, the Bose-Hubbard model—a simplified description for bosons in a periodic potential—can be studied with Bose-Einstein condensates confined in optical lattices [3,4]. Quantum gases have also gained much attention lately regarding the emergence of an effective space-time [5–19]: their low-energy phase fluctuations obey the same covariant field equations as a scalar quantum field in a certain curved space-time. Hence, the study of phonons in this laboratory system might shed some light on aspects of cosmic quantum effects, e.g., Hawking radiation [8–11,20] or the freezing and amplification of quantum fluctuations in expanding space-times [12–19,21,22]. Although the fluctuations in Bose-Einstein condensates are usually small, it has recently become possible in experiments to go beyond the classical order parameter and resolve signatures of the fluctuations [23–27].

Theoretically, the fluctuations in a Bose-Einstein condensate are usually treated as small perturbations of the mean field. The solution of the coupled field equations is rather demanding and often requires further approximations, especially for time-dependent condensates. The Hartree-Fock-Bogoliubov method (see, e.g., [28]) permits in principle the self-consistent propagation of the mean field and the quantum correlations for arbitrary time dependences of the trap potential or interaction strength. But the scaling of the numerics with system size often limits the actual calculations to a low number of dimensions, certain symmetries, or a short time interval. Thermal condensates might be studied using the projected Gross-Pitaevskii equation [29], where the low-energy part of the fluctuations is expanded into an arbitrary basis and, in view of the large thermal occupation, treated classically; higher excitations as well as the vacuum effect are omitted. On the other hand, studies in the context of

expanding space-times, e.g., [6,7,12–16], indeed focus on the quantum fluctuations but often assume a homogeneous background or start with the hydrodynamic action, which is only valid on scales longer than the healing length.

In this paper, I will discuss the evolution of the quantum fluctuations in trapped time-dependent Bose-Einstein condensates. The linear field operators will be expanded into their eigenmodes, thus permitting the diagonalization of the (initial) evolution equations. Although basis expansions of the field operator are frequently used (e.g., in [29,30]), these references usually consider the harmonic-oscillator eigenfunctions, a large number of which must be used in order to describe the excitations properly. By adopting the eigenmodes, a much smaller part of the basis needs to be considered and many more situations will become numerically feasible. (Note, however, that in order to obtain the fluctuation eigenmodes, a relatively large number of oscillator functions must be employed—but they need not be propagated.)

This paper is organized as follows. Section II reviews the field equations and their linearization. The Bogoliubov–de Gennes equation for the linear quantum fluctuations can be diagonalized by an eigenmode expansion, which will be performed in Sec. III. However, as soon as trap frequency or interaction strength is varied, off-diagonal terms appear. This coupling of different modes is negligible for excitations with energies much smaller than the chemical potential even in time-dependent condensates, as will be shown in Sec. IV, where the order parameter is treated in the Thomas-Fermi approximation. It is then also possible to establish the analogy between phase fluctuations and a massless scalar field in a certain curved space-time. If the excitation energies are of the same order as the chemical potential, the coupling of different modes might lead to a population transfer, which will be illustrated in an example in Sec. V.

## II. FIELD EQUATIONS

### A. Scaling transformation

In dimensionless units, the field operator  $\hat{\Psi}$  of a trapped (quasi-)D-dimensional Bose-Einstein condensate obeys the nonlinear Schrödinger field equation [31]

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$$i\frac{\partial}{\partial t}\hat{\Psi} = \left[ -\frac{\nabla^2}{2} + \omega^2(t)\frac{\mathbf{r}^2}{2} + g(t)\hat{\Psi}^\dagger\hat{\Psi} \right]\hat{\Psi}, \quad (1)$$

with  $D$ -dimensional coupling strength  $g(t) \propto a_s$ . By changing the  $s$ -wave scattering length  $a_s$  through Feshbach resonances [32] or by varying the trap frequency  $\omega(t)$ , an external time dependence can be prescribed on the condensate. The gas cloud will adapt to these changes and it will either expand or contract and with it the quasiparticle excitations residing upon it. A part of this background motion can be accounted for by transforming to new coordinates  $\mathbf{x}=\mathbf{r}/b(t)$  with scale factor  $b(t)$ . The field operator then reads [33]

$$\hat{\Psi}(\mathbf{r},t) = e^{i\Phi}\frac{\hat{\psi}(\mathbf{x},t)}{b^{D/2}}. \quad (2)$$

The phase  $\Phi=(\mathbf{r}^2/2)\dot{b}/b$  is chosen so as to generate an isotropic velocity field  $\nabla\Phi=\mathbf{r}\dot{b}/b$  which (at least partially) describes the expansion/contraction of the condensate. If the scale factor  $b(t)$  obeys

$$f^2(t) = b^3\frac{\partial^2 b}{\partial t^2} + b^4\omega^2(t) = \frac{g(t)}{g_0}b^{2-D}, \quad (3)$$

with  $g_0$  being the initial value of the coupling strength, a scaled field equation follows:

$$ib^2\frac{\partial}{\partial t}\hat{\psi} = \left[ -\frac{\nabla_x^2}{2} + f^2\left(\frac{\mathbf{x}^2}{2} + g_0\hat{\psi}^\dagger\hat{\psi}\right) \right]\hat{\psi}, \quad (4)$$

where trapping and interaction terms have acquired the same time-dependent prefactor  $f^2(t)$  and all other coefficients are time independent. [The scale factor  $b(t)$  on the left-hand side might be included into a redefined time  $d\tau=dt/b^2$ ; see Sec. IV B.]

### B. Linearization

For large particle numbers  $N$ , one might formally expand the field operator into inverse powers of  $\sqrt{N}$  [34],

$$\hat{\psi} = (\psi_0 + \hat{\chi} + \hat{\xi})\frac{\hat{A}}{\sqrt{\hat{N}}}. \quad (5)$$

Here,  $\hat{A}$  and  $\hat{N}=\hat{A}^\dagger\hat{A}$  are the atomic operators. They commute with the linear  $\hat{\chi}=\mathcal{O}(N^0)$  and higher-order quantum excitations  $\hat{\xi}$  and thus yield the exact conservation of particle number. The order parameter  $\psi_0=\mathcal{O}(\sqrt{N})$  in the center of the trap but diminishes toward the edge of the condensate. Insertion of expansion (5) into scaled Heisenberg equation (4) yields the Gross-Pitaevskii equation for the classical background  $\psi_0$  [35],

$$ib^2\frac{\partial}{\partial t}\psi_0 = \left[ -\frac{\nabla_x^2}{2} + f^2\left(\frac{\mathbf{x}^2}{2} + g_0|\psi_0|^2\right) \right]\psi_0. \quad (6)$$

The linear quantum fluctuations  $\hat{\chi}$  obey the Bogoliubov–de Gennes equation [36]

$$ib^2\frac{\partial}{\partial t}\hat{\chi} = \left[ -\frac{\nabla_x^2}{2} + f^2\left(\frac{\mathbf{x}^2}{2} + 2g_0|\psi_0|^2\right) \right]\hat{\chi} + f^2g_0\psi_0^2\hat{\chi}^\dagger, \quad (7)$$

and the residual terms comprise the equation of motion for  $\hat{\xi}$ ,

$$i\frac{b^2}{f^2}\frac{\partial}{\partial t}\hat{\xi} = \left[ -\frac{\nabla_x^2}{2} + \frac{\mathbf{x}^2}{2} + 2g_0|\psi_0|^2 \right]\hat{\xi} + g_0\psi_0^2\hat{\xi}^\dagger + g_0(2\psi_0\hat{\chi}^\dagger\hat{\chi} + \psi_0^*\hat{\chi}^2 + \hat{\chi}^\dagger\hat{\chi}^2) + \mathcal{O}(g_0\hat{\xi}). \quad (8)$$

These higher orders  $\hat{\xi}$  must remain small in order for mean-field expansion (5) to be valid, i.e., for the linearized Eq. (7) to be applicable. This means that the terms involving products of  $\hat{\chi}$  must remain small because they act as source terms for higher orders  $\hat{\xi}$ .

From Gross-Pitaevskii equation (6), I can infer when the evolution of the order parameter  $\psi_0$  is solely described by the scale factor  $b(t)$ : apart from the trivial case  $f^2=\text{const}$ , this occurs only when the spatial derivatives can be neglected with respect to the interaction and trapping terms,  $\nabla_x^2\psi_0 \ll f^2(\mathbf{x}^2 + 2g_0|\psi_0|^2)\psi_0$ , i.e., in the Thomas-Fermi approximation. Density and phase of the order parameter  $\psi_0=e^{i\phi_0}\sqrt{\varrho_0}$  then assume the forms

$$\varrho_0^{\text{TF}} = \frac{\mu_0 - \mathbf{x}^2/2}{g_0}\Theta(\mu_0 - \mathbf{x}^2/2),$$

$$\phi_0^{\text{TF}} = -\mu_0 \int^t dt' \frac{f^2(t')}{b^2(t')}, \quad (9)$$

where the Heaviside step function  $\Theta(\mu_0 - \mathbf{x}^2/2)$  is 1 for  $\mu_0 > \mathbf{x}^2/2$  and 0 elsewhere. In this approximation, the motion of the classical background becomes stationary and the scaled coordinates  $\mathbf{x}$  are comoving with the condensate.

Bogoliubov–de Gennes equation (7) can be tackled by introducing self-adjoint operators

$$\hat{\chi}_+ = e^{-i\phi_0}\hat{\chi} + e^{i\phi_0}\hat{\chi}^\dagger,$$

$$\hat{\chi}_- = \frac{1}{2i}(e^{-i\phi_0}\hat{\chi} - e^{i\phi_0}\hat{\chi}^\dagger), \quad (10)$$

with  $\phi_0=\arg\psi_0$  being the phase of the order parameter. These operators resemble (relative) density and phase fluctuations  $\delta\hat{\varrho}/\varrho_0=\hat{\chi}_+/\sqrt{\varrho_0}$  and  $\delta\hat{\phi}=\hat{\chi}_-/\sqrt{\varrho_0}$  up to the prefactor  $1/\sqrt{\varrho_0}$ . Since this prefactor eventually becomes large near (and beyond) the surface of the condensate, the smallness of  $\delta\hat{\varrho}/\varrho_0$  and  $\delta\hat{\phi}$  cannot be ensured. Therefore, I will stick to  $\hat{\chi}_\pm$  in the following, but still refer to them as density and phase fluctuations. They obey

$$2\left[b^2\frac{\partial}{\partial t} + \mathbf{v}_0\nabla_x + \frac{1}{2}(\nabla_x\mathbf{v}_0)\right]\hat{\chi}_- = -\mathcal{K}_+\hat{\chi}_+,$$

$$\frac{1}{2}\left[b^2\frac{\partial}{\partial t} + \mathbf{v}_0\nabla_x + \frac{1}{2}(\nabla_x\mathbf{v}_0)\right]\hat{\chi}_+ = \mathcal{K}_-\hat{\chi}_-, \quad (11)$$

where the velocity field  $\mathbf{v}_0=\nabla_x\phi_0$  results from the residual background phase beyond Thomas-Fermi approximation (9).

However,  $\mathbf{v}_0$  is small for  $g_0 > 0$  and if  $f^2$  does not change too swiftly because an almost homogeneous phase will develop with only small deviations near the boundary of the condensate. On the other hand, attractive  $g_0 < 0$  invalidate the Thomas-Fermi approximation and generally  $\mathbf{v}_0 \neq 0$  even in the center of the trap. The differential operators on the right-hand sides,

$$\begin{aligned}\mathcal{K}_+ &= -\frac{\nabla_x^2}{2} + \frac{\mathbf{v}_0^2}{2} + f^2 \left( \frac{\mathbf{x}^2}{2} + 3g_0 \varrho_0 + \frac{b^2}{f^2} \dot{\phi}_0 \right), \\ \mathcal{K}_- &= -\frac{\nabla_x^2}{2} + \frac{\mathbf{v}_0^2}{2} + f^2 \left( \frac{\mathbf{x}^2}{2} + g_0 \varrho_0 + \frac{b^2}{f^2} \dot{\phi}_0 \right),\end{aligned}\quad (12)$$

generally do not commute for inhomogeneous condensates  $\nabla_x \varrho_0 \neq 0$ ,

$$[\mathcal{K}_+, \mathcal{K}_-] = fg_0 \{ (\nabla_x^2 \varrho_0) + 2(\nabla_x \varrho_0) \nabla_x \}. \quad (13)$$

### III. EIGENMODE EXPANSION

In order to define the initial state unambiguously, I will assume that the condensate is at rest before  $t=t_{\text{in}}$ . Then  $b=1$ ,  $\dot{b}=0$ ,  $f^2=1$ , and  $\mathbf{v}_0=0$  such that the left-hand sides of Eq. (11) reduce to partial time derivatives and the initial eigenmode equations for  $\hat{\chi}_\pm$  follow

$$\begin{aligned}\frac{\partial^2}{\partial t^2} \hat{\chi}_+ &= -\mathcal{K}_- \mathcal{K}_+ \hat{\chi}_+, \\ \frac{\partial^2}{\partial t^2} \hat{\chi}_- &= -\mathcal{K}_+ \mathcal{K}_- \hat{\chi}_-.\end{aligned}\quad (14)$$

Because  $\mathcal{K}_+ \mathcal{K}_- \neq \mathcal{K}_- \mathcal{K}_+$  for inhomogeneous condensates [cf. Eq. (13)], density and phase fluctuations of each mode must have different space dependences. This leads to the expansions [37] (I will adopt the sum convention throughout this paper for brevity; any indices appearing only on one side of the equation are to be summed)

$$\begin{aligned}\hat{\chi}_+(\mathbf{x}, t) &= h_n^+(\mathbf{x}) \hat{X}_n^+(t), \\ \hat{\chi}_-(\mathbf{x}, t) &= h_n^-(\mathbf{x}) \hat{X}_n^-(t)\end{aligned}\quad (15)$$

of  $\hat{\chi}_\pm$  into different eigenmode bases  $\{h_n^+\}$  and  $\{h_n^-\}$ ; see Appendix A for more details on how to obtain  $h_n^\pm$ . Usually, these two bases are neither orthogonal nor normalized,  $\int h_n^+ h_m^+ \neq \delta_{nm} \neq \int h_n^- h_m^-$ , but instead can be chosen to be dual to each other,

$$\int d^D x h_n^+(\mathbf{x}) h_m^-(\mathbf{x}) = \delta_{nm}. \quad (16)$$

Note that this condition does not fix the norm of  $h_n^\pm$  but still permits the multiplication by an arbitrary factor,  $h_n^+ \rightarrow \Lambda_n h_n^+$  and  $h_n^- \rightarrow (1/\Lambda_n) h_n^-$ . Observables must be unaffected by this ambiguity; see Appendix B.

Insertion of eigenmode expansion (15) into linear field equations (11) yields a set of coupled first-order differential equations,

$$\begin{aligned}b^2 \frac{\partial}{\partial t} \hat{X}_n^- &= -\frac{1}{2} \mathcal{A}_{nm}(t) \hat{X}_m^+ - \mathcal{V}_{nm}(t) \hat{X}_m^-, \\ b^2 \frac{\partial}{\partial t} \hat{X}_n^+ &= 2\mathcal{B}_{nm}(t) \hat{X}_m^- + \mathcal{V}_{mn}(t) \hat{X}_m^+, \end{aligned}\quad (17)$$

with time-dependent coefficients. The symmetric matrices

$$\begin{aligned}\mathcal{A}_{nm}(t) &= \int d^D x h_n^+ \mathcal{K}_+ h_m^+, \\ \mathcal{B}_{nm}(t) &= \int d^D x h_n^- \mathcal{K}_- h_m^-\end{aligned}\quad (18)$$

are initially diagonal,  $\mathcal{A}_{nm}(t_{\text{in}}) = A_n(t_{\text{in}}) \delta_{nm}$  and  $\mathcal{B}_{nm}(t_{\text{in}}) = B_n(t_{\text{in}}) \delta_{nm}$ . At later times, they acquire off-diagonal elements because of the different-time commutators  $[\mathcal{K}_+(t), \mathcal{K}_+(t')] \neq 0$  and  $[\mathcal{K}_-(t), \mathcal{K}_-(t')] \neq 0$  when  $\mathcal{K}_\pm(t) \neq \mathcal{K}_\pm(t')$ . The velocity coupling matrix

$$\mathcal{V}_{nm} = \int d^D x h_n^+ \left[ \mathbf{v}_0 \nabla_x + \frac{1}{2} (\nabla_x \mathbf{v}_0) \right] h_m^- \quad (19)$$

is not symmetric, but vanishes for homogeneous phases  $\phi_0$  of the order parameter, e.g., initially or in Thomas-Fermi approximation (9). For repulsive  $g_0 > 0$  and slow variations in interaction strength  $g(t)$  and trap frequency  $\omega(t)$ , the order-parameter phase is homogeneous except for small ripples near the boundary of the condensate such that the matrix  $\mathcal{V}_{nm}$  is usually negligible.

From evolution equations (17) with the initially diagonal coupling matrices (18), the introduction of bosonic operators  $\hat{a}_n^\dagger$  and  $\hat{a}_n$  creating or annihilating an initial quasiparticle is straightforward,

$$\begin{aligned}\hat{X}_n^-(t) &= F_n^m(t) \hat{a}_m + \bar{F}_n^m(t) \hat{a}_m^\dagger, \\ \hat{X}_n^+(t) &= G_n^m(t) \hat{a}_m + \bar{G}_n^m(t) \hat{a}_m^\dagger.\end{aligned}\quad (20)$$

Here, a bar shall denote complex conjugation, e.g.,  $\bar{F}_n^m = (F_n^m)^*$ . The coefficients obey the initial values

$$F_n^m(t_{\text{in}}) = \sqrt{\frac{A_m}{2\Omega_m}} \delta_{nm}, \quad G_n^m(t_{\text{in}}) = i \sqrt{\frac{\Omega_m}{2A_m}} \delta_{nm}, \quad (21)$$

where the phase has been appropriately chosen and the frequencies  $\Omega_m = \sqrt{A_m B_m}$ . The upper index of the coefficients  $F_n^m$  and  $G_n^m$  labels the mode, while the lower index denotes the component of this particular mode when expanded in a certain basis, e.g., the initial eigenfunctions  $\{h_n^\pm\}$ .

Since coupling matrices (18) become nondiagonal even for slow (adiabatically) variation in the trap frequency  $\omega(t)$  or coupling strength  $g(t)$ , the initial bases  $\{h_n^\pm\}$  cannot represent the eigenmodes at later times. Although  $h_n^\pm$  might be employed in order to calculate the spatial correlation functions (see Appendix B), the use of these functions might be misleading regarding the correlations between different modes. Furthermore, when probing the excitations using,

e.g., the scheme proposed in [39], the proper particles defined at the time of measurement will be detected and not the initial ones.

The particle definition in the time-dependent background is a nontrivial task; see, e.g., [22]. Nonetheless, it is always possible to expand  $\hat{\chi}_{\pm}$ ,

$$\hat{\chi}_{\pm}(\mathbf{x}, t) = h_{n;t_1}^{\pm}(\mathbf{x}) \hat{X}_{n;t_1}^{\pm}(t), \quad (22)$$

into bases  $\{h_{n;t_1}^{\pm}\}$ , which are defined such that the coupling matrices  $\mathcal{A}_{nm;t_1} = \int d^D x h_{n;t_1}^+ \mathcal{K}_+ h_{m;t_1}^+$  and  $\mathcal{B}_{nm;t_1} = \int d^D x h_{n;t_1}^- \mathcal{K}_- h_{m;t_1}^-$  [cf. Eq. (18)] become diagonal at any particular instant  $t_1$ ,

$$\mathcal{A}_{nm;t_1}(t_1) = A_{n;t_1} \delta_{nm}, \quad \mathcal{B}_{nm;t_1}(t_1) = B_{n;t_1} \delta_{nm}. \quad (23)$$

Of course, the velocity term  $\mathcal{V}_{nm;t_1}$  is then generally nondiagonal and the evolution equations of the different modes will not exactly decouple at this particular instant  $t_1$ . But one should bear in mind that measurement occurs usually in an adiabatic region, where the external parameters are only slowly varying functions of time. Then, the background phase is approximately homogeneous and the velocity  $\mathbf{v}_0 \approx 0$ . Hence, quasiparticle creators and annihilators might be introduced that are analogous to Eq. (20),

$$\begin{aligned} \hat{X}_{n;t_1}^-(t_1) &= \sqrt{\frac{A_{n;t_1}}{2\Omega_{n;t_1}}} (\hat{b}_{n;t_1} + \hat{b}_{n;t_1}^\dagger), \\ \hat{X}_{n;t_1}^+(t_1) &= i \sqrt{\frac{\Omega_{n;t_1}}{2A_{n;t_1}}} (\hat{b}_{n;t_1} - \hat{b}_{n;t_1}^\dagger), \end{aligned} \quad (24)$$

where  $\Omega_{n;t_1} = \sqrt{A_{n;t_1} B_{n;t_1}}$ .

#### IV. THOMAS-FERMI APPROXIMATION AND EFFECTIVE SPACE-TIME

In Secs. II and III, I made no approximations except for linearization (5) and the assumption of an isotropic trap. The formalism is, in principle, applicable for arbitrary variations in trap frequency  $\omega(t)$  and interactions  $g(t)$ . To this end, it would be necessary to solve Gross-Pitaevskii equation (6) and linear evolution equations (17) simultaneously. The numerical solution is complicated by the fact that coupling matrices (18) and (19) need to be calculated at each time step. Some of the numerical difficulties can be circumvented by adopting Thomas-Fermi profile (9), where density and phase of the background become time independent (in the coordinates  $\mathbf{x}$ ) and thus require the calculation of the coupling matrices only once. Despite some shortcomings regarding the dynamics of the order parameter, this approximation is usually applicable for repulsive interactions and in the center of the trap, but becomes inaccurate toward the surface of the condensate, where the quantum pressure  $\propto (\nabla \sqrt{\varrho_0})^2$  is relevant.

##### A. Coupled evolution equations

Within Thomas-Fermi approximation (9), the coordinate transformation  $\mathbf{r} \rightarrow \mathbf{x}$  associated with scaling transformation

(2) renders the background density time independent,  $\dot{\varrho}_0^{\text{TF}} = 0$ , while the phase becomes homogeneous,  $\nabla_x \phi_0^{\pm \text{TF}} = 0$  and thus  $\mathcal{V}_{nm}^{\text{TF}} = 0$ . The integrals of coupling matrices (18) simplify considerably and the evolution equations can be cast into the forms

$$\begin{aligned} -2b^2 \frac{\partial}{\partial t} \hat{X}_n^+ &= A_n \hat{X}_n^+ + (f^2 - 1) \mathcal{M}_{nm} \hat{X}_m^+, \\ \frac{1}{2} b^2 \frac{\partial}{\partial t} \hat{X}_n^- &= B_n \hat{X}_n^- + (f^2 - 1) \mathcal{N}_{nm} \hat{X}_m^-, \end{aligned} \quad (25)$$

i.e.,  $\mathcal{A}_{nm}$  and  $\mathcal{B}_{nm}$  can be split into time-independent diagonal parts [cf. Eq. (18)],

$$\begin{aligned} A_n \delta_{nm} &= \int d^D x h_n^+ \left( -\frac{\nabla_x^2}{2} + \frac{\mathbf{x}^2}{2} + 3g_0 \varrho_0^{\text{TF}} - \mu_0 \right) h_m^+, \\ B_n \delta_{nm} &= \int d^D x h_n^- \left( -\frac{\nabla_x^2}{2} + \frac{\mathbf{x}^2}{2} + g_0 \varrho_0^{\text{TF}} - \mu_0 \right) h_m^-, \end{aligned} \quad (26)$$

and constant, symmetric, nondiagonal coupling matrices,

$$\begin{aligned} \mathcal{M}_{nm} &= \int d^D x h_n^+ \left( \frac{\mathbf{x}^2}{2} + 3g_0 \varrho_0^{\text{TF}} - \mu_0 \right) h_m^+, \\ \mathcal{N}_{nm} &= \int d^D x h_n^- \left( \frac{\mathbf{x}^2}{2} + g_0 \varrho_0^{\text{TF}} - \mu_0 \right) h_m^-, \end{aligned} \quad (27)$$

with time-dependent prefactors  $f^2(t) - 1$ . The external variation in trap frequency  $\omega(t)$  and coupling strength  $g(t)$  is solely encoded in the scale factor  $b(t)$  and the scalar function  $f^2(t) = g(t) b^{2-D}$ . Note also that coefficients (27) and thus also evolution equations (25) for the fluctuations are independent of the initial coupling strength  $g_0$ . The addend  $g_0 \varrho_0^{\text{TF}}$  appearing in the parentheses in Eqs. (26) and (27) can be expressed through the chemical potential  $g_0 \varrho_0^{\text{TF}} = (\mu_0 - \mathbf{x}^2/2) \Theta(x_{\text{TF}} - x)$ , where  $x_{\text{TF}} = \sqrt{2\mu_0}$ ; cf. Thomas-Fermi equation (9).

##### B. Low energies and effective space-time metric

The mode functions  $h_n^{\pm}$  of excitations with low energies,  $\Omega_n \ll \mu_0$ , are localized inside the condensate. Hence, changing the bounds of integrals (26) and (27) from infinity to the Thomas-Fermi radius  $x_{\text{TF}}$  will not alter these matrix elements significantly. Bearing further in mind that  $\mathbf{x}^2/2 + g_0 \varrho_0^{\text{TF}} - \mu_0 = 0$  for  $x < x_{\text{TF}}$ , it follows that  $\mathcal{N}_{nm} = 0$  and  $\mathcal{M}_{nm} = \int d^D x h_n^+ 2g_0 \varrho_0 h_m^+$ . Also  $A_n \delta_{nm} = \int d^D x h_n^+ 2g_0 \varrho_0 h_m^+$  because of the restriction to low energies,  $\Omega_n = \sqrt{A_n B_n} \ll \mu_0 \approx g_0 \varrho_0$ , and one gets  $\mathcal{M}_{nm} = A_n \delta_{nm}$ . Hence, the evolution equations of different eigenmodes approximately decouple and I obtain second-order equations of motion for phase [40]

$$\left[ \frac{\partial^2}{\partial \tau^2} - 2 \frac{\partial \ln f}{\partial \tau} \frac{\partial}{\partial \tau} + f^2 A_n B_n \right] \delta \hat{\phi}_n = 0, \quad (28)$$

and density fluctuations,

$$\left[ \frac{\partial^2}{\partial \tau^2} + f^2 A_n B_n \right] \delta \hat{\mathcal{Q}}_n = 0, \quad (29)$$

where I also introduced proper time  $d\tau = dt/b^2$ . Equation (28) is the evolution equation of a mode of a minimally coupled massless scalar field in a Friedman-Lemaître-Robertson-Walker space-time [21,22], provided the scale factor  $a_{\text{FLRW}}$  of the space-time is identified with  $1/f$  (cf. [17]),

$$a_{\text{FLRW}} = \frac{1}{f}. \quad (30)$$

[Note that the prefactor of the damping term is 2 in Eq. (28), while it is  $D$  in a  $D+1$ -dimensional Friedman-Lemaître-Robertson-Walker space-time.]

Analogy (28) is not restricted to the evolution equations in mode expansion but applies in the low-energy limit of field equations (11) as well: within the Thomas-Fermi approximation,  $\mathbf{x}^2/2 + g_0 \varrho_0 - \mu_0 = 0$  for  $x < x_{\text{TF}}$  and  $\mathbf{v}_0 = 0$ , and for low excitations,  $\nabla_x^2 \hat{\chi}_+ \ll 4g_0 \varrho_0 \hat{\chi}_+$ , the phase fluctuations obey a second-order field equation [17],

$$\left[ \frac{\partial^2}{\partial \tau^2} - 2 \frac{\partial \ln f}{\partial \tau} \frac{\partial}{\partial \tau} - f^2 g_0 \varrho_0 \nabla_x^2 \right] \delta \hat{\phi} = 0, \quad (31)$$

which is similar to that of a minimally coupled scalar field in a Friedman-Lemaître-Robertson-Walker space-time.

Having established this kinematical analogy [cf. Eq. (31)], some of the concepts of general relativity can be applied to time-dependent Bose-Einstein condensates. Sonic analogs of horizons [21,41–46] are of particular interest for the study of nonequilibrium effects because they give a rough estimate of whether and when adiabaticity will be violated and the (quantum) fluctuations freeze and get amplified, i.e., (quasi)particle production occurs. An effective particle horizon occurs if a phonon emitted at a time  $\tau_0$  can only travel a finite (comoving) distance, i.e., if the integral

$$\Delta = \int_{\tau_0}^{\tau} c_s(\tau') d\tau' = \sqrt{g_0 \varrho_0} \int_{\tau_0}^{\tau} f(\tau') d\tau' \quad (32)$$

converges to a finite value  $\Delta_{\text{Horizon}}$  for  $\tau(t \rightarrow \infty)$ . Wave packets emitted at time  $\tau_0$  at the origin  $\mathbf{x} = 0$  can reach only points within the horizon,  $x < \Delta_{\text{Horizon}}$ , in a finite time. All other points are concealed by the horizon. [For simplicity, I assumed in Eq. (32) a homogeneous sound velocity  $c_s$ .]

### C. Particle production in static traps

In order to point out the analogy of phase fluctuations to cosmic quantum fields, I formulated evolution equations (28) and (31) using proper time  $\tau$ . On the other hand, experiments are usually performed in the laboratory and thus the variations in trap frequency  $\omega$  and coupling strength  $g$  are prescribed in laboratory time  $t$ . Since  $\tau$  is a complicated function of  $t$ , it is not quite obvious whether or not the quantum fluctuations will experience nonadiabatic evolution for a given modulation of  $\omega(t)$  or  $g(t)$ . In laboratory time  $t$ , Eq. (29) reads

$$\left[ \frac{\partial^2}{\partial t^2} + 2 \frac{\dot{b}(t)}{b(t)} \frac{\partial}{\partial t} + \Omega_n^2(t) \right] \delta \hat{\mathcal{Q}}_n = 0, \quad (33)$$

which is the evolution equation of a damped harmonic oscillator with time-dependent coefficients  $2\dot{b}/b$ , and

$$\Omega_n^2(t) = \left( \omega^2 + \frac{\ddot{b}}{b} \right) A_n B_n = \frac{g(t)}{b^{2+D}(t)} A_n B_n. \quad (34)$$

Initially, when  $\dot{b} = 0$ , the field modes perform free oscillations. Upon the gradual increase in the damping term  $2\dot{b}/b$  with respect to the oscillation frequencies  $\Omega_n$ , the nonadiabatic evolution of the quantum fluctuations slowly sets in, until they finally freeze and get amplified when both terms  $2\dot{b}/b$  and  $\Omega_n$  are of the same order [6,7,12–17].

Let me discuss the two extremal ways a time-dependent scale factor  $b(t)$  can be achieved: first, only the trap frequency might be varied, while the interaction strength  $g = 1$ . With instantaneous frequencies  $\Omega_n \propto b^{-1-D/2}$ , adiabaticity can be violated for any finite change  $\dot{b}/b \neq 0$  if  $b$  becomes sufficiently large. The quantum fluctuations cannot adapt to the changing background anymore; they freeze and get amplified. Second, for static traps,  $\omega = 1$ , where  $g(t)$  is time dependent. Then, the situation is not so clear because  $\Omega_n^2 \propto g/b^{2+D} = 1 + \ddot{b}/b$ ; cf. Eq. (3). Hence, only the rapid acceleration of the scale factor  $|\ddot{b}/b| \gtrsim \mathcal{O}(1)$  will lead to notable changes in the excitation frequencies. On the other hand, adiabaticity could be violated by increasing the magnitude of the damping term,  $2\dot{b}/b$ . Then, however, a continuous acceleration of  $b$  is required because otherwise, if  $\dot{b}/b$  was constant, the system would equilibrate.

As an example for the absence of particle production inside a static trap,  $\omega = 1$ , I will consider an exponential sweep of the coupling coefficient

$$g(t) = \exp\{\gamma t\}, \quad (35)$$

with  $\gamma > 0$ . In this case, Eq. (3) for the scale factor permits an analytic solution,

$$b(t) = \left[ \frac{(2+D)^2}{\gamma^2 + (2+D)^2} \right]^{1/(2+D)} \exp\left\{ \frac{\gamma t}{2+D} \right\}. \quad (36)$$

The coefficients of Eq. (33) become time independent,

$$2 \frac{\dot{b}}{b} = \frac{2\gamma}{2+D}, \quad \Omega_n^2 = \left( 1 + \frac{\gamma^2}{(2+D)^2} \right) A_n B_n, \quad (37)$$

and the density eigenmodes are just damped harmonic oscillators with solutions

$$\delta \hat{\mathcal{Q}}_n = e^{-\gamma t/(2+D)} \delta \hat{\mathcal{Q}}'_n, \quad (38)$$

where  $\delta \hat{\mathcal{Q}}'_n$  is some residual oscillating function with frequency  $\Omega_n$ . Hence, the density-density fluctuations diminish,  $\langle (\hat{X}_{n;t}^+)^2 \rangle \propto e^{-2\gamma t/(2+D)}$ , and, consequently, the phase-phase fluctuations increase. But this is just the adiabatic evolution because  $A_{n;t} \propto e^{2\gamma t/(2+D)}$  and thus  $\langle \hat{X}_{n;t}^+ \rangle_{\text{ad}} = \Omega_{n;t}/(2A_{n;t}) \propto e^{-2\gamma t/(2+D)}$ ; cf. Appendix B. This means that no quasiparti-

cle production occurs for dynamics (35) in the hydrodynamic regime, i.e., for low-energy excitations with order parameter treated in the Thomas-Fermi approximation. These findings can also be inferred from (the absence of) effective particle horizon (32). For particular shape (36) of the scale factor  $b$  follows

$$\Delta \propto \int_0^\infty f(t) \frac{dt}{b^2(t)} = \int_0^\infty dt \rightarrow \infty, \quad (39)$$

because  $f(t) \propto \exp\{2\gamma t/(2+D)\} \propto b^2(t)$ .

Note that the presented solution assumes  $g=e^{\gamma t}$  at all times, especially also for  $t < t_{\text{in}}$ . Hence  $g(t') \neq 1$  at some time  $t'$  when  $b(t')=1$ ; cf. Eqs. (35) and (36). On the other hand,  $b(t_{\text{in}})=1$  and  $g(t_{\text{in}})=1$  for a condensate at rest; see Eq. (3). Since both solutions (static initial state and exponential sweep) cannot be matched at  $t_{\text{in}}$  such that  $g$  and  $b$  are both continuous, the switching on of the exponential sweep would excite breathing oscillations. These oscillations, however, generally affect particle production, e.g., through parametric resonance.

## V. QUASI-ONE-DIMENSIONAL CONDENSATE

The simplest application of the presented formalism consists of a quasi-one-dimensional condensate. In highly anisotropic traps, where the perpendicular trap frequency  $\omega_\perp$  is much larger than the chemical potential, the motion in the perpendicular directions is restricted to the ground state and might be integrated out. An effectively one-dimensional field equation (1) follows, where the interaction strength

$$g_{1D} = \frac{g_{3D}}{2\pi a_\perp^2} = 2a_s \omega_\perp \quad (40)$$

can be varied through Feshbach resonance or by changing  $\omega_\perp$ . However, one should bear in mind that the transversal extent of the condensate  $a_\perp = 1/\sqrt{m\omega_\perp}$  has to be much larger than the  $s$ -wave scattering length  $a_s$  such that the interaction of different atoms can still be described through three-dimensional scattering theory. For simplicity, I will adopt in this section Thomas-Fermi approximation (9) for the order parameter but will permit arbitrary energies for the excitations.

### A. Spectrum

In Fig. 1, the excitation frequencies  $\Omega_n = \sqrt{A_n B_n}$  of the lowest modes are plotted versus the chemical potential  $\mu_0$ . For  $\mu_0=0$ , one has the equidistant spectrum of the harmonic oscillator  $\Omega_n = (n+1/2)\omega_0$ , whereas for high  $\mu_0 \gg \Omega_n$ , the frequencies become almost independent of the chemical potential. In particular, the frequency  $\Omega_1$  of the lowest excitation with odd parity tends for  $\mu_0 \rightarrow \infty$  toward the Thomas-Fermi breathing frequency  $\Omega_{\text{breath}}^{\text{TF}} = \sqrt{3}\omega_0$  obtained from Eq. (3).

The discrepancy between these two frequencies  $\Omega_1$  and  $\Omega_{\text{breath}}^{\text{TF}}$  for finite values of the chemical potential  $\mu_0$  hints at shortcomings of the Thomas-Fermi approximation. In particular, Eq. (3) does not describe the breathing motion of the background properly and care must be taken when employ-

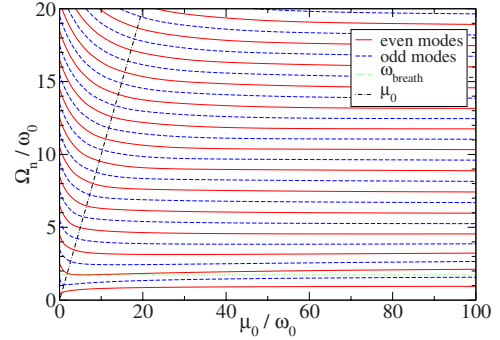


FIG. 1. (Color online) Frequencies of even (solid curves) and odd (dashed curves) excitations versus the chemical potential  $\mu_0$  calculated using 200 harmonic-oscillator basis functions. The frequencies equal the chemical potential at the intersections with the dash-dotted line. The lowest odd mode converges for  $\mu_0 \rightarrow \infty$  to the classical Thomas-Fermi breathing frequency  $\Omega_{\text{breath}}^{\text{TF}} = \sqrt{3}\omega_0$  (dotted curve), which follows from Eq. (3).

ing Eq. (9) for the order parameter. Nonetheless, Eq. (3) still predicts the correct order of magnitude of the characteristic response time of the background,  $1/\omega_{\text{breath}} = \mathcal{O}(1/\omega_{\text{breath}}^{\text{TF}})$ , to variations in trap frequency or interaction strength.

Hence, it is still possible to discuss several cases where the difference between  $\omega_{\text{breath}}$  and  $\omega_{\text{breath}}^{\text{TF}}$  is either small or does not matter: first, if the shape of the condensate varies only slowly, i.e., if  $\dot{b} \ll \omega_{\text{breath}}$  and the condensate can adapt to changes in  $\omega$  and/or  $g$  immediately; second, if no breathing oscillations are excited, e.g., because the condensate expands or contracts [see also Eqs. (35) and (36)]; and third, for very large chemical potentials,  $\mu_0 \rightarrow \infty$ , the Thomas-Fermi approximation becomes exact. The quantum fluctuations are in the hydrodynamic regime and their effective evolution equations (28) decouple. In order to address cases where breathing of the background occurs, it would be necessary to abandon Thomas-Fermi approximation (9) and to solve Gross-Pitaevskii equation (6), which could, e.g., be done by expanding the order parameter  $\psi_0$  into oscillator functions [47].

### B. Exponential sweep in stationary condensate

A stationary condensate, i.e.,  $b=1$ , can be accomplished through simultaneous variations in trap frequency  $\omega$  and interaction strength  $g$  [cf. Eq. (3)],

$$f^2(t) = \frac{\omega^2(t)}{\omega_0^2} = \frac{g(t)}{g_0}, \quad (41)$$

though one should be aware that this only holds within the Thomas-Fermi approximation: the instantaneous chemical potential must at all times be much larger than the trap frequency,

$$\mu_{\text{inst}}(t) = \mu_0 f^2(t) \gg \omega(t) = \omega_0 f(t). \quad (42)$$

If both were of the same order, the kinetic term  $-\nabla_x^2 \psi_0/2$  in Gross-Pitaevskii equation (6) becomes important; no stationary background could be realized even for simultaneous variations in  $\omega$  and  $g$ .

### I. Analytical effective space-time solution

For an exponential sweep

$$f^2(t) = e^{-2\gamma t}, \quad \gamma > 0, \quad (43)$$

the effective second-order Eq. (31) for the phase fluctuations in the hydrodynamic limit,

$$\left[ \frac{\partial^2}{\partial t^2} + 4\gamma \frac{\partial}{\partial t} - f^2 g_0 \mathcal{Q}_0 \nabla_x^2 \right] \delta \hat{\phi} = 0, \quad (44)$$

is that of a massless scalar field in a de Sitter space-time with exponentially growing scale factor  $a_{\text{FLRW}} = 1/f = e^{\gamma t}$  [cf. Eq. (30)]—which is believed to describe the universe during the epoch of cosmic inflation [21,41]. For time dependence (43), integral (32) is finite and an effective sonic horizon occurs; the quantum fluctuations freeze and get amplified.

Instead of solving Eq. (44) for  $\delta \hat{\phi}$ , I will consider the evolution equation of the density fluctuations  $\hat{X}_n^+$ ,

$$\left[ \frac{\partial^2}{\partial t^2} + \gamma^2 e^{-2\gamma(t-t_n)} \right] \hat{X}_n^+ = 0, \quad (45)$$

where  $A_n B_n = \gamma^2 e^{2\gamma t_n}$ . Obviously, all modes undergo the same evolution, just at different times. Equation (45) can be solved analytically in terms of Bessel functions [48]

$$\hat{X}_n^+ = \sqrt{\frac{\pi B_n}{2\gamma}} \{ \hat{a}_n H_0^{(1)}(e^z) + \hat{a}_n^\dagger H_0^{(2)}(e^z) \}, \quad (46)$$

where  $z = -\gamma(t - t_n)$ . The Hankel functions  $H_0^{(1/2)}$  have the proper asymptotics for early times  $t \rightarrow -\infty$  such that the operators  $\hat{a}_n$  annihilate the initial vacuum state. The phase fluctuations  $\hat{X}_n^- = (1/2B_n) \partial \hat{X}_n^+ / \partial t$  read

$$\hat{X}_n^- = \sqrt{\frac{\pi\gamma}{8B_n}} \{ \hat{a}_n e^z H_1^{(1)}(e^z) + \hat{a}_n^\dagger e^z H_1^{(2)}(e^z) \}. \quad (47)$$

From these expressions (46) and (47), I can infer the correlations of each mode. At late times it follows that

$$\begin{aligned} \langle (\hat{X}_n^+)^2 \rangle (t \rightarrow \infty) &= \frac{2\gamma B_n}{\pi} (t - t_n)^2, \\ \langle (\hat{X}_n^-)^2 \rangle (t \rightarrow \infty) &= \frac{\gamma}{2\pi B_n}. \end{aligned} \quad (48)$$

Comparison with the adiabatic values  $\langle (\hat{X}_n^\pm)^2 \rangle_{\text{ad}}$  [cf. Eq. (B8)] yields the quasiparticle number at late times,

$$N_n(t \rightarrow \infty) = \frac{1}{\pi} e^{\gamma(t-t_n)} - 1. \quad (49)$$

The occupation number of all modes grows exponentially though at different times  $t - t_n$ , where the shift  $t_n$  is determined by the excitation frequencies  $\sqrt{A_n B_n} = \gamma e^{\gamma t_n}$ . However, one should bear in mind that Eq. (45) is only valid for a limited time before leaving the hydrodynamic regime.

### 2. Numerical results

In order to go beyond the effective space-time description and thus analytical findings (46)–(49), I will now consider

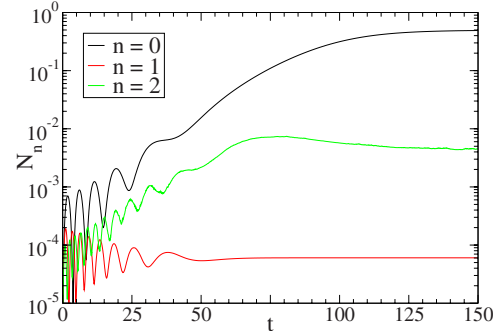


FIG. 2. (Color online) Instantaneous particle number in the lowest three quasiparticle modes for an exponential sweep (43). The initial chemical potential is  $\mu_0 = 50$  and the sweep rate  $\gamma = 0.1$ . At  $t \approx 80$ , the chemical potential equals the trap frequency. For the calculation of the eigenmodes  $h_n^\pm$  and coupling matrices (18), I used the harmonic-oscillator functions  $h_\alpha(x)$  up to  $\alpha = 79$ ; cf. Appendix A. The lowest 20 modes  $\hat{X}_n^\pm$  were then propagated. The numerical accuracy was set to  $10^{-6}$ .

full evolution equations (25). The sweep rate  $\gamma = 0.1$  shall be chosen such that all modes evolve adiabatically at first, i.e.,  $\gamma \ll \Omega_n(t_{\text{in}})$  for all  $n$ . When subsequently reducing trap potential and coupling strength, the excitation frequencies  $\Omega_n(t)$  decrease and nonadiabatic evolution sets in at different times for each mode. The fluctuations freeze and get amplified.

Figure 2 shows the instantaneous particle numbers of the lowest three modes for initial chemical potential  $\mu_0 = 50$  and  $\gamma = 0.1$ . The lowest excitation,  $n = 0$ , which becomes nonadiabatic first, acquires the largest particle number. The next two modes,  $n = 1, 2$ , experience less squeezing, though, remarkably,  $N_2 > N_1$ —an unexpected result, which can be explained by the coupling of different modes: the second even mode,  $n = 2$ , gets populated from the principle excitation,  $n = 0$ , whereas the coupling matrix elements between  $n = 0$  and  $n = 1$  are zero because of different parity.

## VI. SUMMARY

The main objective of this paper was the investigation of quantum fluctuations in time-dependent harmonically trapped Bose-Einstein condensates with repulsive interactions. To this end, the linear fluctuations were expanded into their initial eigenmodes and the field equations were diagonalized. This diagonal form, however, persists only as long as the condensate is at rest; as soon as trap frequency or interaction strength is varied, the coupling of different modes sets in (only part of which can be accounted for by transforming to the instantaneous eigenmodes, though the definition of instantaneous eigenmodes is a nontrivial task).

Two regimes were identified: first, for energies much smaller than the chemical potential, the coupling of different modes is negligible and an effective space-time metric might be introduced for the phase fluctuations. This, however, necessitates a redefinition of the time coordinate such that the required change in trap frequency and/or interaction strength for a certain dynamics of this effective space-time is not obvious. It turned out that the sole variation in the interaction

coefficient  $g(t)$  in a smooth monotonic way is hardly sufficient to render the evolution of the quantum fluctuations nonadiabatic, since the expansion/contraction of the background might compensate for changes in  $g$  such that the sound velocity (in comoving coordinates) remains constant. Breathing oscillations of the background, excited, e.g., by the sudden change in the interaction coefficient  $g$ , might still yield a notable amount of quasiparticles. Second, if the excitation energy is of the same order as the chemical potential, different quasiparticle modes couple.

The amplification and freezing of the fluctuations and also the coupling of different modes were illustrated in an example, where the trap frequency and interaction strength were exponentially ramped down such that the shape of the condensate remains constant. For the considered parameters, a quasiparticle number of 0.5 was obtained in the lowest mode, though higher occupation numbers could be achieved by faster sweep rates or by starting with a higher chemical potential. The inversion of the occupation number in the next two modes could be attributed to the intermode coupling: although the third excitation,  $n=2$ , experiences a much briefer period of nonadiabatic evolution than the second mode,  $n=1$ , only the former couples to the lowest mode,  $n=0$ , and gets populated from it.

### ACKNOWLEDGMENTS

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### APPENDIX A: EIGENMODES

The aim of this appendix is the derivation of the initial eigenfunctions  $h_n^\pm$  of density and phase fluctuations. To this end, let me expand the field operator into any orthonormal basis  $\{h_\alpha(x)\}$  of the underlying Hilbert space  $L^2(\mathbb{R}^D)$  with some operator-valued coefficients  $\hat{\chi}_\alpha^\pm(t)$ ,

$$\hat{\chi}_\pm(\mathbf{x}, t) = h_\alpha(\mathbf{x}) \hat{\chi}_\alpha^\pm(t). \quad (\text{A1})$$

Lowercase Greek indices ( $\alpha, \beta, \dots$ ) shall denote the components in this arbitrary basis, while lowercase Latin indices ( $m, n, \dots$ ) label the initial eigenmodes.

If the condensate is initially at rest, the scale factor  $b=1$  and the phase of the order parameter is homogeneous,  $\nabla_x \phi_0 = 0$ , such that  $\mathbf{v}_0 = 0$  and thus  $\mathcal{V}_{nm} = 0$ ; cf. Eq. (19). Evolution equations (11) simplify considerably and can be expanded into the basis  $\{h_\alpha\}$ . It follows that [cf. Eq. (17)]

$$\begin{aligned} -2 \frac{\partial}{\partial t} \hat{\chi}_\alpha^- &= \mathcal{A}_{\alpha\beta} \hat{\chi}_\beta^+ = \int d^D x h_\alpha \mathcal{K}_+ h_\beta \hat{\chi}_\beta^+, \\ \frac{1}{2} \frac{\partial}{\partial t} \hat{\chi}_\alpha^+ &= \mathcal{B}_{\alpha\beta} \hat{\chi}_\beta^- = \int d^D x h_\alpha \mathcal{K}_- h_\beta \hat{\chi}_\beta^-, \end{aligned} \quad (\text{A2})$$

where  $\mathcal{K}_\pm$  are defined in Eq. (12). The matrices  $\mathcal{A}_{\alpha\beta}$  and  $\mathcal{B}_{\alpha\beta}$  are real and symmetric but do not commute due to  $[\mathcal{K}_+, \mathcal{K}_-] \neq 0$ ; cf. Eq. (13). From Eq. (A2), I obtain second-order evolution equations for phase and density fluctuations,

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \hat{\chi}_\alpha^- &= -\mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} \hat{\chi}_\gamma^-, \\ \frac{\partial^2}{\partial t^2} \hat{\chi}_\alpha^+ &= -\mathcal{B}_{\alpha\beta} \mathcal{A}_{\beta\gamma} \hat{\chi}_\gamma^+, \end{aligned} \quad (\text{A3})$$

which can be diagonalized by a transformation to the eigenvectors.

Because the matrix  $\mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} \neq \mathcal{B}_{\alpha\beta} \mathcal{A}_{\beta\gamma}$  is not symmetric, phase and density fluctuations obey different eigenvalue equations,

$$\begin{aligned} \mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} v_\gamma^n &= \lambda_n v_\alpha^n, \\ \mathcal{B}_{\alpha\beta} \mathcal{A}_{\beta\gamma} \tilde{v}_\gamma^n &= \tilde{\lambda}_n \tilde{v}_\alpha^n, \end{aligned} \quad (\text{A4})$$

where  $n$  denotes the vectors and  $\alpha$  merely counts the components in the particular basis representation (A1). For brevity, I will only discuss the eigenvectors of  $\mathcal{A}\mathcal{B}$  in the following, though the same applies for those of  $\mathcal{B}\mathcal{A}$  as well. The eigenvectors are generally not orthogonal,

$$\sum_\alpha v_\alpha^n v_\alpha^m \neq \delta_{nm}, \quad (\text{A5})$$

but  $\{v_\alpha^n\}$  usually still forms a basis. [Note, however, that the eigenvectors of a nonsymmetric matrix do not always span the entire vector space. But if the  $v^n$  were no basis, evolution equation (A3) could not be diagonalized. This would mean that there existed some fluctuations, which have constant losses to some eigenmodes—rather unphysical in view of the stationary initial state considered here. Therefore, I will not discuss this case any further.] With the  $v^n = h_\alpha v_\alpha^n$  forming a basis of  $L^2(\mathbb{R}^D)$ , there must exist a dual basis  $\{v^{d,n}\}$  in the space of linear functionals (the dual) on  $L^2(\mathbb{R}^D)$ , which obeys

$$\int d^D x v^{d,n} v^m = \sum_\alpha v_\alpha^{d,n} v_\alpha^m = \delta_{nm}. \quad (\text{A6})$$

[Roughly speaking, the elements of  $L^2(\mathbb{R}^D)$  become column vectors in basis expansion (A1), whereas row vectors correspond to the functionals on  $L^2(\mathbb{R}^D)$ , i.e., the elements of the dual. Since the eigenvectors  $v^n$  form a basis, the matrix with the  $v^n$  as columns must be invertible. The rows of the (left) inverse matrix then comprise the elements of the dual basis  $\{v^{d,n}\}$  in the particular basis expansion (A1).]

After the multiplication of the first equation in Eq. (A4) with  $v_\alpha^{d,m}$  from the left and summation over  $\alpha$  follows

$$v_\alpha^{d,m} \mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} v_\gamma^n = v_\alpha^{d,m} v_\alpha^n \lambda_n = \delta_{nm} \lambda_m, \quad (\text{A7})$$

which, because  $\{v^n\}$  is a basis, implies that the  $v^{d,n}$  are the left eigenvectors of  $\mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma}$  with the same eigenvalues  $\lambda_n$ ,

$$v_\alpha^{d,n} \mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} = v_\gamma^{d,n} \lambda_n. \quad (\text{A8})$$

Transposition yields

$$\mathcal{B}_{\gamma\beta} \mathcal{A}_{\beta\alpha} v_\alpha^{d,n} = \lambda_n v_\gamma^{d,n}, \quad (\text{A9})$$

i.e., the eigenvalue equation of  $\mathcal{B}\mathcal{A}$ ; cf. Eq. (A4). Hence,  $\tilde{v}^n \propto v^{d,n}$  and  $\mathcal{A}\mathcal{B}$  and  $\mathcal{B}\mathcal{A}$  (i.e., density and phase fluctua-



tions) must have the same spectrum,  $\{\lambda_n\}=\{\tilde{\lambda}_n\}$ . For simplicity,  $\tilde{v}^n=v^{d,n}$ , which can be achieved by renormalization of  $\tilde{v}^n$ , in the following.

Note that the spectrum of a real nonsymmetric matrix might contain pairs of complex conjugate eigenvalues  $\lambda_n$  and  $\lambda_n^*$ , which can be seen when taking the complex conjugate of Eq. (A4). Complex eigenvalues  $\Im\lambda_n \neq 0$  are associated with exponentially growing solutions, i.e., unstable modes. Since I am interested in the quantization of the stationary initial state, I will not discuss this case but instead assume  $\lambda_n \in \mathbb{R} \forall n$ . (As can be easily verified, the components of the eigenvectors  $v_\alpha^n$  and  $\tilde{v}_\alpha^n$  must be real valued as well.) Nonetheless, complex eigenvalues might still occur in dynamical situations, e.g., during the signature change event proposed in [13] or during (quantum) phase transitions [49] (see also Ref. [50] for an illustrative example).

Initial evolution equations (A3) can be diagonalized by multiplication with the left eigenvectors,

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \tilde{v}_\alpha^n \hat{\chi}_\alpha^- &= -\tilde{v}_\alpha^n \mathcal{A}_{\alpha\beta} \mathcal{B}_{\beta\gamma} \hat{\chi}_\gamma^- = -\lambda_n \tilde{v}_\alpha^n \hat{\chi}_\alpha^-, \\ \frac{\partial^2}{\partial t^2} v_\alpha^n \hat{\chi}_\alpha^+ &= -v_\alpha^n \mathcal{B}_{\alpha\beta} \mathcal{A}_{\beta\gamma} \hat{\chi}_\gamma^+ = -\lambda_n v_\alpha^n \hat{\chi}_\alpha^+, \end{aligned} \quad (\text{A10})$$

which leads to the definition of the density and phase fluctuation eigenmodes  $\hat{X}_n^\pm$  through

$$\hat{X}_n^- = \tilde{v}_\alpha^n \hat{\chi}_\alpha^-, \quad \hat{X}_n^+ = v_\alpha^n \hat{\chi}_\alpha^+, \quad (\text{A11})$$

where the spatial mode functions

$$h_n^- = v_\alpha^n h_\alpha, \quad h_n^+ = \tilde{v}_\alpha^n h_\alpha \quad (\text{A12})$$

follow from comparison of Eq. (A1) with Eq. (15) and duality (A6) of the  $v_\alpha^n$  and  $\tilde{v}_\alpha^n$  implies Eq. (16) for the  $h_n^\pm$ .

In view of Eq. (A11), initial evolution equations (A2) can be transformed to the new basis [cf. Eq. (17)],

$$-2 \frac{\partial}{\partial t} \hat{X}_n^- = A_n \hat{X}_n^-, \quad \frac{1}{2} \frac{\partial}{\partial t} \hat{X}_n^+ = B_n \hat{X}_n^+, \quad (\text{A13})$$

where the transformed matrices

$$\begin{aligned} A_{nm} &= \tilde{v}_\alpha^n \mathcal{A}_{\alpha\beta} \tilde{v}_\beta^m = A_n \delta_{nm}, \\ B_{nm} &= v_\alpha^n \mathcal{B}_{\alpha\beta} v_\beta^m = B_n \delta_{nm} \end{aligned} \quad (\text{A14})$$

have become diagonal. Note that transformation (A11) is not orthogonal because the matrices comprising the eigenvectors  $v_\alpha^n$  and  $\tilde{v}_\alpha^n$  are not orthogonal. Hence, the commutators are *not* preserved, in particular,

$$\begin{aligned} A_{\alpha\beta} \mathcal{B}_{\beta\gamma} &\neq \mathcal{B}_{\alpha\beta} A_{\beta\gamma}, \\ A_{nm} B_{ml} &= B_{nm} A_{ml}, \end{aligned} \quad (\text{A15})$$

where the latter can be inferred from Eq. (A7),

$$\begin{aligned} \lambda_n \delta_{nm} &= \tilde{v}_\alpha^n \mathcal{A}_{\alpha\beta} \tilde{v}_\beta^k v_\gamma^k \mathcal{B}_{\gamma\delta} v_\delta^m = \mathcal{A}_{nk} \mathcal{B}_{km} = (\lambda_n \delta_{nm})^T \\ &= v_\alpha^n \mathcal{B}_{\alpha\beta} v_\beta^k \tilde{v}_\gamma^k \mathcal{A}_{\gamma\delta} \tilde{v}_\delta^m = \mathcal{B}_{nk} \mathcal{A}_{km}. \end{aligned} \quad (\text{A16})$$

Since  $\mathcal{A}_{nm}$  and  $\mathcal{B}_{nm}$  also commute with their product,  $\mathcal{A}_{nk} \mathcal{B}_{km} = \lambda_n \delta_{nm}$ , which is diagonal, they must be diagonal, too.

## APPENDIX B: OBSERVABLES

Duality condition (16) does not fix the norm of the eigenvectors but still permits the multiplication with an arbitrary factor  $\Lambda_n$ ,

$$h_n^+ \rightarrow \Lambda_n h_n^+, \quad h_n^- \rightarrow \frac{1}{\Lambda_n} h_n^-. \quad (\text{B1})$$

This renormalization then leads to a stretching/shrinking of the operator-valued coefficients [cf. Eq. (15)],

$$\hat{X}_n^+ \rightarrow \frac{1}{\Lambda_n} \hat{X}_n^+, \quad \hat{X}_n^- \rightarrow \Lambda_n \hat{X}_n^-. \quad (\text{B2})$$

Since this is merely a basis transformation, the time evolution of the quantum fluctuations  $\hat{X}_n^\pm$  must be unaffected. To see this, recall definitions (18) and (19) of the coupling matrices: they are the matrix elements of the operators  $\mathcal{K}_\pm$  and of  $\mathbf{v}_0 \nabla_x + (\nabla_x \mathbf{v}_0)/2$  with respect to the basis functions  $h_n^\pm$  and thus acquire additional factors as well. As expected, all of these factors cancel such that the time evolution remains unchanged. In particular, the eigenfrequencies are invariant,  $\Omega_n = \sqrt{A_n B_n} \rightarrow \sqrt{\Lambda_n^2 A_n B_n / \Lambda_n^2} = \Omega_n$ .

### 1. Correlation functions

Furthermore the observables should not depend on the particular normalization of the basis functions. The relative density-density correlations at time  $t$  read

$$\begin{aligned} \frac{\langle \delta \hat{\rho}(\mathbf{x}) \delta \hat{\rho}(\mathbf{x}') \rangle}{\langle \hat{\rho}(\mathbf{x}) \rangle \langle \hat{\rho}(\mathbf{x}') \rangle} &= \frac{h_{n;t}^+(\mathbf{x}) h_{m;t}^+(\mathbf{x}')}{\sqrt{\varrho_0(\mathbf{x}) \varrho_0(\mathbf{x}')}} \langle \hat{X}_{n;t}^+ \hat{X}_{m;t}^+ \rangle \\ &= \frac{h_{n;t}^+(\mathbf{x}) h_{m;t}^+(\mathbf{x}')}{\sqrt{\varrho_0(\mathbf{x}) \varrho_0(\mathbf{x}')}} G_{n;t}^k \bar{G}_{m;t}^k, \end{aligned} \quad (\text{B3})$$

where I used the instantaneous eigenmode basis  $h_{n;t}^+$ ; see Eq. (22). Obviously, the factors  $\Lambda_n$  and  $1/\Lambda_n$  contributed by  $h_{n;t}^+$  and  $\hat{X}_{n;t}$  cancel each other and the spatial correlations are independent of the normalization  $\Lambda_n$ . Similarly, the expression for the spatial phase-phase correlations,

$$\begin{aligned} \langle \delta \hat{\phi}(\mathbf{x}) \delta \hat{\phi}(\mathbf{x}') \rangle &= \frac{h_{n;t}^-(\mathbf{x}) h_{m;t}^-(\mathbf{x}')}{\sqrt{\varrho_0(\mathbf{x}) \varrho_0(\mathbf{x}')}} \langle \hat{X}_{n;t}^- \hat{X}_{m;t}^- \rangle \\ &= \frac{h_{n;t}^-(\mathbf{x}) h_{m;t}^-(\mathbf{x}')}{\sqrt{\varrho_0(\mathbf{x}) \varrho_0(\mathbf{x}')}} F_{n;t}^k \bar{F}_{m;t}^k, \end{aligned} \quad (\text{B4})$$

yields the same result regardless of the employed basis.

Hence, the factors  $\Lambda_n$  can be chosen at will. There exist, however, several convenient choices for  $\Lambda_n$ : first, the density modes might be normalized to unity,  $\int d^D x (h_n^+)^2 = 1$ . This is advantageous if left and right eigenvectors are the same, i.e., if  $\mathcal{A}\mathcal{B}$  is symmetric. The drawback is that if  $\mathcal{A}\mathcal{B}$  is not symmetric and therefore  $h_n^+ \neq h_n^-$ , only one of the mode functions,

$h_n^+$ , can be normalized to unity, whereas the norm of the  $h_n^-$  follows from Eq. (16). Second, these factors  $\Lambda_n$  can be fixed by demanding that  $A_n=B_n=\Omega_n$ . In this case, the prefactors of Eq. (20) initially obey  $F_n^m(t_{\text{in}})=\delta_{nm}/\sqrt{2}$  and  $G_n^m(t_{\text{in}})=i\delta_{nm}/\sqrt{2}$  and both quadratures  $\langle(\hat{X}_n^\pm)^2\rangle(t_{\text{in}})=1/2$ . However, one should note that neither of the eigenfunctions  $h_n^\pm$  is generally normalized to unity,  $\int d^Dx(h_n^\pm)^2 \neq 1$ .

Of course, the coefficients  $\langle\hat{X}_{n;t}^+\hat{X}_{m;t}^+\rangle$  and  $\langle\hat{X}_{n;t}^-\hat{X}_{m;t}^-\rangle$  do depend on the particular choice of the basis functions  $h_{n;t}^\pm$  and thus also on the factors  $\Lambda_n$ . For instance, the adiabatic density-density and phase-phase correlations of a particular mode read

$$\begin{aligned}\langle\hat{X}_{n;t}^-\hat{X}_{n;t}^-\rangle_{\text{ad}}(t) &= \frac{A_{n;t}}{2\Omega_{n;t}} \propto \frac{1}{\Lambda_n^2}, \\ \langle\hat{X}_{n;t}^+\hat{X}_{n;t}^+\rangle_{\text{ad}}(t) &= \frac{\Omega_{n;t}}{2A_{n;t}} \propto \Lambda_n^2.\end{aligned}\quad (\text{B5})$$

The dependence on the factor  $\Lambda_n$  becomes important regarding the low excitations in the Thomas-Fermi approximation (cf. Sec. IV): the modes decouple and it is not necessary to introduce a new spatial basis  $h_n^\pm$  at the time of measurement. One has instead  $A_{n;t}=f^2A_{n;t_0}$  and  $B_{n;t}=B_{n;t_0}$  such that  $\langle\hat{X}_{n;t}^+\hat{X}_{n;t}^+\rangle_{\text{ad}} \propto f$ , while  $\langle\hat{X}_{n;t}^-\hat{X}_{n;t}^-\rangle_{\text{ad}} \propto 1/f$ , i.e., the phase and density fluctuations apparently increase or decrease even for adiabatic evolution. In view of this  $\Lambda_n$  ambiguity, the (absolute) density-density or phase-phase correlations provide no adequate measure for the squeezing (i.e., nonadiabaticity) of a single mode. The Fourier transforms of Eqs. (B3) and (B4) on the other hand are independent of  $\Lambda_n$  but do not represent the excitation eigenmodes.

## 2. Bogoliubov transformation and particle production

As will be shown in the following, the (instantaneous) quasiparticle number measures the relative deviation of density and phase fluctuations from their adiabatic values. In view of the different expansions (15) and (22) of density and phase fluctuations into their initial and adiabatic eigenfunctions, the corresponding creation and annihilation operators  $\hat{a}_n^\dagger, \hat{a}_n$  and  $\hat{b}_{n;t}^\dagger, \hat{b}_{n;t}$ , respectively, can be transformed by virtue of a Bogoliubov transformation. For the annihilators  $\hat{b}_{n;t}$  follows in particular

$$\hat{b}_{n;t} = \sqrt{\frac{\Omega_{n;t}}{2A_{n;t}}}\hat{X}_{n;t}^- - i\sqrt{\frac{A_{n;t}}{2\Omega_{n;t}}}\hat{X}_{n;t}^+ = \alpha_{nm}(t)\hat{a}_m + \beta_{nm}(t)\hat{a}_m^\dagger, \quad (\text{B6})$$

with the Bogoliubov coefficients  $\alpha_{nm}$  and  $\beta_{nm}$ . The first equality follows from inversion of Eq. (24) and the second

equality can be inferred after transforming Eq. (20) to the new basis  $h_{n;t}^\pm$ . Since  $\beta_{nm} \neq 0$  for nonadiabatic evolution, the quasiparticle number operator  $\hat{N}_n(t)=\hat{b}_{n;t}^\dagger\hat{b}_{n;t}$  will have a non-zero expectation value as well,

$$\begin{aligned}N_n(t) &= \langle\hat{b}_{n;t}^\dagger\hat{b}_{n;t}\rangle \\ &= \sum_m |\beta_{nm}(t)|^2 \\ &= \frac{\Omega_{n;t}}{2A_{n;t}}\langle(\hat{X}_{n;t}^-)^2\rangle + \frac{A_{n;t}}{2\Omega_{n;t}}\langle(\hat{X}_{n;t}^+)^2\rangle + \frac{i}{2}\langle[\hat{X}_{n;t}^+, \hat{X}_{n;t}^-]\rangle,\end{aligned}\quad (\text{B7})$$

where the commutator  $[\hat{X}_{n;t}^+, \hat{X}_{n;t}^-]=-i$ . Noting that the prefactors  $\Omega_{n;t}/2A_{n;t}$  and  $A_{n;t}/2\Omega_{n;t}$  are just the adiabatic density-density and phase-phase correlations [see Eq. (B5)], the particle number can be rewritten as

$$N_n(t) = \frac{\langle(\hat{X}_{n;t}^-)^2\rangle - \frac{A_{n;t}}{2\Omega_{n;t}}\langle(\hat{X}_{n;t}^+)^2\rangle - \frac{\Omega_{n;t}}{2A_{n;t}}}{2A_{n;t}/\Omega_{n;t}} + \frac{\langle(\hat{X}_{n;t}^+)^2\rangle - \frac{\Omega_{n;t}}{2A_{n;t}}\langle(\hat{X}_{n;t}^-)^2\rangle - \frac{A_{n;t}}{2\Omega_{n;t}}}{2\Omega_{n;t}/A_{n;t}}, \quad (\text{B8})$$

i.e., the instantaneous particle number gives just the relative deviation of the density and phase correlations from their adiabatic values. Note that expression (B8) does not contain the correlations between different modes. To this end, it would be necessary to evaluate  $\langle\hat{N}_n\hat{N}_m\rangle$ , which is fourth order in the  $\hat{b}_{n;t}$ . This observable can be reduced to expectation values quadratic in the  $\hat{b}_{n;t}$  by virtue of Wick's theorem; see, e.g., [51].

## 3. Scaling

Another interesting aspect is with regard to the scaling of correlation functions (B3) and (B4) with the interaction strength: within the Thomas-Fermi approximation [see Sec. IV], evolution equations (25) are independent of  $g_0$ . All properties of the linear excitations are determined by the chemical potential  $\mu_0$  and the variations  $g(t)/g_0$  and  $\omega(t)/\omega_0$ , in particular, the expectation values  $\langle\hat{X}_{n;t}^\pm\hat{X}_{m;t}^\pm\rangle$ . Only the normalization factor

$$\sqrt{\varrho_0(\mathbf{x})\varrho_0(\mathbf{x}')} = (1/g_0)\sqrt{[\mu - V(\mathbf{x})][\mu_0 - V(\mathbf{x}')]}$$

in Eqs. (B3) and (B4) depends on  $g_0$ . Hence, the relative density-density and phase-phase correlations in the Thomas-Fermi approximation are both proportional to  $g_0$  for fixed  $\mu_0$ .

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