Entanglement dynamics of multimode Gaussian states coupled to the same reservoir

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We investigate the entanglement dynamics of a generic multimode Gaussian state interacting with a common environmental noise by means of the characteristic function method and the defined operators. We find that such operators obey the rather simple commutation relations. This enables us to exactly solve the Fokker-Planck equation and to establish the general form of the input and output covariance matrices of this state. As an application, we consider the separability and entanglement of three-mode squeezed Gaussian state in the common thermal and squeezed thermal bath.

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I. INTRODUCTION

In the theory of open quantum systems, a conventional way to deal with the decoherence and dissipation processes is that the system-environment interaction can be described by the operator master equation of the density operator of the system under consideration, through which one may directly derive the equations of motion for the expectation values of system operators. Unfortunately, in most cases, the exact solution to this equation is not easily obtained, especially for multipartite and higher-dimensional Gaussian continuous-variable (CV) states. To proceed in such situations, using the operator correspondences [1,2] the operator master equation can be mapped into an equivalent c-number Fokker-Planck equation, which can indeed be expanded in any one of the three phase-space representations: P, Q, and W representations. It has been shown that the main problem of the Q function is that not all positive normalizable Q functions correspond to positive definite normalizable density operators [3], while in the P representation, no nonsingular P function exists for a squeezed state [2]. Therefore, the Fokker-Planck equation for the Wigner function has been more frequently used [4–9]. For instance, within this method, Prauzner-Bechcicki [6] made a precise study of the Markovian time evolution of entanglement of a two-mode continuous-variable state in a common reservoir. Daffer *et al.* [7] investigated the nonlocality, separability, and purity of the diffused two-mode squeezed state; Serafini *et al.* [8] studied the evolution of purity, entanglement, and total correlations of general two-mode continuous-variable Gaussian states in arbitrary uncorrelated Gaussian environments; McAneney *et al.* [9] studied the decoherence mechanism of an entangled two-mode system where a thermal environment is modeled by an array of beam splitters. However, in these works, the equation was only solved subjected to some special initial conditions, e.g., two-mode squeezed vacuum state. Very recently, Serafini *et al.* [10] examined the decoherence of generic states of continuous-variable systems by using optical master equations and proposed a formal solution for Gaussian states with arbitrary initial covariance matrix, but it dealt with a general Gaussian uncorrelated environment. The relevant investigation for a general multimode Gaussian state in a same environment has not been made yet. In addition, one finds that the Fokker-Planck equation for Wigner function is a second-order differential one from which it is extremely difficult to reveal the special structure hidden in the system of interest. More recently, Lu *et al.* [11] proposed an elegant method to solve the master equation with superoperator generators of su(1,1) and su(2) Lie algebras. But their approach cannot easily be extended to a case in which multimode bosonic fields are embedded in a common noisy environment. Nevertheless, our recent research shows that the problem of quantum entanglement dynamics for a generic two-mode Gaussian state in a common squeezed bath has been solved by using the Fokker-Planck equation for the characteristic function rather than the Wigner function [12]. A main advantage of our choice is that by presenting the operators, we can easily establish the rather simple commutation relations between them, and then obtain the explicit solution to this equation for a generic two-mode Gaussian state. Following this idea, the aim of the present paper is to further investigate quantum entanglement dynamics of a generic multimode Gaussian state embedded in a common bath and work out the input-output relation of the covariance matrix.

Another active issue in quantum information theory is the quantification and characterization of quantum entanglement. Up to now, the qualitative characterization of the entanglement for arbitrary two-mode Gaussian states has been completely established by determining the necessary and sufficient criterion for their separability [13]. A computable measure of entanglement for such states is available as well, being provided by negativity or logarithmic negativity [14] and, in the symmetric instance, by the entanglement of formation [15]. How to quantify the amount of entanglement contained in more general multimode or multipartite Gaussian states, in contrast, to our best knowledge is a rather complicated and still unsolved problem, which should be taken into account in evaluating the influence of a noisy environment on the performance of quantum information processing. Thanks to the recent work of Adesso *et al.* [16,17], now one can exploit the residual Gaussian contangle to quantify the entanglement contained in some symmetric multimode Gaussian states, e.g., CV Greenberger-Horne-Zeilinger

(GHZ) and W states, noisy GHZ and W states, and tripartite entanglement (T) states [16]. In this paper, we will use this measure to investigate the entanglement dynamics of a special class of three-mode Gaussian states embedded in a common environment.

The paper is organized as follows. In Sec. II, we provide the master equation to describe the system of n-mode fields embedded in a common environment. By presenting the operators, we establish a compact input-output formula of the covariance matrix for such a Gaussian state in a noisy environment. In Sec. III, we investigate the entanglement dynamics of three-mode squeezed state in the common-reservoir model by means of the residual Gaussian contangle. Finally, a brief summary is given in Sec. IV.

II. CHARACTERISTIC FUNCTION APPROACH TO MASTER EQUATION FOR MULTIMODE GAUSSIAN STATES

Let us consider a quantum register composed of n noninteracting single-mode bosonic fields, which may be referred to as harmonic oscillators, radiation modes, motional states of trapped ions, cavity fields, and related systems. All the fields are assumed to be coupled to the same noisy bath described by an ensemble of harmonic oscillators with annihilation (and creation) operators $b_{\mathbf{k}}$ (and $b_{\mathbf{k}}^{\dagger}$) and frequencies $\omega_k = ck$. For such a n-mode CV system, its Hilbert space is a tensor product of all the subsystems' Hilbert spaces, $\mathcal{H} = \otimes_{i=1}^{n}\mathcal{H}_i$. We denote by a_i the annihilation operator acting on the space $\mathcal{H}_i$ and by $\hat{x}_i = (a_i^{\dagger} + a_i)/\sqrt{2}$ and $\hat{p}_i = i(a_i^{\dagger} - a_i)/\sqrt{2}$ the quadrature phase operators related to the mode i of the field. These operators obey the standard commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ and $[a_i, a_j^{\dagger}] = i\delta_{ij}$. For the sake of simplicity, we assume that all the fields have the same frequency, i.e., $\omega_i = \omega_j = \omega$ and should act homogeneously with the environment. Under the rotating-wave approximation, the Hamiltonian describing them embedded in the same radiation bath is given by ($\hbar = 1$),

$$H = \sum_{i=1}^{n} \omega a_i^{\dagger} a_i + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{i=1}^{n} (\Gamma a_i^{\dagger} + \Gamma^{\dagger} a_i), \tag{1}$$

where the heat bath operators $\Gamma = \Sigma_{\mathbf{k}} g_{\mathbf{k}} b_{\mathbf{k}}$ and $\Gamma^{\dagger} = \Sigma_{\mathbf{k}} g_{\mathbf{k}}^{*} b_{\mathbf{k}}^{\dagger}$ with $g_{\mathbf{k}}$ being the system-environment coupling.

Without loss of generality, the reservoir is assumed to be a squeezed bath with the following correlations [18]:

$$\langle \Gamma^{\dagger}(t)\Gamma(t')\rangle = \gamma N \delta(t - t'), \tag{2a}$$

$$\langle \Gamma(t)\Gamma^{\dagger}(t')\rangle = \gamma (N+1) \delta(t - t'), \tag{2b}$$

$$\langle \Gamma(t)\Gamma(t')\rangle = \gamma M \delta(t - t'), \tag{2c}$$

$$\langle \Gamma^{\dagger}(t)\Gamma^{\dagger}(t')\rangle = \gamma M^{*} \delta(t - t'), \tag{2d}$$

where γ is the decay constant of each mode, N represents the mean photon number of the squeezed reservoir, and M is a parameter related to the phase correlations of the squeezed reservoir. The Heisenberg uncertainty relation imposes the constraint $|M|^2 \leq N(N+1)$.

In the interaction picture and Markovian approximation, we can obtain the master equation for the reduced density matrix of the n-mode field as

$$\begin{aligned}\frac{\partial}{\partial t}\rho = \frac{\gamma}{2}\sum_{i,j=1}^{n} \{&(N+1)(2a_i\rho a_j^{\dagger} - a_i^{\dagger}a_j\rho - \rho a_i^{\dagger}a_j) + N(2a_i^{\dagger}\rho a_j \\ &- a_i a_j^{\dagger}\rho - \rho a_i a_j^{\dagger}) + M(2a_i^{\dagger}\rho a_j^{\dagger} - a_i^{\dagger}a_j^{\dagger}\rho - \rho a_i^{\dagger}a_j^{\dagger}) \\ &+ M^{*}(2a_i\rho a_j - a_i a_j\rho - \rho a_i a_j)\}.\end{aligned} \tag{3}$$

It should be pointed out that the $i=j$ terms in Eq. (3) describe the individual dissipations in each mode due to the environment, while the $i \neq j$ terms denote the couplings between the modes induced by the common bath.

The n-mode Weyl characteristic function is defined as

$$\chi(\{\beta\}_n) \coloneqq \chi(\beta_1, \beta_2, \cdots, \beta_n) = \mathrm{Tr}[\rho D(\beta_1, \beta_2, \cdots, \beta_n)], \tag{4}$$

where $D(\beta_1, \beta_2, \cdots, \beta_n) = \otimes_{k=1}^{n} D_k(\beta_k)$ is the n-mode displacement operator and where

$$D_k(\beta_k) = \exp(\beta_k a_k^{\dagger} - \beta_k^{*} a_k) \tag{5}$$

is the single-mode displacement operator.

Using the standard operator correspondence we find that the characteristic function obeys

$$\begin{aligned}\frac{\partial}{\partial t}\chi(\{\beta\}_n, t) = -\frac{\gamma}{2}\sum_{i,j=1}^{n}\Bigg[&(2N+1)\beta_i\beta_j^{*} + M\beta_i^{*}\beta_j^{*} + M\beta_i\beta_j \\ &+ \beta_i\frac{\partial}{\partial\beta_j} + \beta_i^{*}\frac{\partial}{\partial\beta_j^{*}}\Bigg]\chi(\{\beta\}_n, 0).\end{aligned} \tag{6}$$

To solve this equation, we need to introduce the operators as follows:

$$O^n = \sum_{i=1}^{n}\left(\beta_i\frac{\partial}{\partial\beta_i} + \beta_i^{*}\frac{\partial}{\partial\beta_i^{*}}\right), \quad P^n = \sum_{i,j=1,i\neq j}^{n}\left(\beta_i\frac{\partial}{\partial\beta_j} + \beta_i^{*}\frac{\partial}{\partial\beta_j^{*}}\right),$$

$$K^n = \sum_{i,j=1}^{n}\beta_i\beta_j^{*}, \quad D^n = \sum_{i,j=1}^{n}(\beta_i\beta_j + \beta_i^{*}\beta_j^{*}), \tag{7}$$

which obey the commutation relations as follows:

$$[O^n + P^n, K^n] = 2nK^n, \quad [O^n + P^n, D^n] = 2nD^n, \quad [O^n, P^n] = 0, \tag{8}$$

so that we get the corresponding formal solution

$$\begin{aligned}\chi(\{\beta\}_n, t) = \exp&\left\{-\frac{1}{2n}[(2N+1)K^n + MD^n](1 - e^{-n\gamma t})\right\} \\ &\times\exp\left(-\frac{\gamma t}{2}O^n\right)\exp\left(-\frac{\gamma t}{2}P^n\right)\chi(\{\beta\}_n, 0),\end{aligned} \tag{9}$$

where we have assumed that $M \geq 0$. This assumption is reasonable for the real environmental noise, e.g., the squeezed vacuum bath. We see from Eq. (9) that, to obtain the characteristic function at a time t, we need to further disentangle the express $\exp(-\frac{\gamma t}{2}P^n)$; for this purpose we introduce the $2n \times 2n$ matrix of the form

$$\boldsymbol{B}=\begin{pmatrix} 1 & 0 & -\frac{1}{n-1} & 0 & -\frac{1}{n-1} & \cdots & 0 & -\frac{1}{n-1} & 0 & -\frac{1}{n-1} \\ 0 & 1 & 0 & -\frac{1}{n-1} & 0 & \cdots & -\frac{1}{n-1} & 0 & -\frac{1}{n-1} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{n-2} & \cdots & 0 & -\frac{1}{n-2} & 0 & -\frac{1}{n-2} \\ 0 & 0 & 0 & 1 & 0 & \cdots & -\frac{1}{n-2} & 0 & -\frac{1}{n-2} & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & -\frac{1}{n-3} & 0 & -\frac{1}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{10}$$

so that the new and old variables are related according to

$$\boldsymbol{\xi}=\boldsymbol{B}\boldsymbol{\beta}, \tag{11}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_1^*,\xi_2,\xi_2^*,\cdots,\xi_n,\xi_n^*)^T$ and $\boldsymbol{\beta}=(\beta_1,\beta_1^*,\beta_2,\beta_2^*,\cdots,\beta_n,\beta_n^*)^T$.

After performing such a transformation, we can obtain

$$\exp\left(-\frac{\gamma t}{2}P^n\right)=\exp\left\{\frac{\gamma t}{2}\left[\sum_{i=1}^{n-1}\left(\xi_i\frac{\partial}{\partial\xi_i}+\xi_i^*\frac{\partial}{\partial\xi_i^*}\right)-(n-1)\times\left(\xi_n\frac{\partial}{\partial\xi_n}+\xi_n^*\frac{\partial}{\partial\xi_n^*}\right)\right]\right\}. \tag{12}$$

It is seen that the new variables are uncoupled to each other; moreover we can calculate exactly the evolved covariance matrix of the n-mode Gaussian state in a common bath.

Assuming that all the modes are initially prepared in a pure n-mode Gaussian state with the characteristic function represented by [19]

$$\chi(\{\beta\}_n,0)=\exp\left[-\frac{1}{2}\boldsymbol{\beta}^T\mathbf{V}_n(0)\boldsymbol{\beta}\right], \tag{13}$$

where $\boldsymbol{V}_n(0)$ is the initial covariance matrix satisfying the uncertainty principle $\boldsymbol{V}_n(0)+i\boldsymbol{\Omega}/2\geq 0$. Here $\boldsymbol{\Omega}$ is the standard symplectic form

$$\boldsymbol{\Omega}:=\oplus_{i=1}^n\boldsymbol{J},\boldsymbol{J}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{14}$$

After a straightforward but lengthy calculation, we can obtain the covariance matrix of n-mode fields subjected to the initial condition (13) as

$$\boldsymbol{V}_n(t)=\boldsymbol{G}\boldsymbol{B}^T\boldsymbol{W}(\boldsymbol{B}^{-1})^T\boldsymbol{V}_n(0)\boldsymbol{B}^{-1}\boldsymbol{W}\boldsymbol{B}\boldsymbol{G}-\boldsymbol{G}^n\boldsymbol{R}\boldsymbol{G}^n+\boldsymbol{R}, \tag{15}$$

where the superscript T denotes the transposition, $\boldsymbol{G}=e^{-(1/2)\gamma t}\boldsymbol{I}_{2n\times 2n}$, $\boldsymbol{W}=e^{(1/2)\gamma t}\boldsymbol{I}_{(2n-2)\times(2n-2)}\oplus e^{-(1/2)(n-1)\gamma t}\boldsymbol{I}_{2\times 2}$, and

$$\boldsymbol{R}=\frac{1}{n}\begin{pmatrix} 2N+1 & 2M & \cdots & 2N+1 & 2M \\ 2M & 2N+1 & \cdots & 2M & 2N+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2N+1 & 2M & \cdots & 2N+1 & 2M \\ 2M & 2N+1 & \cdots & 2M & 2N+1 \end{pmatrix}. \tag{16}$$

That is, we can obtain the relationship between the input and output covariance matrices of a generic multimode Gaussian state embedded in a common noisy environment, which is the main formula of the present paper. Therefore, once the initial characteristic function or covariance matrix of n-mode Gaussian state is known, one can easily obtain its time-dependent characteristic function or covariance matrix using Eq. (15) and further study its properties of decoherence and quantum entanglement under the influence of a common reservoir.

III. SEPARABILITY AND ENTANGLEMENT OF THREE-MODE SQUEEZED STATE IN A SAME ENVIRONMENT

To illustrate the advantage of the proposed method, we will investigate how the environmental noise influences on the entanglement dynamic behavior in a tripartite system. Since the separability and entanglement measures of a generic multimode and multipartite Gaussian state, especially mixed Gaussian states, are still lacking, we shall mainly focus our attention on a simple but interesting case of the modes being initially in a pure, symmetric three-mode Gaussian state.

Very recently, the n-mode nondegenerate optical parametric amplifier (NOPA) has been introduced [20] and its nonlocality of such a n-mode squeezed state by violation of the Zukowski-Brukner n-qubit Bell inequality [21] has been investigated. It has shown that the n-NOPA field modes are equivalent to an entangled state of n oscillators. When $n=3$, the corresponding covariance matrix can be written as

$$\mathbf{V}_3(0) = \frac{1}{2}\begin{pmatrix} n_1 & 2m_1 & 0 & m_{c12} & 0 & m_{c13} \\ 2m_1 & n_1 & m_{c12} & 0 & m_{c13} & 0 \\ 0 & m_{c12} & n_2 & 2m_2 & 0 & m_{c23} \\ m_{c12} & 0 & 2m_2 & n_2 & m_{c23} & 0 \\ 0 & m_{c13} & 0 & m_{c23} & n_3 & 2m_3 \\ m_{c13} & 0 & m_{c23} & 0 & 2m_3 & n_3 \end{pmatrix}, \tag{17}$$

where $n_i=n=\cosh(2r)$, $m_i=m=\frac{1}{6}\sinh(2r)$, $m_{cij}=m_c=-\frac{2}{3}\sin(2r)$ with r being the squeeze parameter.

On the other hand, Adesso *et al.* [16,17] have proven that the *bona fide* quantification of tripartite entanglement is provided by the residual Gaussian contangle (G_τ^{res}):

$$G_\tau^{\text{res}} := G_\tau^{i|j|k} := \min_{(i,j,k)}[G_\tau^{i|(jk)} - G_\tau^{i|j} - G_\tau^{i|k}], \tag{18}$$

where the symbol (i,j,k) denotes all the permutations of the three-mode indexes, which is nonzero only in the fully inseparable region. Note that the *minimum* in the above Eq. (18) is physically meaningful and mathematically necessary.

After some algebra, one finds the G_τ^{res} for this state:

$$G_\tau^{\text{res}} = \text{arcsinh}^2\left[\frac{2}{3}\sqrt{2(\cosh^2 2r - 1)}\right]. \tag{19}$$

It is clear to see that this state is a genuine tripartite entangled state only for nonvanishing squeezing parameter, i.e., $r>0$.

We have obtained the evolving covariance matrix via Eqs. (15)–(17),

$$\mathbf{V}_3(t) = \frac{1}{6}\begin{pmatrix} e & f & (\tilde{N}-n)\tau & g & (\tilde{N}-n)\tau & g \\ f & e & g & (\tilde{N}-n)\tau & g & (\tilde{N}-n)\tau \\ (\tilde{N}-n)\tau & g & e & f & (\tilde{N}-n)\tau & g \\ g & (\tilde{N}-n)\tau & f & e & g & (\tilde{N}-n)\tau \\ (\tilde{N}-n)\tau & g & (\tilde{N}-n)\tau & g & e & f \\ g & (\tilde{N}-n)\tau & g & (\tilde{N}-n)\tau & f & e \end{pmatrix}, \tag{20}$$

where $e=(\tilde{N}-n)\tau+3n$, $f=2(M-m-m_c)\tau+6m$, and $g=2(M-m-m_c)\tau+3m_c$. Hereinafter such a state is called the noisy state.

In the following, we shall discuss the separability and entanglement properties of the noisy state (20). It has been shown that three-mode Gaussian states can be divided into five distinct separability classes according to the positive partial transpose (PPT) criterion [22]. But in our case, if we introduce the following effective squeezing degree:

$$s = \frac{1}{2}\sqrt{\frac{3(3a^2+\sqrt{9a^4-10a^2b^2+b^4})}{b^2} - 5}, \tag{21}$$

with b being the global purity and a being local purity of one-mode reduced state, then the evolving state can be classified in three different categories depending on the defining parameters s and b [23],

$$s > u = \frac{\sqrt{9b^4-2b^2+9+3(b^2-1)\sqrt{9b^4+14b^2+9}}}{4b}$$
$$\Rightarrow \text{class } 1, \tag{22}$$

$$b < s < u \Rightarrow \text{class } 2, \tag{23}$$

$$s \leq b \Rightarrow \text{class } 3. \tag{24}$$

These inequalities show that noisy states which fulfill Eq. (22) are fully inseparable (i.e., genuine tripartite entanglement), while in the range defined by Eq. (23), noisy states are three-mode biseparable, that is, they exhibit tripartite *bound entanglement*. Finally, noisy states in class 3 are fully separable ones, containing no entanglement at all. Nevertheless, due to the symmetry and the homogeneous interaction of three modes with the environment, no two-mode or three-mode biseparable three-mode Gaussian states in our investigation are allowed during the evolution of three field modes in a common bath, so the evolving state is either fully inseparable or fully separable. For the convenience of description, we introduce the quality $Y=u-s$ or $b-s$ to judge whether this state is separable or not. From Eqs. (22) and (24), we see that if $Y<0$, the state is fully inseparable; otherwise, the state is fully separable.

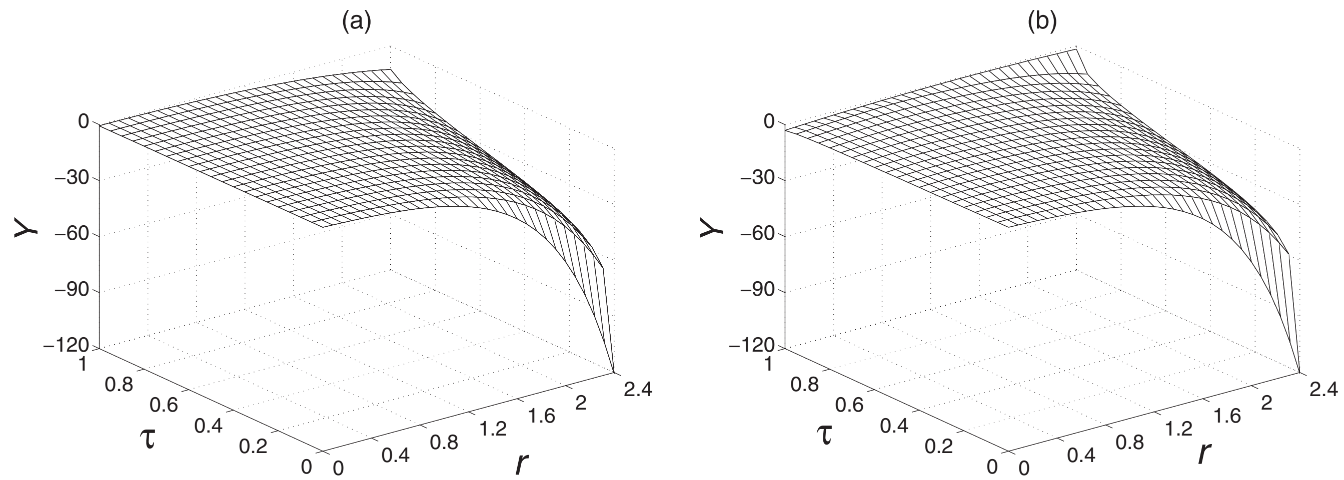

FIG. 1. The quality Y is plotted as a function of $\tau=1-e^{-3\gamma t}$ and r in two different noise models, (a) common thermal bath with $N=4$, and (b) common squeezed bath with $N=4$ and $M=2\sqrt{5}$.

Figure 1 shows the quality Y as a function of the dimensionless time τ and the squeezing parameter r. Figure 1(a) is plotted for a case in which three modes are coupled to a common thermal environment. In this case, we see that the Y is always less than zero once the three mode fields begin to interact with the environment, indicating that the evolving state behaves still its inseparable properties when immersed in a common thermal-equilibrium environment. Figure 1(b) shows the quality Y for the case where three modes are embedded in a common squeezed vacuum bath. We find that for small and very large values of the initial squeezing parameters, Eq. (22) is always satisfied during the evolution process, while for the case of moderate squeezing, the steady-state value of the Y is close to zero, implying that the three modes evolve into a product of three independent thermal states from the initially genuine tripartite entanglement in this case. It is also interesting to see from Fig. 1 that the three modes can be entangled via the interaction with the common environmental noise even though they are initially in the nonsqueezed state.

When Eq. (22) is satisfied, a tripartite entanglement of such a noisy state is given by [17]

$$G_{\tau}^{\mathrm{res}}=\frac{1}{4}\ln\left\{\frac{b^2[4s^4+s^2+4-2(s^2-1)\sqrt{4s^4+10s^2+4}]}{9s^2}\right\}-2\left[\max\left\{0,-\ln\left(\frac{b\sqrt{s^2+2}}{\sqrt{3}s}\right)\right\}\right]^2, \tag{25}$$

and for other case, $G_{\tau}^{\mathrm{res}}=0$.

The residual Gaussian contangle, G_{τ}^{res}, of the noisy state is plotted with respect to the dimensionless interaction time τ and the squeezing parameter r in Fig. 2 choosing the environmental parameters (a) $N=4$, $M=0$ and (b) $N=4$, $M=2\sqrt{5}$. From this figure, we can observe a general trend. First, we see that for weak squeezing, the three modes *do not lose* their initial entanglement, and, on the contrary, the amount of their entanglement continues to increase by the interaction with the surrounding environment. This implies that these squeezed states with small squeezing parameters

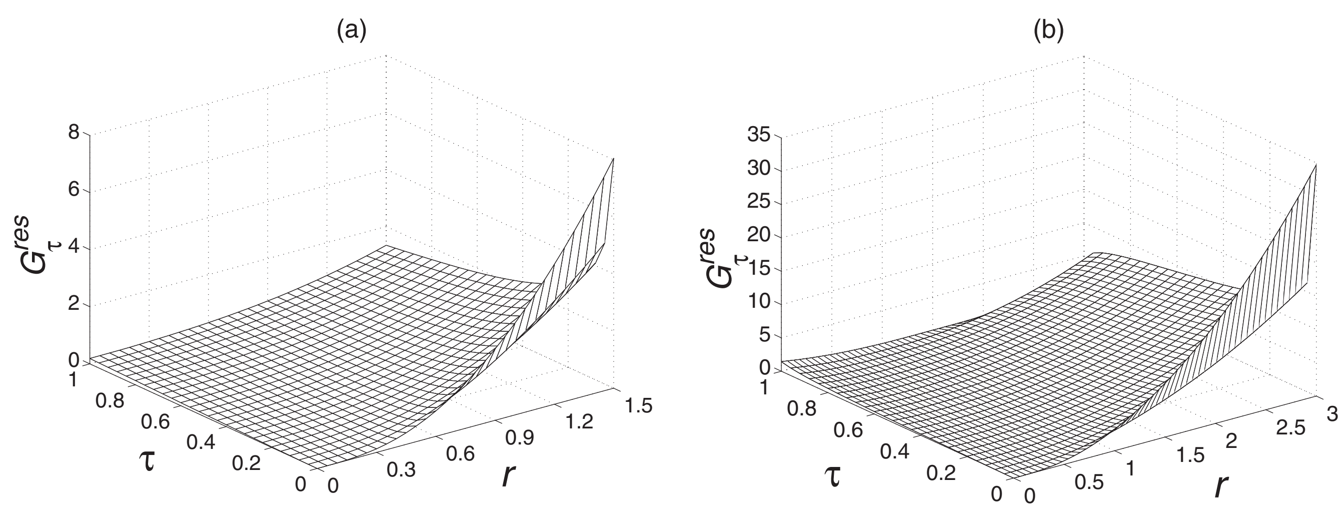

FIG. 2. The residual Gaussian contangle G_{τ}^{res} of the three-mode squeezed state embedded in (a) common thermal bath with $N=4$ and (b) common squeezed bath with $N=4$ and $M=2\sqrt{5}$ as a function of the dimensionless time τ and the squeezing parameter r. $\tau=1-e^{-3\gamma t}$.

are rather robust against the type of noise. Second, for larger squeezing values, the initial entanglement will be degraded rapidly by the external environment, as we expected. Nevertheless, for the asymptotical behavior of the entanglement, we see that its entanglement attains a nonzero value at $t \to \infty$, which is in quite good agreement with the evolution of a two-mode entanglement case [6,12]. This shows that in this case, such a common environment (thermal or squeezed bath) still serves the Gaussian property of quantum state and makes a three-mode Gaussian state evolve to a three-mode Gaussian entangled mixed state. So, the squeezed states with higher squeezing can be regarded as good entanglement source for various quantum information processing applications. Third, within a range of moderate squeezing, the noise behavior of three-mode squeezed state is rather sophisticated. For the common thermal environment, we see that the initial entanglement is beginning to fall off, reaches a minimum, and then increases to a nonzero steady value, which is much less than the amount of the initial entanglement. The same trend is found for the case in which three modes are embedded in a common squeezed vacuum bath. But for some parameters satisfying the expression $Y=0$, we can see that $G_{\tau}^{\text{res}}=0$, implying that an initial pure, fully inseparable three-mode Gaussian state evolves eventually into a fully separable noisy three-mode Gaussian one.

IV. CONCLUSIONS

In summary, we have studied the problem of a generic multimode Gaussian state embedded in a common environmental noise. We have derived a general input-output formula of the covariance matrix for a generic multimode Gaussian state by using the Fokker-Planck equation for the characteristic function and the introduced operators. We have also investigated the separability and entanglement of three-mode squeezed state in such a noisy environment. Our analysis shows that the entanglement dynamics behaves differently for the different squeezing parameters. In particular, we find that if the squeezing r is sufficiently small or large, then the three-mode squeezed state may serve as a good entanglement source for some applications in quantum communication and quantum computation.

From a broader theoretical standpoint, further research stemming from the present work should probably be directed along two main directions. The first direction is considered providing the time-dependent characteristic function or input-output formula of the covariance matrix for multipartite and multimode Gaussian states embedded in other environmental noise, especially in a complex environment. The second one is how to apply it to the treatment of non-Markovian dynamics in open quantum system. This is due to the fact that strong system-environment coupling can lead to long memory times and to a failure of the Born-Markov approximation, while the non-Markovian process of continuous-variable entanglement is currently an interesting open problem [24].

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