

# Implementation of Clifford gates in the Ising-anyon topological quantum computer

André Ahlbrecht,<sup>1</sup> Lachezar S. Georgiev,<sup>1,2</sup> and Reinhard F. Werner<sup>1</sup>

<sup>1</sup>*Institut für Mathematische Physik, Technische Universität Braunschweig, Mendelssohnstrasse 3, 38106 Braunschweig, Germany*

<sup>2</sup>*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria*

(Received 12 December 2008; published 11 March 2009)

We give a general proof for the existence and realizability of Clifford gates in the Ising topological quantum computer. We show that all quantum gates that can be implemented by braiding of Ising anyons are Clifford gates. We find that the braiding gates for two qubits exhaust the entire two-qubit Clifford group. Analyzing the structure of the Clifford group for  $n \geq 3$  qubits we prove that the image of the braid group is a nontrivial subgroup of the Clifford group so that not all Clifford gates could be implemented by braiding in the Ising topological quantum computation scheme. We also point out which Clifford gates cannot in general be realized by braiding.

DOI: [10.1103/PhysRevA.79.032311](https://doi.org/10.1103/PhysRevA.79.032311)

PACS number(s): 03.67.Lx, 03.67.Ac, 71.10.Pm, 73.43.-f

## I. INTRODUCTION

Quantum computation (QC) [1] development encountered tremendous difficulties in storing and manipulating quantum information in real physical systems because of the overwhelming decoherence and noise. Topological quantum computation (TQC) [2] is a branch of QC in which both information storage and processing are protected by the topological nature of the quantum computer. In the TQC approach the quantum information is encoded in nonlocal topological degrees of freedom and is therefore inaccessible to noise and decoherence which are mainly due to local interactions. Moreover, the quantum gates are implemented by nontrivial topological operations which are once again protected against decoherence. The purpose of this topological protection of qubits and quantum gates is to improve the quantum computation hardware to such an extent that quantum-information processing becomes more feasible in real physical systems. The feasibility of this approach depends on the detailed noise structure [3–5]. In this paper we work entirely in the noiseless case and ask about the possibility of realizing certain key gates in a specific scheme.

One of the most promising TQC schemes [6] employs the anticipated non-Abelian braid statistics of the lowest energy excitations, called Ising anyons, of the fractional quantum Hall state at filling factor  $\nu=5/2$ , which is believed to belong to the universality class of the Moore-Read (MR) Pfaffian state [7]. Due to the topological degeneracy in the two-dimensional critical Ising model, representing the neutral degrees of freedom in the MR state, it becomes possible to realize  $n$  qubits by  $2n+2$  Ising anyons: the states of  $2n+2$  Ising anyons are represented by conformal field theory (CFT) correlation functions (more precisely, chiral CFT blocks) which happen to belong to one of the two inequivalent spinor irreducible representations (IRs) [8–13] of the covering group  $\text{Spin}(2n+2)$  of the rotation group  $\text{SO}(2n+2)$ . There are two inequivalent spinor IRs [8,9,12,13] of  $\text{SO}(2n+2)$ , of dimension  $2^n$ , which differ by their total fermion parity [12,13]. Despite being mathematically inequivalent they appear to be equivalent from the computational point of view [13], i.e., the set of matrices that could be obtained by braiding Ising anyons in both representations is the same.

Clifford-group gates, which are defined as those unitary operations that preserve the Pauli group, play a central role in quantum-information theory. Although the Clifford group is not sufficient for universal QC and its computational power cannot exceed classical computers [1,14,15], experimental realization of the Clifford group with sufficient scalability would be a cornerstone of QC. Indeed, the Clifford group is of significance for quantum error correction [16] and allows the generation of entangled states, such as Greenberger-Horne-Zeilinger (GHZ) [17] or cluster states, the latter being a prerequisite for universal quantum computation in the measurement-based scheme [18].

Using the explicit representation of the braid-group generators for the exchanges of Ising anyons one of us has constructed [10,11] the entire Clifford groups for one and two qubits in terms of braid generators for  $\mathcal{B}_4$  and  $\mathcal{B}_6$ , respectively. However, this approach encountered serious difficulties for embedding some Clifford gates in systems with three or more qubits [10,11]: the topological entanglement between distant Ising anyons induces additional phases when exchanging anyon pairs which are in the state  $|1\rangle$ , i.e., when exchanging pairs sharing Majorana fermions. Therefore, it was possible to construct only a part of the Clifford group for three qubits. In this paper we shall address the question whether all Clifford-group gates could be realized by braiding of Ising anyons in the Ising TQC scheme or not. Because, as we shall prove below, the  $n$ -qubit Pauli group coincides with the monodromy subgroup representation for  $2n+2$  Ising anyons and because in general the monodromy group is a normal subgroup of the braid group, it naturally follows that the Ising-model braid-group representation is a subgroup of the Clifford group. In other words, all quantum gates that could be implemented by braiding of Ising anyons are actually Clifford gates. Unfortunately, it also appears that not all Clifford gates could be realized by braiding for three or more Ising qubits and we shall try to explain why. The Clifford gates that cannot be implemented by braiding are typically the embeddings of the two-qubit SWAP gate [1] into larger systems.

The rest of this paper is organized as follows: in Sec. II we describe how anyonic states of matter could be labeled by fusion paths in Bratteli diagrams and how this could be used to determine the dimension of the computational space. In

Sec. III we summarize the explicit representation of the elementary Ising-model exchange matrices as proposed by Nayak and Wilczek [8]. In Sec. IV we explain the symplectic description of the Clifford group for  $n$  Ising qubits and estimate the order of the Clifford group in order to compare it with the order of the braid-group representation. In Sec. V we analyze the relation between the Pauli group and the monodromy subgroup of the braid group and explain why all quantum gates that could be realized by braiding of Ising anyons are in fact Clifford gates. In Sec. VII we give the explicit braid construction of the two-qubit SWAP gate which allows us to construct the entire two-qubit Clifford group by braiding. Some important technical details are collected in several appendixes.

## II. FUSION PATHS: LABELING THE ANYONIC STATES OF MATTER

The anyonic states of matter (should they really exist in nature) differ from the ordinary fermionic and bosonic states in that we need to give additional nonlocal information in order to specify the quantum state. In TQC this extra information is expressed in terms of topological quantum numbers that are eventually used to encode quantum information nonlocally in order to gain topological protection. An important ingredient of any TQC scheme is the necessary degeneracy of ground states of the multianyon system in presence of trapping potentials [19,20] (the potentials that keep our computational anyons at fixed positions). This is equivalent to a degeneracy of the multianyon states, considered as excitations corresponding to having a number of anyons at fixed positions in the plane, over the ground state (this time without trapping potentials) [19]. In the CFT language, that we will use to characterize the anyonic states as CFT correlators, the second point of view is more appropriate and we shall speak about the degeneracy of the CFT correlation functions corresponding to  $2n+2$  Ising anyons at fixed positions. In more detail, the CFT correlation functions of  $2n+2$  Ising anyons at fixed positions inside a Pfaffian droplet [8] span a Hilbert space of dimension  $2^n$  and are therefore appropriate for representing  $n$  Ising qubits. The nonlocal internal quantum numbers, for non-Abelian anyons, that would allow us to distinguish between the different states in this *computational space* are the so-called *fusion channels*. In order to make this notion more transparent let us try to explain the fusion rules of the Ising model: the most relevant quasiparticle excitations in the low-temperature, low-energy regime are described by the chiral Ising spin field  $\sigma$  with dimension  $1/16$ . If two such quasiparticles, having nontrivial topological properties, are fused, i.e., taken to the same point in the coordinate plane, the composite object would look (from far away) like another topological excitation and this is symbolically expressed by the fusion rule

$$\sigma \times \sigma = \mathbb{1} + \psi. \quad (1)$$

This rule means that there are two distinct channels in the fusion process and if we consider a large number of identically prepared experiments some pairs of  $\sigma$  would behave collectively as the vacuum [this is the  $\mathbb{1}$  in Eq. (1)] and this is

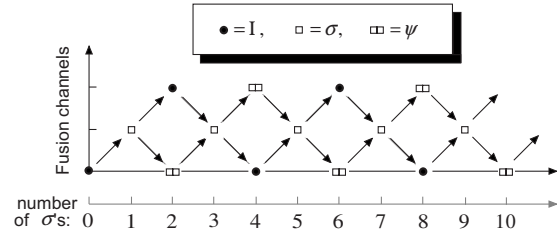


FIG. 1. Bratteli diagram for the Ising model in the  $su(2)$  notation.

called the vacuum channel, while some others would look like the Majorana fermion [i.e., the  $\psi$  in Eq. (1)] and this is called the Majorana channel. In other words, the combined quasiparticle can be considered as a mixed state of the vacuum and a Majorana fermion. Localized particlelike collective excitations, such as the field  $\sigma$  in Eq. (1), which have more than one available fusion channels are called non-Abelian anyons. There are in general superselection rules which forbid creation of coherent superpositions of anyons, such as the vacuum and the Majorana fermion in the Ising model, belonging to different superselection sectors and this has to be taken into account when using non-Abelian anyons for TQC.

The important point is that the topological properties of the anyon pairs are persistent (under some reasonable assumptions), i.e., if two quasiparticles, which are in a state characterized by a definite fusion channel, are pulled away the pair still possesses the properties of their corresponding fusion channel and, e.g., if they are fused again after some time, they will produce the same result as that determined by the original fusion channel.

When we have many Ising anyons, which we will assume to be ordered on a line, we could represent the string of  $2n+2$  Ising anyons  $\sigma$  into  $n+1$  pairs  $(\sigma, \sigma)_c$  and characterize each pair by its fusion channel  $c$ . Then the sequential composition of the fusion channels for all pairs can be described by a *fusion path* in the corresponding fusion diagram. The fusion diagrams that we will use are known as Bratteli diagram [19–21] and represent graphically the possible results of fusion of a single basic non-Abelian anyon to an array of other non-Abelian anyons usually of the same type. One way to understand this diagram is to represent the Ising model as the affine coset [22]  $\widehat{su(2)}_2 / u(1)$  and then use the fact that the Ising-model fusion rules are the same as those for the  $\widehat{su(2)}_2$  Wess-Zumino-Witten model [22]. Then the CFT primary fields  $\sigma$  and  $\psi$  (of CFT dimensions  $1/16$  and  $1/2$ , respectively) could be labeled by the reduced Young tableaux for the admissible [22]  $\widehat{su(2)}_2$  representations as shown in Fig. 1 and define (together with the vacuum) the three different superselection sectors (anyons) of the Ising model. In this way the fusion rules for the Ising model can be inferred from the tensor product decomposition of the  $su(2)$  representations. Each step to the right in Fig. 1 denotes fusing one more fundamental anyon  $\sigma$  to the existing string of  $\sigma$  anyons (whose length is determined by the  $x$  coordinate) and the arrows represent the possible fusion channels for this process, which are listed along the vertical axis. Note that the

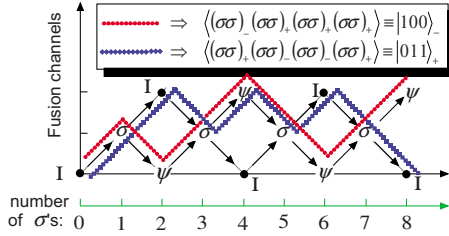


FIG. 2. (Color online) Fusion paths in Brattelli diagram: red circles display a path corresponding to the three-qubit state  $|100\rangle_-$ , while the path of blue squares corresponds to the state  $|011\rangle_+$ . One horizontal step to the right represents adding one more  $\sigma$  field and the arrows point to the corresponding fusion channel. The black circle denotes the vacuum channel  $I$ . The subscript  $\pm$  of the computational states denotes the positive- or negative-parity representations.

Brattelli diagram in Fig. 1 is finite in vertical direction which expresses the important property of rationality: any rational CFT, such as the critical two-dimensional Ising model, contains a finite number of topologically inequivalent superselection sectors (i.e., finite number of distinct anyons) which are “closed” under fusion. In order to build the Brattelli diagram in Fig. 1 we have to supplement Eq. (1) with the two other fusion rules,  $\psi \times \sigma = \sigma$  and  $I \times \sigma = \sigma$ . All anyons, such as  $I$  and  $\psi$ , from which originates only one arrow pointing to the right are Abelian, while those, such as  $\sigma$ , with more than one (two in this case) arrows to the right are non-Abelian.

In order to gain some intuition about labeling the states in the Ising model we show in Fig. 2 two distinct three-qubit states realized as fusion paths in the Brattelli diagram for eight Ising anyons. Notice that there are two inequivalent representations of the multianyon states with eight anyons [8,12,13]: besides the eight  $\sigma$  fields inside the CFT correlation function there are also a number of Majorana fermions present in there [not shown explicitly in the correlation functions in Fig. 2 but see Eq. (4) below]—when this number is even the state belongs to the positive-parity representation (in which case the eight  $\sigma$  fields must fuse to the identity in order for the CFT correlator to be nonzero), while when it is odd the state is in the negative-parity representation (and the eight  $\sigma$  fields must fuse to the Majorana fermion). The Brattelli diagrams also appear to be a very useful graphical tool for computing the dimensions of the computational spaces for TQC with anyons. For example, the space dimension of Ising-model correlation functions with eight anyons at fixed positions could be read off from Fig. 2 as follows: each new step to the right defines a single new fusion channel if the current number of  $\sigma$  is even and two fusion channels if the number of  $\sigma$  is odd. Therefore, the number of distinct fusion paths is just  $2^{(\# \sigma / 2) - 1}$ , the power of 2 with an exponent equals the number of the odd  $\sigma$  steps and finally divided by 2 because the last step is always fixed to fuse to the vacuum or to the Majorana fermion depending on the parity of the representation we deal with. We see from Fig. 2 that the number of independent correlation functions of eight fields  $\sigma$  is  $2^{4-1} = 8$  which could be easily generalized to the case of  $2n$   $\sigma$  fields where the correlation functions (for fixed positions of the anyons) span [8] a computational space of dimension  $2^{n-1}$ .

Thus we can say that the state of a multianyon system can be ultimately characterized by its fusion path in the Brattelli diagram. Having specified a multianyon state of matter as a fusion path in a Brattelli diagram any TQC scheme requires physical processes that could initialize the multianyon system into a given  $n$ -qubit state and this has to be further supplemented by procedures for measuring the states of the individual qubits. For the TQC scheme based on Ising anyons the initialization as well as the measurement by a Fabry-Pérot or Mach-Zehnder interferometer has been discussed in Refs. [6,19,20].

Now that we have got an idea about how to encode qubits using non-Abelian anyons we could think about executing quantum gates which are unitary transformations acting within the computational space of multianyon states labeled by fusion paths. A central issue in this context is that the multianyon states, which we would like to use to encode qubits, are degenerate in energy (at least approximately, see Refs. [19,20] for more detailed explanation) and separated from the rest of the excitation spectrum by a gap. This allows us to apply a version of the adiabatic theorem which is appropriate for the degenerate case. In simple words the adiabatic theorem in this case states that if the initial state is in the degenerate subspace that is separated by a gap from the rest of the spectrum and we consider the adiabatic evolution, when some of the anyons are transported along complete loops around others, the final multianyon state would be again a member of the same degenerate subspace (e.g., of ground states in presence of trapping potentials). Then the validity of the adiabatic approximation would guarantee that the transformation of the initial state into the final one is described by the action of a unitary operator which includes the Berry phase and the explicit monodromies of the (typically multivalued) multianyon states. It is, however, possible to choose a basis of CFT blocks [8] in which the Berry phases are trivial and the entire effect of the adiabatic evolution is contained in the monodromies of the CFT correlation functions. Thus we see that it might be possible to execute quantum gates over our topologically protected qubits by adiabatically transporting some anyons around others and these quantum operations are naturally protected against noise and decoherence.

Using the CFT-correlator representation we can describe the  $n$ -qubit Ising system by a CFT correlation function including  $2n+2$  Ising spin fields  $\sigma$  and the quantum-information encoding rule could be chosen to be, e.g.,

$$|c_1, \dots, c_i, \dots, c_n\rangle \rightarrow \langle (\sigma\sigma)_{c_1} \cdots (\sigma\sigma)_{c_i} \cdots (\sigma\sigma)_{c_n} (\sigma\sigma)_{c_0} \rangle_{\text{CFT}},$$

where  $c_0 = c_1 c_2 \cdots c_n$ . Notice that the  $n$  qubits are encoded from left to right starting with the first pair of  $\sigma$  fields and there is always one extra pair, which we shall take as the rightmost one, which is inert, i.e., it contains no independent information because its fusion channel  $c_0$  is determined by the product of the fermion parities of the individual qubit states. In other words, the physical meaning of  $c_i$  is the fermion parity of the pair representing the  $i$ th qubit; the role of the inert pair is to compensate the combined fermion parity of the first  $n$  pairs so that the correlation function is nonzero.

Since we shall use the fermion parity of the non-Abelian  $\sigma$  fields to encode information in the Ising qubits it is worth saying a few words about it. The chiral spin fields  $\sigma$  are identified with the primary fields of the Ising CFT that intertwine between the vacuum sector [or the Neveu-Schwarz (NS) sector] and the so-called Ramond (R) sector of the Ising model [23,24]. The conservation of the fermion parity, combined with the fact that the Majorana fermion has a zero mode in the R sector, implies that Ramond sector in the Ising model is double degenerate [23,24]. This means that there must exist two Abelian chiral spin fields  $\sigma_+$  and  $\sigma_-$  of CFT dimension  $1/16$  with fermion parity  $+$  and  $-$ , respectively. However, this double degeneration of the R sector is incompatible with the modular invariance which is at the heart of the fusion rules for the anyonic model. In order to guarantee the modular invariance of the model we need to choose only one linear combination [24] of  $\sigma_+$  and  $\sigma_-$  which is conventionally taken as

$$\sigma = \frac{\sigma_+ + \sigma_-}{\sqrt{2}}. \tag{2}$$

As a result of this Gliozzi-Scherk-Olive (GSO) projection [24] the chiral fermion parity is spontaneously broken which is also obvious in the fusion rules (1). Note that conservation of chiral fermion parity requires that the fields  $\sigma_{\pm}$  with definite parity are Abelian, i.e., their fusion rules must be  $\sigma_+ \times \sigma_+ = \mathbb{1}$ ,  $\sigma_- \times \sigma_- = \mathbb{1}$ , and  $\sigma_+ \times \sigma_- = \psi$ , and the non-Abelian statistics appears only when this symmetry is broken (which could only be spontaneous because the generator of the fermion parity symmetry commutes with the conformal Hamiltonian).

Although the only field which is believed to appear in the physical system is Eq. (2) and the chiral fermion parity is broken by the non-Abelian fusion rule (1) the fields  $\sigma_{\pm}$  are still convenient for labeling our computational basis in terms of Ising-model correlation functions. In other words, even if  $\sigma_{\pm}$  are unphysical they could efficiently label the independent functions spanning the degenerate space of correlation functions because the product  $e_1 e_2 = c = \pm 1$  of the two indices in a pair  $\sigma_{e_1} \sigma_{e_2}$  determines the fusion channel  $c = +1$  for the vacuum channel and  $c = -1$  for the Majorana one. Thus, the general qubit encoding scheme for Ising qubits can be represented as

$$\begin{aligned} |0\rangle &\Leftrightarrow (\sigma\sigma)_{+1} \Leftrightarrow \sigma_+ \sigma_+, \\ |1\rangle &\Leftrightarrow (\sigma\sigma)_{-1} \Leftrightarrow \sigma_+ \sigma_-. \end{aligned} \tag{3}$$

More precisely, using for instance the CFT description of Ising qubits in the positive-parity representation and introducing one extra pair as illustrated in Fig. 3, we could write the single-qubit computational basis as

$$|0\rangle_+ \equiv \left\langle \sigma_+ \sigma_+ \sigma_+ \sigma_+ \prod_{j=1}^{2N} \psi(z_j) \right\rangle_{\text{CFT}},$$

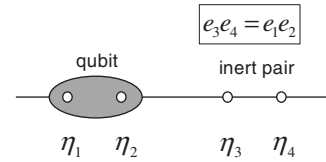


FIG. 3. Single-qubit configuration in terms of four Ising quasi-holes corresponding to the positive-parity representation  $S_+$  of the braid group  $\mathcal{B}_4$ .

$$|1\rangle_+ \equiv \left\langle \sigma_+ \sigma_- \sigma_+ \sigma_- \prod_{j=1}^{2N} \psi(z_j) \right\rangle_{\text{CFT}}, \tag{4}$$

where the first pair of  $\sigma$  fields represents the qubit while the second one is the noncomputational inert pair of Ising anyons compensating the fermion parity. The total topological charge of the four  $\sigma$  fields is  $+1$  which corresponds to the positive-parity spinor representation of  $\text{SO}(4)$  (cf. Sec. III) and is denoted as a subscript of the computational basis states in Eq. (4). The computational basis for two Ising qubits is explicitly written in Sec. VII and for more qubits in Refs. [12,13].

### III. ELEMENTARY EXCHANGE MATRICES: SPINOR APPROACH

As we have already mentioned the  $n$ -qubit system could be represented by  $2n+2$  Ising anyons because the degeneracy of the states containing  $2n+2$  anyons is [8]  $2^{\lfloor (2n+2)/2 \rfloor - 1} = 2^n$ . The most natural approach to determine the braid matrices representing the exchanges of adjacent anyons over the (degenerate) space of correlation functions is to use the analytic continuation of these multivalued analytic functions that have been used to define the qubits. The results in Ref. [8] suggested that the multianyon Ising wave functions span a spinor representation of the rotation group  $\text{SO}(2n+2)$  [more precisely, one of the inequivalent spinor representations of its covering group  $\text{Spin}(2n+2)$ ]; however, this important conjecture has not been completely proven in Ref. [8]. It has later been proven rigorously in Ref. [11] for the positive-parity representation of the braid group  $\mathcal{B}_4$  and in Ref. [12] for both positive- and negative-parity representations of the braid group  $\mathcal{B}_{2n+2}$  representing the exchanges in the  $n$ -qubit states defined as CFT correlation functions of  $2n+2$  Ising anyons. One of the main results in Ref. [12] is that the  $n$ -qubit representations of the braid-group generators  $B_j^{(2n+2, \pm)}$ , obtained by analytic continuation of the CFT correlation functions, are *completely* equivalent to those, denoted as  $R_j^{(n+1, \pm)}$ , obtained from the standard Clifford-algebra realization of the generators of the rotation group  $\text{SO}(2n+2)$ , i.e.,

$$B_j^{(2n+2, \pm)} = (C^{(2n+2, \pm)})^{-1} R_j^{(n+1, \pm)} C^{(2n+2, \pm)}.$$

Notice that this is not completely trivial since the matrices which establish the equivalence between the braid generators in the wave-function representations with definite parity and those in the Clifford-algebra realizations with the same parity are special nonsingular entangling matrices (i.e., matrices

which cannot be written as factorized tensor products of smaller ones) which are explicitly constructed in Ref. [12]. For example, the equivalence matrices for three qubits in the two representations are  $C^{(8,+)} = \text{diag}(-1, 1, 1, 1, 1, 1, 1, -1)$  while  $C^{(8,-)} = i \text{diag}(1, 1, 1, -1, 1, -1, -1, -1)$ . Thus, as proven in Ref. [12], it is completely legitimate to use the Clifford-algebra construction of the braid-group generators for those representing the coordinate exchanges in the CFT correlation functions. The advantage of using the Clifford-algebra representation of the braid generators comes from the possibility to express the Pauli gates in terms of the squares of elementary braid generators (cf. Sec. V) and then use the group-theoretic approach to find the stabilizing Clifford group.

That is why we shall review below the general Clifford-algebra construction of the two inequivalent spinor representations of the braid group  $\mathcal{B}_{2n}$  which will be used later to implement various quantum operations over the Ising multi-qubit systems. Nayak and Wilczek suggested [8] that the elementary matrices representing the coordinate exchanges of  $2n$  Ising quasiparticles in the Pfaffian FQH wave functions can be expressed in terms of the gamma matrices  $\gamma_i^{(n)}$ ,  $1 \leq i \leq 2n$ , which satisfy the anticommutation relations of the Clifford algebra

$$\{\gamma_i^{(n)}, \gamma_j^{(n)}\} = 2\delta_{ij}, \quad 1 \leq i, j \leq 2n. \quad (5)$$

In more detail, the elementary operations for the exchange of the  $j$ th and  $(j+1)$ th quasiparticles could be expressed (including the correct phase) as [8,12,13,25]

$$R_j^{(n)} = e^{i\pi/4} \exp\left(-\frac{\pi}{4} \gamma_j^{(n)} \gamma_{j+1}^{(n)}\right) \equiv \frac{e^{i\pi/4}}{\sqrt{2}} (\mathbb{I} - \gamma_j^{(n)} \gamma_{j+1}^{(n)}). \quad (6)$$

The  $2n$  matrices  $\gamma_j^{(n)}$  have dimension  $2^n \times 2^n$  and can be defined recursively in terms of the Pauli matrices  $\sigma_k$  ( $k=1, 2, 3$ ) as follows [26]:

$$\begin{aligned} \gamma_j^{(n+1)} &= \gamma_j^{(n)} \otimes \sigma_3, \quad 1 \leq j \leq 2n, \\ \gamma_{2n+1}^{(n+1)} &= \mathbb{I}_{2^n} \otimes \sigma_1, \\ \gamma_{2n+2}^{(n+1)} &= \mathbb{I}_{2^n} \otimes \sigma_2. \end{aligned} \quad (7)$$

Starting with  $n=0$  as a base, where  $\gamma_1^{(1)} = \sigma_1$  and  $\gamma_2^{(1)} = \sigma_2$  we could write the gamma matrices explicitly as follows [26,27]:

$$\begin{aligned} \gamma_1^{(n)} &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \\ \gamma_2^{(n)} &= \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \\ &\vdots \\ \gamma_{2j-1}^{(n)} &= \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{j-1} \otimes \sigma_1 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-j}, \\ \gamma_{2j}^{(n)} &= \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{j-1} \otimes \sigma_2 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-j}, \end{aligned}$$

$$\vdots$$

$$\gamma_{2n-1}^{(n)} = \mathbb{I}_{2^{n-1}} \otimes \sigma_1,$$

$$\gamma_{2n}^{(n)} = \mathbb{I}_{2^{n-1}} \otimes \sigma_2. \quad (8)$$

The ‘‘gamma-five’’ matrix  $\gamma_F^{(n)}$  could be constructed by

$$\gamma_F^{(n)} = (-i)^n \gamma_1^{(n)} \cdots \gamma_{2n}^{(n)} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_n.$$

Because  $\gamma_F^{(n)}$  anticommutes with all  $\gamma_j^{(n)}$  it commutes with all  $R_j$  and therefore this representation of the braid group is reducible. Since  $\gamma_F^{(n)}$  has only two distinct eigenvalues,  $\pm 1$ , it splits into two irreducible representations that could be obtained with the help of the two projectors

$$P_{\pm}^{(n)} = \frac{\mathbb{I}_{2^n} \pm \gamma_F^{(n)}}{2}, \quad \text{where } (P_{\pm}^{(n)})^2 = P_{\pm}^{(n)} = (P_{\pm}^{(n)})^\dagger. \quad (9)$$

In other words, the generators of the two irreducible spinor representations of the braid group  $\mathcal{B}_{2n}$  can be obtained by simply projecting Eq. (6), i.e.,

$$\begin{aligned} R_j^{(n,\pm)} &= e^{i\pi/4} P_{\pm}^{(n)} \exp\left(-\frac{\pi}{4} \gamma_j \gamma_{j+1}\right) P_{\pm}^{(n)} \\ &= \frac{e^{i\pi/4}}{\sqrt{2}} (\mathbb{I} - \gamma_j^{(n)} \gamma_{j+1}^{(n)}) P_{\pm}^{(n)}. \end{aligned} \quad (10)$$

It would be useful for our analysis to give more detailed expressions for the braid generators

$$R_{2j-1}^{\pm} = \frac{e^{i\pi/4}}{\sqrt{2}} (\mathbb{I}_2 - i\sigma_3^{(j)}) P_{\pm}^{(n)}, \quad 1 \leq j \leq n,$$

$$R_{2j}^{\pm} = \frac{e^{i\pi/4}}{\sqrt{2}} (\mathbb{I}_4 - i\sigma_2^{(j)} \otimes \sigma_2^{(j+1)}) P_{\pm}^{(n)}, \quad 1 \leq j \leq n-1,$$

where we have used the shorthand notation  $\sigma_k^{(j)}$  to denote  $\sigma_k$  acting on qubit  $j$ . A more explicit form of the matrices can be found in Ref. [13].

#### IV. CLIFFORD GROUP FOR $n$ QUBITS: SYMPLECTIC DESCRIPTION

Quantum operations belonging to the Clifford group are characterized in the Heisenberg picture as the group of unitary operators which stabilize the Pauli group, i.e., they map an arbitrary factorized tensor product of Pauli operators to another factorized tensor product of Pauli operators. This property has important consequences, extending a suitable description of the Pauli group [cf. Eq. (17)] to the Clifford group equips us with an efficient way of writing down such unitaries [cf. Eq. (15)] as opposed to general unitaries where the number of parameters typically grows exponentially with the number of qubits. Although, this eliminates exponential speed up, the Clifford group is of great relevance for quantum computation. On the one hand we can study basic properties of quantum-mechanical systems, such as entanglement

generation, explicitly and on the other hand there are ingredients of quantum computation, e.g., error correction [16], which are based on the Clifford group. Detailed descriptions of Clifford-group quantum computations are given in Refs. [1,14,15]. In the following we will introduce isomorphism (15) between a factor of the Clifford group and the symplectic group  $\text{Sp}_{2n}(2)$ , which will be used to further analyze the structure and computational power of the Ising-model TQC (cf. Appendixes B and C).

The Pauli group  $\mathcal{P}_n$  for  $n$  qubits is generated by the Pauli operators  $\sigma_i$  acting on each of the qubits. The elements of  $\mathcal{P}_n$  can thus be written as tensor products of the Pauli matrices multiplied by overall phases, which are powers of  $i$ ,

$$\mathcal{P}_n = \{i^m \sigma_{\alpha(1)} \otimes \cdots \otimes \sigma_{\alpha(n)} | \alpha(j), m \in \{0,1,2,3\}\}, \quad (11)$$

where we have defined  $\sigma_0 = \mathbb{I}_2$ . The Clifford group  $\mathcal{C}_n$  for  $n$  qubits is defined as the normalizer of the Pauli group, i.e., the group of those  $n$ -qubit unitary transformations  $U$  which map the Pauli group  $\mathcal{P}_n$  to itself under conjugation

$$\mathcal{C}_n = \{U \in \text{SU}(2^n) | U^* \mathcal{P}_n U \subset \mathcal{P}_n\}. \quad (12)$$

Clearly,  $\mathcal{C}_n$  is an infinite group since if  $U$  satisfies Eq. (12) so does  $e^{i\phi}U$  for any  $\phi \in \mathbb{R}$ . However, the overall phases  $e^{i\phi}$  are in general irrelevant for quantum computation and therefore we could remove the center  $Z$  of the Clifford group, which is generated by these phase factors. The resulting factor group

$$\mathcal{PC}_n = \mathcal{C}_n / Z,$$

which we will call the *projective Clifford group*, is already a finite group whose order will be derived later in this section. Notice that the minimal center of the Clifford group, which is required by the group multiplication law, is isomorphic to  $\mathbb{Z}_8$  and is generated by  $e^{i\pi/4}$  as shown in Ref. [28]. The extension of the projective Clifford group by the center  $\mathbb{Z}_8$  defines the minimal unprojected Clifford group which is also finite.

For  $U \in \mathcal{C}_n$  we will denote the corresponding equivalence class in  $\mathcal{PC}_n$  by  $[U]$ . A frequently used discrete set of Clifford unitaries, whose images  $[\cdot]$  generate  $\mathcal{PC}_n$  if we allow them to act on arbitrary qubits, is

$$P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I}_2 + |1\rangle\langle 1| \otimes \sigma_1. \quad (13)$$

It is obvious that the Pauli group is a subgroup of the Clifford group because

$$\sigma_i^* \sigma_j \sigma_i = \begin{cases} \sigma_j, & i=j \\ -\sigma_j, & i \neq j \end{cases} \Rightarrow \mathcal{P}_n \subset \mathcal{C}_n. \quad (14)$$

The equivalence classes of the elements of  $\mathcal{P}_n$ , which differ only by a phase, form a factor group of  $\mathcal{P}_n$ , which we call the *projective Pauli group*,

$$[\mathcal{P}_n] \equiv \mathcal{P}_n / \mathbb{Z}_4,$$

and it should play an important role in what follows. Because of Eq. (14) the projective Pauli group is also a normal subgroup of the projective Clifford group.

The main results obtained in this paper are based on the isomorphism between the projective Clifford group, factorized by its projective Pauli subgroup  $[\mathcal{P}_n]$ , and the symplectic group  $\text{Sp}_{2n}(2)$  over the finite field  $\mathbb{F}_2$ , i.e.,

$$\mathcal{PC}_n / [\mathcal{P}_n] \cong \text{Sp}_{2n}(2). \quad (15)$$

Although this is a well-known result we would like to explain it in more detail as it is of central importance for estimating the computational power of our topological quantum computer. The group  $\text{Sp}_{2n}(2)$  is defined as the set of invertible  $2n \times 2n$  matrices with entries from the finite field  $\mathbb{F}_2$  satisfying the relation

$$S \in \text{Sp}_{2n}(2) \Leftrightarrow S^T M S = M, \quad (16)$$

where  $M$  is the  $2n \times 2n$  block-diagonal matrix,

$$M = \oplus^n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathbb{I}_n \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In order to understand isomorphism (15) we first note that the Pauli operators  $\sigma_i$  “commute” up to phase factors. This means that, if we ignore these phase factors and keep track only of the multiplication rules, we obtain an additive group with a “modulo-2” arithmetic, which is isomorphic to  $\mathbb{Z}_2^2$ . Indeed, if we identify the Pauli matrices with the vectors

$$\mathbb{I} \rightarrow (0,0), \quad \sigma_1 \rightarrow (1,0),$$

$$\sigma_2 \rightarrow (0,1), \quad i\sigma_3 \rightarrow (1,1),$$

we could see that the Pauli multiplication rules are translated into vector summation over the finite field  $\mathbb{F}_2$ . In other words, the one-qubit projective Pauli group is isomorphic to the Abelian group  $\mathbb{Z}_2^2$ ,

$$[\mathcal{P}_1] \cong (\mathbb{Z}_2)^2.$$

This idea extends naturally to the  $n$ -qubit Pauli group  $\mathcal{P}_n$ , where we identify the  $n$ -qubit Pauli gates with vectors in  $(\mathbb{F}_2)^{2n}$ , i.e., the vector space of the  $2n$ -component vectors over the field  $\mathbb{F}_2$ , as illustrated by the example

$$\sigma_1 \otimes \cdots \otimes i\sigma_3 \rightarrow (1,0, \dots, 1,1).$$

Therefore we conclude that the projective  $n$ -qubit Pauli group is isomorphic to the Abelian group  $\mathbb{Z}_2^{2n}$ ,

$$[\mathcal{P}_n] \cong (\mathbb{Z}_2^2)^n \equiv \mathbb{Z}_2^{2n}. \quad (17)$$

Taking into account that the center of the  $n$ -qubit Pauli group is  $\mathbb{Z}_4$ , generated by the powers of  $i$ , we could compute from Eq. (17) the order of the complete Pauli group  $\mathcal{P}_n$  to be

$$|\mathcal{P}_n| = 2^{2n+2}. \quad (18)$$

Recall that the Clifford-group elements are structure preserving invertible mappings (automorphisms) of the Pauli group, i.e., they map  $\mathcal{P}_n$  to itself and respect products of Pauli operators. In more detail, if  $U \in \mathcal{C}_n$  and we denote by  $\underline{p}$  the multi-index of the Pauli operator  $\sigma_{\underline{p}} \in \mathcal{P}_n$  then

$$U^* \sigma_{\underline{p}} U = i^{f_U(\underline{p})} \sigma_{S_U(\underline{p})}, \quad (19)$$

where  $f_U(\underline{p})$  is the power exponent of the phase, and we have denoted the mapping of the Pauli indices, corresponding to the action of the Clifford operator  $U$  by

$$S_U: \mathbb{F}_2^{2n} \rightarrow \mathbb{F}_2^{2n}, \quad \underline{p} \rightarrow S_U(\underline{p}). \quad (20)$$

While index mapping (20) essentially expresses the action of the Clifford gate  $U$  inside of the projective Pauli group  $[\mathcal{P}_n]$ , the pair  $(f_U, S_U)$  in Eq. (19) describes unambiguously this action on the complete Pauli group  $\mathcal{P}_n$ . Mapping (20) is furthermore an automorphism of  $\mathbb{F}_2^{2n}$  which could be characterized by the following properties:

(1)  $S_U$  is linear. It follows from the definition (19) that

$$[\sigma_{S_U(\underline{p})}] = [U^* \sigma_{\underline{p}} U],$$

where the brackets mean taking the projective equivalence class. Therefore we can write, using the sign  $\oplus$  for modulo-2 addition of vectors,

$$\begin{aligned} [\sigma_{S_U(\underline{p} \oplus \underline{q})}] &= [U^* \sigma_{\underline{p} \oplus \underline{q}} U] = [U^* \sigma_{\underline{p}} \sigma_{\underline{q}} U] = [U^* \sigma_{\underline{p}} U][U^* \sigma_{\underline{q}} U] \\ &= [\sigma_{S_U(\underline{p})}][\sigma_{S_U(\underline{q})}] = [\sigma_{S_U(\underline{p} \oplus \underline{q})}] \end{aligned}$$

from which we could derive the relation

$$S_U(\underline{p} \oplus \underline{q}) = S_U(\underline{p}) \oplus S_U(\underline{q}), \quad \forall \underline{p}, \underline{q} \in \mathbb{F}_2^{2n}.$$

This means that  $S_U \in \text{GL}_{2n}(2)$ .

(2)  $S_U$  is symplectic. Working with the projective Pauli group  $[\mathcal{P}_n] = \mathbb{F}_2^{2n}$  is convenient, however, we lose the information about commutation or anticommutation of the Pauli operators, e.g., it is impossible to distinguish the two cases,

$$\sigma_2 i \sigma_3 = \sigma_1,$$

$$i \sigma_3 \sigma_2 = -\sigma_1.$$

Nevertheless, it is possible to keep track of the corresponding commutation relations. If we define the inner product, denoted by  $*$ , of two vectors  $\underline{p} = (p_1, p_2), \underline{q} = (q_1, q_2) \in \mathbb{F}_2^2$  to be  $\underline{p} * \underline{q} = p_2 q_1$ , it is easy to check that

$$\sigma_{\underline{p}} \sigma_{\underline{q}} = (-1)^{\underline{p} * \underline{q}} \sigma_{\underline{p} \oplus \underline{q}},$$

where  $\sigma_a$  denotes the Pauli operator associated with  $a \in \mathbb{F}_2^2$ . This generalizes in a straightforward way to the  $n$ -qubit case where  $\underline{p} = (p_1, \dots, p_{2n}), \underline{q} = (q_1, \dots, q_{2n})$ , and the inner product is given by

$$\underline{p} * \underline{q} = \sum_{i=1}^n p_{2i} q_{2i-1}.$$

Whether two operators  $\sigma_{\underline{p}}, \sigma_{\underline{q}} \in \mathcal{P}_n$  commute or not can be inferred from the symplectic form

$$\omega(\underline{p}, \underline{q}) = \underline{p} * \underline{q} - \underline{q} * \underline{p} = \underline{p}^T M \underline{q},$$

which could only be equal to 0 or 1, where  $M$  is given in Eq. (16) and  $\underline{p}^T$  is the transpose of  $\underline{p}$ . It is easy to see that

$$\sigma_{\underline{p}} \cdot \sigma_{\underline{q}} = (-1)^{\omega(\underline{p}, \underline{q})} \sigma_{\underline{q}} \sigma_{\underline{p}}, \quad (21)$$

from where we can conclude if the two operators  $\sigma_{\underline{p}}$  and  $\sigma_{\underline{q}}$  commute  $[\omega(\underline{p}, \underline{q})=0]$  or anticommute  $[\omega(\underline{p}, \underline{q})=1]$ .

Now we can prove that if the operator  $U$  in Eq. (19) belongs to the Clifford group then  $S_U$  must be symplectic, i.e.,  $\forall \underline{p}, \underline{q} \in \mathbb{F}_2^{2n}$ ,

$$\omega(\underline{p}, \underline{q}) = \omega(S_U(\underline{p}), S_U(\underline{q})). \quad (22)$$

Indeed, let us multiply Eq. (21) by  $U^*$  from the left and by  $U$  from the right. We have

$$U^* \sigma_{\underline{p}} U U^* \sigma_{\underline{q}} U = (-1)^{\omega(\underline{p}, \underline{q})} U^* \sigma_{\underline{q}} U U^* \sigma_{\underline{p}} U.$$

Using Eq. (19) and canceling the phase factor  $i^{f_U(\underline{p})+f_U(\underline{q})}$  on both sides we obtain

$$\sigma_{S_U(\underline{p})} \sigma_{S_U(\underline{q})} = (-1)^{\omega(\underline{p}, \underline{q})} \sigma_{S_U(\underline{q})} \sigma_{S_U(\underline{p})}. \quad (23)$$

On the other side, however, if  $U \in \mathcal{C}_n$  as we assumed, then both  $\sigma_{S_U(\underline{p})}$  and  $\sigma_{S_U(\underline{q})}$  are Pauli operators as well, so they should also satisfy Eq. (21), i.e.,

$$\sigma_{S_U(\underline{p})} \sigma_{S_U(\underline{q})} = (-1)^{\omega(S_U(\underline{p}), S_U(\underline{q}))} \sigma_{S_U(\underline{q})} \sigma_{S_U(\underline{p})}. \quad (24)$$

Comparing Eq. (23) with Eq. (24) we derive Eq. (22) which implies that  $S_U$  is symplectic, i.e.,

$$S_U^T M S_U = M \Rightarrow S_U \in \text{Sp}_{2n}(2).$$

(3) The kernel of  $U \rightarrow S_U$  is the projective Pauli group  $[\mathcal{P}_n]$ . Consider the mapping between the Clifford and the symplectic groups,

$$S: \mathcal{C}_n \rightarrow \text{Sp}_{2n}(2), U \rightarrow S_U,$$

which is a homomorphism between the two groups  $\mathcal{C}_n$  and  $\text{Sp}_{2n}(2)$  both with matrix multiplication as group concatenation. The kernel of this homomorphism, i.e., the group of elements in  $\mathcal{C}_n$  which is mapped to the identity in  $\text{Sp}_{2n}(2)$ , is generated by those unitaries  $U$  which satisfy

$$U^* \sigma_{j_1} \otimes \dots \otimes \sigma_{j_n} U = \pm \sigma_{j_1} \otimes \dots \otimes \sigma_{j_n} \quad (25)$$

for all choices of indices  $j_1, \dots, j_n \in \{0, 1, 2, 3\}$ . Because of Eq. (14) all Pauli operators satisfy this equation, hence

$$[\mathcal{P}_n] \subset \ker S.$$

On the other hand, since the Pauli operators form a basis for the linear operators on  $\mathcal{H}^{\otimes n}$  we can express any unitary  $U$  as a linear combination,

$$U = \sum_{\sigma_x \in \mathcal{P}_n} \alpha_x \sigma_x, \quad \alpha_x \in \mathbb{C}.$$

Now suppose that there is a  $U$  satisfying Eq. (25) with more than one terms in the sum above. Then it is easy to see that there is a Pauli operator  $\sigma_y$  such that the expansion of  $U^* \sigma_y U$  consists of more than one Pauli operators which contradicts to the assumption that Eq. (25) is satisfied. Therefore the projected Pauli group is not only a subgroup of  $\ker S$  but also

$$\ker S \subset [\mathcal{P}_n] \Rightarrow \ker S \equiv [\mathcal{P}_n].$$

Given that the image of the mapping  $S:U \rightarrow S_U$  is  $\text{Sp}_{2n}(2)$  and that  $\ker S \equiv [\mathcal{P}_n]$  we conclude that

$$\mathcal{PC}_n / \ker S \simeq \mathcal{PC}_n / [\mathcal{P}_n] \simeq \text{Sp}_{2n}(2).$$

This important fact allows us to use the results for the order of the finite symplectic group

$$|\text{Sp}_{2n}(2)| = 2^{n^2} \prod_{j=1}^n (4^j - 1), \quad (26)$$

which is proven in Appendix A, to derive the order of the projective Clifford group. It follows from Eq. (15), and the fact that  $[\mathcal{P}_n]$  has order  $2^{2n}$ , that the order of the projective Clifford group is

$$|\mathcal{PC}_n| = 2^{n^2+2n} \prod_{j=1}^n (4^j - 1). \quad (27)$$

This result will be used in Sec. VI to estimate the computational power of the Ising-anyon topological quantum computer.

### V. BRAIDING GATES AS CLIFFORD GATES: MONODROMY ANALYSIS

A crucial observation for the results obtained in this paper, as well as for topological quantum computation in general, is that the Pauli group  $\mathcal{P}_n$  for  $n$  qubits realized by  $2n+2$  Ising anyons coincides with the representation of the monodromy subgroup of the braid group

$$\mathcal{P}_n \equiv \text{image}(\mathcal{M}_{2n+2}).$$

Because the monodromy group is in general a normal subgroup of the braid group it follows that all braiding gates, which by definition leave the monodromy (hence the Pauli group) invariant, are actually Clifford gates, i.e.,

$$\text{image}(\mathcal{B}_{2n+2}) \subset \mathcal{C}_n.$$

In order to give a more comprehensive explanation of this important result we shall recall the standard algebraic definition of the monodromy subgroup in terms of the exact sequence of group homomorphisms

$$1 \rightarrow \mathcal{M}_n \rightarrow \mathcal{B}_n \rightarrow \mathcal{S}_n \rightarrow 1,$$

where  $\mathcal{M}_n$  is the monodromy group (also called the pure braid group),  $\mathcal{B}_n$  is the full braid group, and  $\mathcal{S}_n$  is the symmetric group. In more detail, recall that a braid can be described geometrically as a bunch of strands connecting the initial configuration of some indistinguishable particles (arranged for simplicity along a line) with the final configuration. If we forget about the strands and look at the final positions of the particles, arranged along the same line, then the final configuration will be simply a permutation of the original particles. As an illustration we show in Fig. 4 the action of the projection map on the two-qubit controlled-NOT (CNOT) gate in Refs. [10,11], which is an element of the

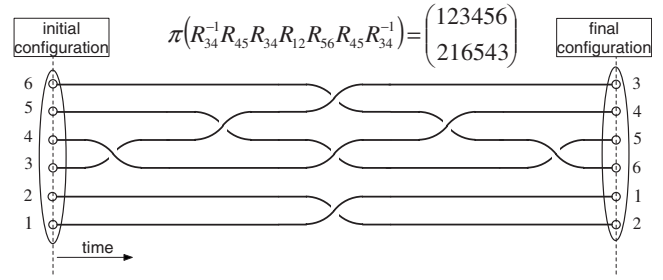


FIG. 4. Example of the projection mapping  $\pi$  acting on the specific braid element  $\text{CNOT} = R_{34}^{-1}R_{45}R_{34}R_{12}R_{56}R_{45}R_{34}^{-1} \in \mathcal{B}_6$ , where  $R_i \equiv R_{i,i+1}$  in the notation of Ref. [11], producing the permutation  $(216543) \in \mathcal{S}_6$ .

Ising-model representation of  $\mathcal{B}_6$ . Therefore we can define the natural ‘‘onto’’ mapping

$$\pi: \mathcal{B}_n \rightarrow \mathcal{S}_n,$$

which could be easily seen to be a group homomorphism. Next, as is well known, the kernel of any group homomorphism is a normal subgroup and the factor group is isomorphic to the symmetric group

$$\mathcal{B}_n / \ker \pi \simeq \mathcal{S}_n.$$

Therefore we may define the monodromy group as the kernel of the projection  $\pi$ , i.e.,

$$\mathcal{M}_n \equiv \ker \pi.$$

Geometrically this definition means that a monodromy transformation is any braid transformation which does not permute the final positions of the particles compared to the initial ones. Pictorially this could be described by such braid transformations in which one or more particles are transported along closed loops around one or more other particles (so that each particle arrives at the end at its original position) as shown in Fig. 5. It is not difficult to see that these pure braid transformations can be generated by the elementary monodromy transformations in which one particle, say with index  $j$ , is transported along a complete loop around another particle, say with index  $i$ , where  $i < j$ . Thus the el-

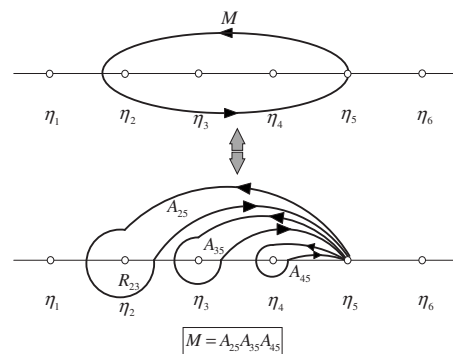


FIG. 5. Decomposition of a monodromy transformation in which the particle with coordinate  $\eta_5$  makes a complete loop around particles with coordinates  $\eta_2$ ,  $\eta_3$ , and  $\eta_4$  in terms of the monodromy generators  $A_{ij}$ .



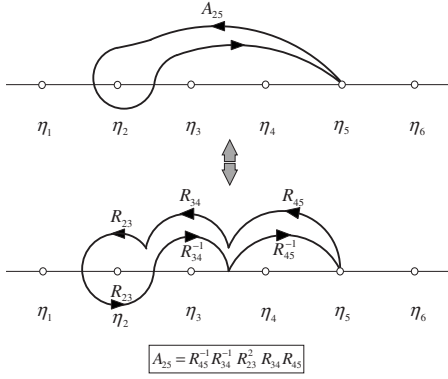


FIG. 6. The monodromy generator  $A_{25}$ , representing the elementary monodromy transformation in which the particle with coordinate  $\eta_5$  is transported in a counterclockwise direction along a complete loop around the particle with coordinate  $\eta_2$ , expressed in terms of the braid generators.

elementary generators of the monodromy group could be given the following presentation [29]:

$$A_{ij} = R_{j-1}^{-1} R_{j-2}^{-1} \cdots R_{i+2}^{-1} R_{i+1}^{-1} R_i^2 R_{i+1} R_{i+2} \cdots R_{j-2} R_{j-1} \\ = U_{ij}^{-1} R_i^2 U_{ij}, \quad \text{where } U_{ij} = \prod_{k=i+1}^{j-1} R_k \quad (28)$$

and  $1 \leq i < j \leq n$ . In other words the monodromy group could be considered as the smallest invariant subgroup of  $\mathcal{B}_n$  which is generated by the squares of the elementary braid generators [30]  $R_i^2$ ,  $i=1, \dots, n-1$ . As an illustration of Eq. (28) we display in Fig. 6 the monodromy transformation in which the particle with coordinate  $\eta_5$  is transported along a complete counterclockwise loop around the particle with coordinate  $\eta_2$ .

Alternatively, the monodromy group could be abstractly defined by the generators  $A_{ij}$ ,  $1 \leq i < j \leq n$ , satisfying certain relations, however we will skip that as we shall only need here the expressions [Eq. (28)] in terms of the braid generators.

It can be seen that in the braid-group representation of the Ising model the Pauli group is represented in terms of monodromy operators. Indeed, the  $\sigma_3$  gate acting on the  $i$ th qubit is simply

$$(R_{2i-1}^{(n+1,+)} )^2 = \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{i-1} \otimes \sigma_3 \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{n-i}, \quad (29)$$

while the squares of the nondiagonal braid generators give

$$(R_{2i}^{(n+1,+)} )^2 = \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{i-1} \otimes \sigma_2 \otimes \sigma_2 \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{n-i-1}, \quad (30)$$

where  $1 \leq i \leq n-1$ . The last two squares are more special,

$$(R_{2n}^{(n+1,\pm)} )^2 = \mp \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1} \otimes \sigma_1, \quad (31)$$

$$(R_{2n+1}^{(n+1,\pm)} )^2 = \pm \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_n. \quad (32)$$

It is obvious from the above equations that all  $n$ -qubit Pauli gates can be expressed (up to phases which are powers of  $i$ ) in terms of products of squares of the elementary braid generators. For example, the  $\sigma_2$  gate acting on the last qubit is

$$\sigma_2^{(n)} = i(R_{2n}^{(n+1,+)} )^2 (R_{2n+1}^{(n+1,+)} )^2,$$

and the  $\sigma_2$  gate acting on qubit  $(n-j)$  can be expressed as

$$\sigma_2^{(n-j)} = i(R_{2n-2j}^{(n+1,+)} )^2 \sigma_2^{(n-j+1)}, \quad 1 \leq j \leq n-1.$$

The above equations also suggest that we might need some additional phase factors in order to construct all elements of  $\mathcal{P}_n$ . To clarify this point we will construct the element  $i\mathbb{I}$  explicitly in the positive-parity representation (a similar construction can also be given for the negative-parity representation because both representations are computationally equivalent, as shown in Ref. [13]). More precisely, we will prove below that

$$R_{2n}^{(n+1,+)} (R_{2n+1}^{(n+1,+)} )^2 R_{2n}^{(n+1,+)} (R_{2n+1}^{(n+1,+)} )^2 = i\mathbb{I}. \quad (33)$$

The left-hand side (LHS) corresponds to a composition of two braids, where particle  $2n$  moves along a complete loop around particles  $2n+1$  and  $2n+2$  and likewise particle  $2n+1$  moves around  $2n+2$ . Therefore Eq. (33) represents an element of the monodromy subgroup and we conclude that

$$\{\pm \mathbb{I}, \pm i\mathbb{I}\} \subset \text{image}(\mathcal{M}_{2n+2}).$$

To verify Eq. (33) we first observe that the last two unprojected braid matrices  $R_{2n}^{(n+1)}$  and  $R_{2n+1}^{(n+1)}$  satisfy, according to Eq. (6) and the anticommutation relations of the  $\gamma$  matrices, the following relation:

$$R_{2n}^{(n+1)} (R_{2n+1}^{(n+1)})^2 R_{2n} = i(R_{2n+1}^{(n+1)})^2. \quad (34)$$

Indeed, using the properties of the  $\gamma$  matrices, it is easy to see that the squares of the unprojected braid matrices [Eq. (6)] are simply

$$(R_j^{(n+1)})^2 = -i \gamma_j^{(n+1)} \gamma_{j+1}^{(n+1)}, \quad 1 \leq j \leq n-1. \quad (35)$$

Applying Eq. (35) for  $j=2n+1$  and using the anticommutation relations [Eq. (5)] for the  $\gamma$  matrices, we have for the LHS of Eq. (34),

$$\text{LHS} = \frac{i}{2} (\mathbb{I} - \gamma_{2n} \gamma_{2n+1}) (-i \gamma_{2n+1} \gamma_{2n+2}) (\mathbb{I} - \gamma_{2n} \gamma_{2n+1}) \\ = i(-i) \gamma_{2n+1} \gamma_{2n+2} = i(R_{2n+1}^{(n+1)})^2.$$

The fourth power of the unprojected braid matrices is always  $\mathbb{I}$  because applying twice Eq. (35) we have

$$(R_j^{(n+1)})^4 = -\gamma_j^{(n+1)} \gamma_{j+1}^{(n+1)} \gamma_j^{(n+1)} \gamma_{j+1}^{(n+1)} = \mathbb{I},$$

which also implies that the fourth power of the projected braid generators [Eq. (10)] is  $\mathbb{I}$ . Finally, the validity of Eq. (33) follows from the fact that the projectors [Eq. (9)] commute with the group multiplication law [see Eq. (38) below].

Now, because the squares  $(R_i^{(n+1,\pm)})^2$  of the elementary braid generators, as well as the element  $i\mathbb{I}$ , belong to the

monodromy group representation, we conclude that the Pauli group  $\mathcal{P}_n$  is naturally represented in the Ising-model TQC as a subgroup of the monodromy group for  $2n+2$  Ising anyons, i.e.,

$$\mathcal{P}_n \subset \text{image}(\mathcal{M}_{2n+2}). \quad (36)$$

On the other hand, as we will show below, the generators [Eq. (28)] of the monodromy group could be entirely expressed in terms of the Pauli operations for  $n$  Ising qubits so that

$$\text{image}(\mathcal{M}_{2n+2}) \subset \mathcal{P}_n \Rightarrow \mathcal{P}_n \equiv \text{image}(\mathcal{M}_{2n+2}). \quad (37)$$

In order to prove group isomorphism (37) we will show that the monodromy generators of the Ising-model representation of  $\mathcal{M}_{2n+2}$  could be expressed (up to powers of  $i$ ) in terms of the squares of the elementary braiding matrices  $(R_i^{(n+1,+)} )^2$ , which, according to Eqs. (29)–(32), are completely expressible in terms of Pauli operators.

Before we prove this let us emphasize once again one important detail of the Ising representation of the braid group. The projectors [Eq. (9)] commute with the group multiplication law, i.e.,

$$R_i^{(n+1,\pm)} R_j^{(n+1,\pm)} = R_i^{(n+1)} R_j^{(n+1)} P_{\pm}^{(n+1)}, \quad (38)$$

so that, if we want, we could work all the time with the unprojected matrices and project them at the end.

Using Eq. (35) as a base we could prove by induction that the unprojected monodromy generators are

$$A_{kl} = i(-1)^{l-k} \gamma_k^{(n+1)} \gamma_l^{(n+1)}, \quad 1 \leq k < l \leq 2n+2. \quad (39)$$

Indeed, assume that Eq. (39) is fulfilled for given  $k$  and  $l$ , where  $l > k$ . Then

$$A_{k,l+1} = R_l^{-1} A_{kl} R_l = \frac{1}{2} (\mathbb{I} + \gamma_l \gamma_{l+1}) [i(-1)^{l-k} \gamma_k \gamma_l] (\mathbb{I} - \gamma_l \gamma_{l+1}),$$

where we have used that the inverse of the unprojected braid generator  $R_l$  is  $R_l^{-1} = e^{-i\pi/4} (\mathbb{I} + \gamma_l \gamma_{l+1}) / \sqrt{2}$ . Then using the anticommutation relations [Eq. (5)], to simplify expressions such as

$$\gamma_l \gamma_{l+1} \gamma_k \gamma_l \gamma_l \gamma_{l+1} = \gamma_l \gamma_{l+1} \gamma_k \gamma_{l+1} = -\gamma_l \gamma_k,$$

we obtain  $A_{k,l+1} = -i(-1)^{l-k} \gamma_k \gamma_{l+1}$  which completes the proof of Eq. (39). Next, plugging between  $\gamma_k$  and  $\gamma_l$  the identities  $\gamma_{k+1} \gamma_{k+1}$ ,  $\gamma_{k+2} \gamma_{k+2}$ , until  $\gamma_{l-1} \gamma_{l-1}$  and reexpressing  $\gamma_k \gamma_{k+1} = i(R_k)^2$  we can write the unprojected monodromy generators as

$$A_{kl} = -(-i)^{l-k+1} (R_k)^2 (R_{k+1})^2 \cdots (R_{l-2})^2 (R_{l-1})^2.$$

Finally, using the fact that projectors commute with the group multiplication, as in Eq. (38), we obtain, e.g., the monodromy generators in the positive-parity representations (and similar for negative parity) as

$$A_{kl}^+ = -(-i)^{l-k+1} (R_k^+)^2 (R_{k+1}^+)^2 \cdots (R_{l-2}^+)^2 (R_{l-1}^+)^2, \quad (40)$$

where  $1 \leq k < l \leq 2n+2$  and  $R_k^+ = R_k^{(n+1)} P_+^{(n+1)}$ . Going back to Eqs. (29)–(32) we conclude that because  $(R_k^+)^2 \in \mathcal{P}_n$  for all  $k$  and the element  $i\mathbb{I} \in \mathcal{P}_n$  as well [see Eq. (33)], the generators of the monodromy representation certainly belong to the Pauli group  $\mathcal{P}_n$  so that

$$A_{kl}^+ \in \mathcal{P}_n \Rightarrow \text{image}(\mathcal{M}_{2n+2}) \subset \mathcal{P}_n. \quad (41)$$

Combining Eq. (41) with Eq. (36) we finally prove that the Ising-model monodromy group representation of  $\mathcal{M}_{2n+2}$  is completely equivalent to the  $n$ -qubit Pauli group  $\mathcal{P}_n$  for the Ising-model topological quantum computer.

*Remark 1.* It follows from Eq. (37) that the center of the monodromy group representation coincides with the center of the  $n$ -qubit Pauli group which is generated by  $i\mathbb{I}$ , i.e.,  $\text{center}(\text{image}(\mathcal{M}_{2n+2})) = \text{center}(\mathcal{P}_n) = \mathbb{Z}_4$ . Combining this with the algebraic fact [29] that the permutation group  $\mathcal{S}_{2n+2}$  is centerless for  $n \geq 1$ , so that the center of the braid group coincides with the center of its monodromy subgroup, and with the fact that  $\mathcal{S}_{2n+2}$  is represented faithfully in the Ising-model representation of  $\mathcal{B}_{2n+2}$  for  $n \geq 2$ , as proven in Appendix C, we conclude that

$$\begin{aligned} \text{center}(\text{image}(\mathcal{M}_{2n+2})) &\equiv \text{center}(\text{image}(\mathcal{B}_{2n+2})) \\ &= \{\pm \mathbb{I}, \pm i\mathbb{I}\}, \quad n \geq 2. \end{aligned} \quad (42)$$

*Remark 2.* Because the monodromy group  $\mathcal{M}_{2n+2}$  in the Ising-model representation of the braid group  $\mathcal{B}_{2n+2}$  exactly coincides with the  $n$ -qubit Pauli group  $\mathcal{P}_n$ , this automatically means that the image of the braid group is a subgroup of the  $n$ -qubit Clifford group  $\mathcal{C}_n$ ,

$$\text{image}(\mathcal{B}_{2n+2}) \subset \mathcal{C}_n,$$

i.e., all quantum gates that could be implemented by braiding in the Ising-model TQC are in fact Clifford gates that stabilize the Pauli group. Unfortunately, the converse is not true, i.e., it appears that not all Clifford gates could be implemented by braiding Ising anyons.

Furthermore, we could use group isomorphism (37) between the monodromy group  $\text{image}(\mathcal{M}_{2n+2})$  and the Pauli group  $\mathcal{P}_n$  to derive the order of the image of the braid group  $\mathcal{B}_{2n+2}$ . Recall that in general the factor group  $\mathcal{B}_{2n+2} / \mathcal{M}_{2n+2}$  is isomorphic to the permutation group  $\mathcal{S}_{2n+2}$ . In our case it appears that for  $n \geq 2$  this permutation group is represented faithfully (see Appendix C for a proof) so that the order of the Ising-model braid-group representation is

$$\begin{aligned} |\text{image}(\mathcal{B}_{2n+2})| &= |\text{image}(\mathcal{M}_{2n+2})| |\text{image}(\mathcal{S}_{2n+2})| \\ &= 2^{2n+2} (2n+2)!, \quad n \geq 2, \end{aligned} \quad (43)$$

where we have used that because of Eq. (37) the order of the monodromy group is equal to the order of the full  $n$ -qubit Pauli group  $\mathcal{P}_n$  [including the center; cf. Eq. (42)], which is  $2^{2n+2}$  and that the order of the faithful representation of  $\mathcal{S}_{2n+2}$  is  $(2n+2)!$ . For  $n=1$  the order of the image of the monodromy group is again given by the order of the Pauli group  $2^{2n+2}=16$ ; however, the representation of the permutation group  $\mathcal{S}_4$  is not faithful. Instead, the third generator is equivalent to the first one, i.e.,  $R_3^{(+)} = R_1^{(+)}$  or  $R_3^{(-)} = i(R_1^{(-)})^{-1}$  and therefore the factor group gives a faithful representation of the permutation group  $\mathcal{S}_3$ . Thus the image of the braid group for four Ising anyons is

$$|\text{image}(\mathcal{B}_4)| = |\text{image}(\mathcal{M}_4)| |\text{image}(\mathcal{S}_3)| = 16 \times 3! = 96,$$

which exactly coincides with the result in Ref. [11] where the order has been directly computed using the Dimino's algo-

TABLE I. Comparing the orders of the image of the braid group  $\mathcal{B}_{2n+2}$ , factorized by its center  $\mathbb{Z}_4$ , and of the projective Clifford group  $\mathcal{PC}_n$ .

No. of qubits $n$	1	2	3	4	5
$ \text{Im}(\mathcal{B}_{2n+2})/\mathbb{Z}_4 $	24	11520	2580480	$\approx 0.9 \times 10^9$	$\approx 0.5 \times 10^{12}$
$ \mathcal{PC}_n $	24	11520	92897280	$\approx 1.2 \times 10^{13}$	$\approx 2.5 \times 10^{19}$

arithm. The result,  $2^{66}! = 46\,080$ , for the order of the image of the braid group  $\mathcal{B}_6$  computed from Eq. (43) also exactly matches that obtained by the Dimino’s algorithm in Ref. [11].

Finding the order of the image of the braid group  $\mathcal{B}_{2n+2}$  is a central result in this paper which is similar to the results in Ref. [9], where the order of the image of the braid group depends on the parity of  $n$ , yet it is different. The reason for the independence of the order of the representation on the parity of  $n$  in our case is the presence of the additional phase factor  $\exp(i\pi/4)$  in Eq. (6), which certainly changes the center of the representation of the braid group.

**VI. PROJECTIVE CLIFFORD GROUP  $\mathcal{PC}_n$  AND THE ISING REPRESENTATION OF THE BRAID GROUP  $\mathcal{B}_{2n+2}$**

In this section we shall prove that not all Clifford-group gates could actually be implemented by braiding of Ising anyons. To this end we shall demonstrate that the order of the projective Clifford group, i.e., the Clifford group factorized by its center, is much bigger and grows much faster with  $n$  than the order of the image of the braid group. Because global phases are irrelevant for quantum computation it makes sense to compare the orders of the two finite groups factorized by their centers.

Let us first compare the single-qubit braid group  $\mathcal{B}_4$  and the single-qubit Clifford group  $\mathcal{PC}_1$ . The order of the Ising-model representation of  $\mathcal{B}_4$  is 96; however if we factorize it by its  $\mathbb{Z}_4$  center we have  $|\text{image}(\mathcal{B}_4)/\mathbb{Z}_4| = 24$ . This precisely coincides with the order of the projective Clifford group for one qubit obtained from Eq. (27).

For two qubits the orders of the representation of the braid group  $\mathcal{B}_6$  and its projective version are

$$|\text{image}(\mathcal{B}_6)| = 46\,080 \Rightarrow |\text{image}(\mathcal{B}_6)/\mathbb{Z}_4| = 11\,520,$$

and again the second number precisely coincides with the order of the two-qubit projective Clifford group obtained from Eq. (27),

$$|\mathcal{PC}_2| = 2^{2^2+4}(4-1)(4^2-1) = 11\,520.$$

While the coincidence of the orders of the one-qubit braid-group representation and the one-qubit Clifford group seems natural, the one for the two-qubit groups is more subtle. It could be understood by the occasional group isomorphism [31,32],

$$\mathcal{PC}_2/[\mathcal{P}_2] \cong \mathcal{S}_6 \equiv \mathcal{B}_6/\mathcal{M}_6.$$

and this remarkable mathematical fact could eventually explain why the entire Clifford group for two qubits can indeed

be realized by braiding, while for any  $n \geq 3$  the Clifford group is much bigger, and grows much faster with  $n$ , than the image of the braid group. Indeed, the orders of the image of the braid group (factored by its center) and of the projective Clifford group for  $n = 1, \dots, 5$  number of qubits are compared in Table I. Already for three qubits the order of the projective Clifford group is about 40 times bigger than the order of the image of the braid group factored by its center. For  $n = 4$  and 5 the discrepancy is even more dramatic: the order of the projective Clifford group is increasing much faster with  $n$  than the order of the image of the braid group. In Fig. 7 we have plotted the logarithm of the ratio between the order of the projective Clifford group and the order of the braid group for number of qubits up to 100. To understand more intuitively Fig. 7 notice that the leading term in the order of the projective Clifford group is

$$|\mathcal{PC}_n| \approx 2^{n^2+2n} \prod_{j=1}^n 2^{2j} \propto 2^{2n^2+3n},$$

while (using the Stirling formula) the order of the image of the (projective) braid group is

$$|\text{image}(\mathcal{B}_{2n+2})| = 2^{2n}(2n+2)! \propto 2^{2n+(2n+2)\log_2(2n+2)},$$

and therefore the leading term in the ratio is

$$|\mathcal{PC}_n|/|\text{image} \mathcal{B}_{2n+2}| \propto 2^{2n^2+n-(2n+2)\log_2(2n+2)} \propto 2^{2n^2},$$

so that the logarithm of this ratio still grows quadratically with  $n$ .

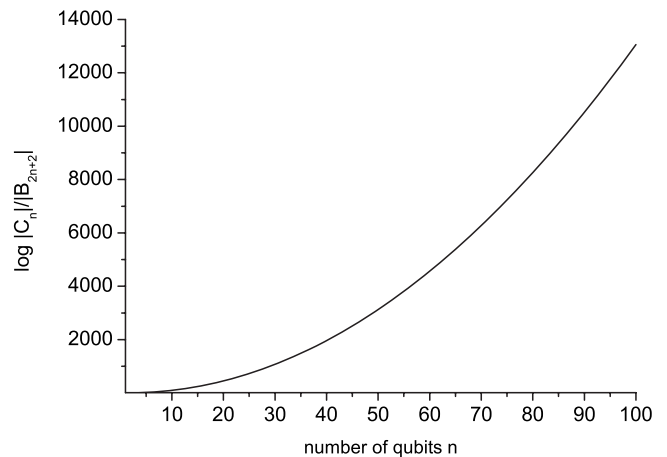


FIG. 7. Comparing the order of the  $n$ -qubit Clifford group  $\mathcal{C}_n$  with that of the image of the  $n$ -qubit braid group  $\mathcal{B}_{2n+2}$ . The natural logarithm of the ratio of the two orders is plotted on the vertical axis versus the number of qubits  $n$ .

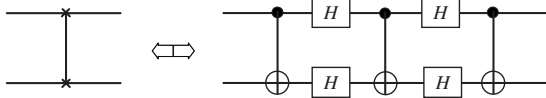


FIG. 8. SWAP gate in terms of three CNOT and four Hadamard gates.

It is not difficult to see which gates are missing, i.e., which are the Clifford gates that cannot be implemented by braiding of Ising anyons. These are the embeddings of the two-qubit SWAP gate into the  $n$ -qubit system. Indeed, if we have them then we could construct by braiding all embeddings of the controlled-Z (CZ) gates because by physically exchanging pairs of Ising anyons, representing the qubits, we readily obtain  $\text{CZ} \times \text{SWAP}$  as illustrated in Eq. (45) below for the two-qubit case. In general, by exchanging the  $j$ th and  $(j+1)$ th pair of anyons, with coordinates  $(\eta_{2j-1}, \eta_{2j})$  and  $(\eta_{2j+1}, \eta_{2j+2})$ , respectively, we obtain [cf. Eq. (45)]

$$R_{2j}^{(n+1,+)} R_{2j+1}^{(n+1,+)} R_{2j-1}^{(n+1,+)} R_{2j}^{(n+1,+)} = i \text{CZ}^{(j,j+1)} \times \text{SWAP}^{(j,j+1)},$$

where  $\text{CZ}^{(j,j+1)}$  is the controlled-Z gate acting on the  $j$ th qubit (the control qubit) and  $(j+1)$ th qubit (the target qubit) and  $\text{SWAP}^{(j,j+1)}$  is the two-qubit SWAP gate exchanging the  $j$ th and  $(j+1)$ th qubits. The phase gate  $P$  in Eq. (13) acting on qubit  $j$  is given by  $R_{2j-1}^{(n+1,+)}$ . Moreover, if we have all the SWAPs then we can construct Hadamard gate acting on arbitrary qubit by swapping the Hadamard gate

$$H^{(n)} = (R_{2n-1}^{(n+1,+)})^2 R_{2n+1}^{(n+1,+)} R_{2n}^{(n+1,+)} (R_{2n+1}^{(n+1,+)})^{-1}$$

acting on the last qubit to the qubit where it is needed.

Thus we see that if we could construct all embeddings of the two-qubit SWAP gate then we could construct the entire Clifford group for  $n$  qubits.

## VII. SOLUTION FOR TWO QUBITS: BRAIDING IMPLEMENTATION OF THE SWAP GATE

The problem with the implementation of all Clifford gates could be most easily illustrated on the example of the two-qubit SWAP gate [33], which is a Clifford gate that can be expressed in terms of CNOT and Hadamard gates [1] as shown in Fig. 8. Imagine a two-qubit system constructed from six Ising anyons, with coordinates  $\eta_1, \dots, \eta_6$  in the plane, in such a way that the first pair (with coordinates  $\eta_1$  and  $\eta_2$ ) forms the first qubit, the second pair (with coordinates  $\eta_3$  and  $\eta_4$ ) forms the second qubit, and the last two qubits with coordinates  $\eta_5$  and  $\eta_6$  form the inert pair as shown in Fig. 9. Without restriction of generality we could

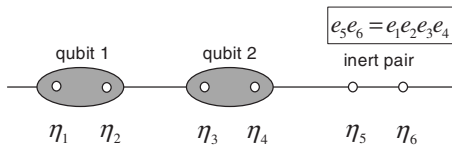


FIG. 9. Two-qubit configurations in terms of six Ising quasi-holes corresponding to the positive-parity representation of the braid group  $\mathcal{B}_6$ .

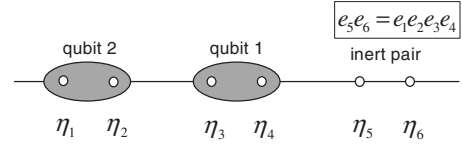


FIG. 10. Two-qubit configurations corresponding to the positive parity, however, with different encoding of the qubits.

choose to work in the spinor representation  $S_+$  with positive total parity because, as shown in Ref. [13], both inequivalent spinor representations  $S_{\pm}$  of the braid group  $\mathcal{B}_{2n+2}$  are computationally equivalent. The two-qubit computational basis in the spinor representation  $S_+$  can be written in terms of Ising-model correlation functions as

$$\begin{aligned} |00\rangle &\equiv \langle \sigma_+ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \rangle, \\ |01\rangle &\equiv \langle \sigma_+ \sigma_+ \sigma_+ \sigma_+ \sigma_- \sigma_- \rangle, \\ |10\rangle &\equiv \langle \sigma_+ \sigma_- \sigma_+ \sigma_+ \sigma_+ \sigma_- \rangle, \\ |11\rangle &\equiv \langle \sigma_+ \sigma_- \sigma_+ \sigma_- \sigma_+ \sigma_- \rangle. \end{aligned} \quad (44)$$

Now, let us try to swap the two qubits by simply exchanging the two pairs comprising the qubits. This could be done by the following braid transformation (see Ref. [13] for the explicit form of the braid matrices in this basis):

$$U_{\#1 \leftrightarrow \#2}^{(2)} = R_2^{(3,+)} R_3^{(3,+)} R_1^{(3,+)} R_2^{(3,+)} = i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (45)$$

The resulting unitary transformation has one extra minus on the last row as compared to the standard SWAP gate [1] (we skip the global phase  $i$ ). This is, however, natural because this minus sign appears in front of the state  $|11\rangle$  that is the state in which both Ising-anyon pairs at  $(\eta_1, \eta_2)$  and at  $(\eta_3, \eta_4)$  share Majorana fermions and therefore exchanging the two qubits is equivalent (especially after fusing  $\eta_1 \rightarrow \eta_2$  and  $\eta_3 \rightarrow \eta_4$ ) to the exchange of two fermions which certainly produces a minus sign. This raises the reasonable question whether the SWAP gates are at all realizable by braiding.

On the other hand, we could have chosen a different basis of Ising  $\sigma$  fields to represent two qubits, e.g., the first pair (with coordinates  $\eta_1$  and  $\eta_2$ ) forms the second qubit, while the second pair (with coordinates  $\eta_3$  and  $\eta_4$ ) forms the first qubit and the last two qubits with coordinates  $\eta_5$  and  $\eta_6$  form the inert pair again as shown in Fig. 10. Then the two-qubit computational basis can be written in terms of Ising-model correlation functions as

$$\begin{aligned} |00\rangle' &\equiv \langle \sigma_+ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \rangle, \\ |01\rangle' &\equiv \langle \sigma_+ \sigma_- \sigma_+ \sigma_+ \sigma_+ \sigma_- \rangle, \\ |10\rangle' &\equiv \langle \sigma_+ \sigma_+ \sigma_+ \sigma_- \sigma_+ \sigma_- \rangle, \end{aligned}$$

$$|11\rangle' \equiv \langle \sigma_+ \sigma_- \sigma_+ \sigma_- \sigma_+ \sigma_+ \rangle. \quad (46)$$

Obviously basis (46) could be obtained from basis (44) by the action of the transformation matrix,

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \text{SWAP}, \quad (47)$$

which simply means that bases (44) and (46) describe equivalent representations of the braid group  $\mathcal{B}_6$ .

The question whether the SWAP gate is implementable by braiding is equivalent to that if the product

$$U_{\#1 \leftrightarrow \#2} A^{(2)} = i \text{diag}(1, 1, 1, -1),$$

which is (up to phase) equal to the controlled-Z gate, is expressible in terms of the braid-group generators. Indeed, using the explicit form of the generators from Ref. [13], we could easily check that

$$R_1^{(3,+)} R_3^{(3,+)} (R_5^{(3,+)})^{-1} = \text{diag}(1, 1, 1, -1) = \text{CZ},$$

which constructively proves that the two-qubit CZ and SWAP gates (hence, all Clifford gates for two qubits) are efficiently realizable by braiding of Ising anyons.

## VIII. CONCLUSION

In this paper we have demonstrated that the Pauli group in the Ising-anyon topological quantum computer exactly coincides with the image of the monodromy group. Therefore, all quantum gates that can be implemented by braiding of Ising anyons are Clifford gates that stabilize the Pauli group. Analyzing the structure of the Clifford group for  $n$  qubits and comparing to the structure of the braid-group representation of the Ising model, we conclude that not all Clifford gates could in general be realized by braiding in this model. The gates that are missing are the embeddings of the two-qubit SWAP gate into a  $n$ -qubit Ising systems. Thus, not only the Ising-anyon topological quantum computer is not universal for topological quantum computation but even the entire Clifford group is not implementable by braiding. Nevertheless, the braid-group representation of the Ising TQC contains all of the entangling gates that could be used to analyze topological entanglement and topological protection of qubits and quantum operations.

## ACKNOWLEDGMENTS

We thank Sergey Bravyi, Holger Vogts, Lyudmil Hadjiivanov, Volkher Scholz, and Johannes Guetschow for useful discussions. A.A. acknowledges support from DFG (Forschergruppe 635). L.S.G. thanks the Galileo Galilei Institute for Theoretical Physics in Firenze, Italy, for hospitality as well as INFN for partial support. L.S.G. has been supported as a Research Fellow by the Alexander von Humboldt foundation and by the BG-NCSR under Contract Nos. F-1406 and DO 02-257.

## APPENDIX A: ORDER OF THE SYMPLECTIC GROUP

Following Ref. [31] we could find the order of the symplectic group  $\text{Sp}_{2n}(q)$  over the field  $\mathbb{F}_q$  by counting the number of different ways to chose a symplectic basis, which we shall order as  $\{e_1, \dots, e_n; f_1, \dots, f_n\}$ , of the vector space  $\mathbb{F}_q^{2n}$ , i.e., the bilinear form in this basis is  $(e_i, f_i) = 1 = -(f_i, e_i)$  and  $(e_i, f_j) = 0 = (f_j, e_i)$  for  $i \neq j$ .

(1) Choose  $e_1$ : any nonzero vector in  $\mathbb{F}_q^{2n}$  could be chosen as  $e_1$ . The number of vectors in  $\mathbb{F}_q^{2n}$  is  $q^{2n}$  so that the number of nonzero vectors is  $\#e_1 = q^{2n} - 1$ .

(2) Choose  $f_1$ : the vector  $f_1$  might be chosen among the vectors which are not orthogonal to  $e_1$ . The dimension of the subspace  $\{v \in \mathbb{F}_q^{2n} | (v, e_1) = 0\}$  is  $2n - 1$  (at this point we could think of  $e_1$  as being an element of an orthonormal basis of  $\mathbb{F}_q^{2n}$ ) so that the number of the orthogonal vectors is  $q^{2n-1}$ .

(3) The number of vectors which are not orthogonal to  $e_1$  is thus  $q^{2n} - q^{2n-1} = (q-1)q^{2n-1}$  and all they come in scalar multiples such as,  $f_1, 2f_1, \dots, (q-1)f_1$ , so that the number of different choices for  $f_1$  is  $\#f_1 = q^{2n-1}$ .

(4) Therefore the number of different choices of  $e_1, f_1$  is

$$\#\{e_1, f_1\} = (q^{2n} - 1)q^{2n-1}.$$

(5) The rest of the space, which is spanned by the basis vectors  $\{e_2, \dots, e_n; f_2, \dots, f_n\}$ , is the vector space  $\mathbb{F}_q^{2n-2}$  of dimension  $2(n-1)$  so we can apply again the above result to obtain

$$\#\{e_2, f_2\} = (q^{2(n-1)} - 1)q^{2(n-1)-1}$$

(after  $\{e_1, f_1\}$  have been chosen).

(6) Induction: continuing in this way we finally obtain

$$|\text{Sp}_{2n}(q)| = \prod_{j=n \downarrow}^1 (q^{2j} - 1)q^{2j-1} = q^{n^2} \prod_{j=1}^n (q^{2j} - 1). \quad (\text{A1})$$

Applying Eq. (A1) for  $q=2$  we obtain Eq. (26).

## APPENDIX B: ISING BRAIDING GATES AS SYMPLECTIC TRANSFORMATIONS

We will determine the symplectic matrices corresponding to the Ising-model exchange matrices via relation (15). To this end we first phrase the Ising-model quantum computer abstractly in three steps:

(i) Starting point is the representation of the Clifford algebra in terms of the  $\gamma_j^{(n+1)}$ . According to Eq. (6) we define operators  $R_j$  which act on  $\mathcal{H}^{\otimes n+1}$ , where  $\mathcal{H} = \mathbb{C}^2$ .

(ii) The subspace of  $\mathcal{H}^{\otimes n+1}$  of positive parity, i.e., the linear span of the set

$$\{|x_1 \cdots x_{n+1}\rangle : x_1 \oplus \cdots \oplus x_{n+1} = 0\},$$

will be denoted by  $\mathcal{H}_+^{\otimes n+1}$ . We will identify  $\mathcal{H}_+^{\otimes n+1}$  with  $\mathcal{H}^{\otimes n}$  via the correspondence

$$|x_1 \cdots x_n z\rangle \leftrightarrow |x_1 \cdots x_n\rangle,$$

where  $z$  is chosen such that the left-hand side is an element of  $\mathcal{H}_+^{\otimes n+1}$ , i.e.,  $z = x_1 \oplus \cdots \oplus x_n$ .

(iii) Finally, the action of the  $R_j^+$  on  $|x_1 \cdots x_n\rangle \in \mathcal{H}^{\otimes n}$  is obtained by applying  $R_j$  to  $|x_1 \cdots x_n z\rangle$  and calculating the cor-

responding vector in  $\mathcal{H}^{\otimes n}$ . Note that this step is only well defined because  $R_j$  is reducible with respect to the decomposition  $\mathcal{H}^{\otimes n+1} = \mathcal{H}_+^{\otimes n+1} \oplus \mathcal{H}_-^{\otimes n+1}$ .

The last two steps describe the action of the projection  $P_{\pm}^{(n)}$  in Sec. III.

Now, instead of focusing on the correspondence between basis vectors, we will concentrate on the correspondence between the Pauli groups associated with  $\mathcal{H}^{\otimes n}$  and  $\mathcal{H}^{\otimes n+1}$ . The operators

$$\tau_j^{(k)} \equiv \mathbb{I}_{k-1} \otimes \sigma_j \otimes \mathbb{I}_{n-k} \otimes \sigma_1 \mathcal{R}_+ \quad (\text{B1})$$

if  $j \neq 3$  and

$$\tau_3^{(k)} \equiv \mathbb{I}_{k-1} \otimes \sigma_3 \otimes \mathbb{I}_{n-k} \otimes \mathbb{I} \mathcal{R}_+, \quad (\text{B2})$$

where we denote by  $\mathcal{R}_+$  the restriction to  $\mathcal{H}_+^{\otimes n+1}$ , are essentially the Pauli operators on  $\mathcal{H}^{\otimes n}$ . This follows from the fact that these operators are also reducible with respect to the decomposition  $\mathcal{H}^{\otimes n+1} = \mathcal{H}_+^{\otimes n+1} \oplus \mathcal{H}_-^{\otimes n+1}$  which allows us to calculate their action on  $\mathcal{H}^{\otimes n}$  just like we do for the  $R_j^+$ .

Since both  $R_j$  and  $R_j^+$  are Clifford unitaries in their respective domains, it is easy to determine the action of  $R_l^+$  on  $\tau_j^{(k)}$  according to the following commutative diagram:

$$\begin{array}{ccc} (R_l^+)^* \tau_j^{(k)} R_l^+ & \rightarrow & \prod_{r=1}^n \tau_{\alpha_r}^{(r)} \\ \downarrow & & \uparrow \\ R_l^* \sigma_j^{(k)} \sigma_{\delta}^{(n+1)} R_l \mathcal{R}_+ & \rightarrow & \prod_{r=1}^n \sigma_{\beta_r}^{(r)} \mathcal{R}_+ \end{array},$$

where we have used the shorthand notation

$$\sigma_i^{(k)} \equiv \mathbb{I}_{k-1} \otimes \sigma_i \otimes \mathbb{I}_{n-k}.$$

Note that the  $\sigma_{\delta}^{(n+1)}$  in the lower left corner depends on the  $\sigma_j^{(k)}$ ; we have  $\delta=1$  if  $j=1,2$  and  $\delta=0$  if  $j=0,3$ .

The action of the operators  $R_j$  on Pauli operators is equivalent [34] to the following Clifford unitaries:

$$\begin{aligned} R_{2j-1} &\simeq P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \\ R_{2j} &\simeq R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

These Clifford gates map Pauli operators according to

$$\begin{aligned} & \begin{array}{cc} P & P \\ \sigma_1 \rightarrow \sigma_2, & \sigma_2 \rightarrow \sigma_1, \end{array} \\ & \begin{array}{cc} R & R \\ \sigma_1 \otimes \mathbb{I} \rightarrow \sigma_3 \otimes \sigma_2, & \sigma_2 \otimes \mathbb{I} \rightarrow \sigma_2 \otimes \mathbb{I}, \end{array} \\ & \begin{array}{cc} R & R \\ \mathbb{I} \otimes \sigma_1 \rightarrow \sigma_2 \otimes \sigma_3, & \mathbb{I} \otimes \sigma_2 \rightarrow \mathbb{I} \otimes \sigma_2. \end{array} \end{aligned}$$

It is straightforward to determine the action of the  $R_j^+$  for all  $j=1, \dots, 2n-1$ . Since the corresponding  $R_j$  do not effect qu-

bit  $n+1$  we get nontrivial relations only for qubits  $j$  and  $j+1$ ,

$$\begin{aligned} R_{2j-1}^+ &: \tau_1^{(j)} \rightarrow \tau_2^{(j)}, \quad \tau_2^{(j)} \rightarrow \tau_1^{(j)}, \\ R_{2j}^+ &: \tau_1^{(j)} \rightarrow \tau_3^{(j)} \tau_2^{(j+1)}, \quad \tau_2^{(j)} \rightarrow \tau_2^{(j)}, \\ & \tau_1^{(j+1)} \rightarrow \tau_2^{(j)} \tau_3^{(j+1)}, \quad \tau_2^{(j)} \rightarrow \tau_2^{(j)}. \end{aligned}$$

The action of the last two operators can be calculated by utilizing

$$\sigma_3^{\otimes n+1} \mathcal{R}_+ \equiv \mathbb{I} \mathcal{R}_+.$$

It follows that

$$\begin{aligned} \tau_1^{(k)} &\xrightarrow{R_{2n+1}^+} \tau_2^{(k)} \prod_{r \neq k} \tau_3^{(r)}, \\ \tau_2^{(k)} &\xrightarrow{R_{2n+1}^+} \tau_1^{(k)} \prod_{r \neq k} \tau_3^{(r)}, \end{aligned}$$

which can be verified as follows:

$$\begin{aligned} (R_{2n+1}^+)^* \tau_1^{(k)} R_{2n+1}^+ &= (R_{2n+1}^+)^* \sigma_1^{(k)} \sigma_1^{(n+1)} R_{2n+1}^+ \mathcal{R}_+ \\ &= \sigma_1^{(k)} (R_{2n+1}^+)^* \sigma_1^{(n+1)} R_{2n+1}^+ \mathcal{R}_+ \\ &= \sigma_1^{(k)} \sigma_2^{(n+1)} \mathcal{R}_+ \\ &= \sigma_1^{(k)} \sigma_2^{(n+1)} \mathcal{R}_+ \mathbb{I} \mathcal{R}_+ \\ &= \sigma_1^{(k)} \sigma_2^{(n+1)} \sigma_3^{\otimes n+1} \mathcal{R}_+ \\ &= \sigma_3^{\otimes k-1} \otimes \sigma_2 \otimes \sigma_3^{\otimes n-k} \otimes \sigma_1 \mathcal{R}_+ \end{aligned}$$

and similarly for  $\tau_2^{(k)}$ . The relations for  $R_{2n}^+$  can be calculated in the same way, but we will skip the proof and give only the result. If  $k \leq n-1$  we have

$$\begin{aligned} \tau_1^{(k)} &\xrightarrow{R_{2n+1}^+} \left( \prod_{r \neq k, n} \tau_3^{(r)} \right) \tau_2^{(k)} \tau_1^{(n)}, \\ \tau_2^{(k)} &\xrightarrow{R_{2n+1}^+} - \left( \prod_{r \neq k, n} \tau_3^{(r)} \right) \tau_1^{(k)} \tau_1^{(n)}. \end{aligned}$$

The case  $k=n$  gives

$$\begin{aligned} \tau_1^{(n)} &\xrightarrow{R_{2n+1}^+} \tau_1^{(n)}, \\ \tau_1^{(k)} &\xrightarrow{R_{2n+1}^+} \prod_{r=1}^n \tau_3^{(r)}. \end{aligned}$$

From these equations we can read off the corresponding symplectic matrices

$$S_{2i-1} = \begin{bmatrix} \mathbb{I}_{2(i-1)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{2(n-i)} \end{bmatrix}$$

for  $1 \leq 2i-1 \leq 2n-1$  and

$$S_{2i} = \begin{bmatrix} \mathbb{I}_{2(i-1)} & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 & \\ & 1 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_{2(n-i-1)} \end{bmatrix}$$

for  $1 \leq 2i \leq 2n-2$ . The last two matrices read

$$S_{2n} = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 0 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 0 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & 0 & 1 \\ \vdots & & & & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

$$S_{2n+1} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 0 & \\ \vdots & \vdots & & & & & \ddots \end{bmatrix}.$$

### APPENDIX C: FAITHFULNESS OF THE REPRESENTATION OF THE PERMUTATION GROUP

In this appendix we will use the symplectic description of the braid generators given in Appendix B to prove that the equivalence classes (i.e., the cosets of the monodromy subgroup) of the elementary braid generators are all different and hence they form a faithful representation of the symmetric group. According to Sec. V we have the relation

$$\text{image}(\mathcal{B}_{2n+2})/\text{image}(\mathcal{M}_{2n+2}) \subset \text{Sp}_{2n}(2)$$

and therefore the advantage of the symplectic description is that symplectic matrices already describe classes of braiding operators with respect to the equivalence relation in which two elements are equivalent if they differ by a monodromy transformation only and therefore it is natural to compare these matrices directly.

Recall that a representation of the permutation group  $\mathcal{S}_{2n+2}$  is by definition generated by  $2n+1$  matrices

$$S_j, \quad 1 \leq j \leq 2n+1,$$

satisfying the relations

$$S_j S_{j+1} S_j = S_{j+1} S_j S_{j+1},$$

$$S_j S_k = S_k S_j, \quad |k-j| > 1,$$

$$(S_j)^2 = \mathbb{I},$$

and this representation is faithful if and only if each generator  $S_j$  is independent of the others with respect to the group multiplication. We denote the elementary generators of the abstract group  $\mathcal{S}_{2n+2}$  by  $\pi_j$ . It is understood that each  $S_j$  represents  $\pi_j$ .

It is easy to see that the matrices  $S_j$ , which we derived in Appendix B, indeed satisfy the above relations. To further analyze the representation given by the  $S_j$  it is convenient to change the basis in the symplectic space according to

$$\underbrace{(0, \dots, 0, 1, 0, 0, \dots, 0)}_{2i} \rightarrow \underbrace{(0, \dots, 0, 1, 0, 1, \dots, 1)}_{2i} \underbrace{, \dots, 1}_{2(n-i-1)},$$

$$\underbrace{(0, \dots, 0, 0, 1, 0, \dots, 0)}_{2i} \rightarrow \underbrace{(0, \dots, 0, 0, 1, 1, \dots, 1)}_{2i} \underbrace{, \dots, 1}_{2(n-i-1)}.$$

To obtain the matrices  $S_j$  in this new basis we have to multiply them from both sides with the (self-inverse over  $\mathbb{F}_2$ ) matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \end{bmatrix},$$

which changes between both bases. The new symplectic matrices  $\tilde{S}_j = T S_j T$  are

$$\tilde{S}_{2i-1} = \begin{bmatrix} \mathbb{I}_{2(i-1)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{2(n-i)} \end{bmatrix}$$

for  $1 \leq 2i-1 \leq 2n-1$ ,

$$\tilde{S}_{2i} = \begin{bmatrix} \mathbb{I}_{2i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{2(n-i)-1} \end{bmatrix}$$

for  $1 \leq 2i \leq 2n-2$ , and the last two are

$$\tilde{S}_{2n} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 1 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 1 & & 0 & 1 \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 1 \end{bmatrix},$$

$$\tilde{S}_{2n+1} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 0 & \\ \vdots & \vdots & & & & & \ddots \end{bmatrix}.$$

It is now obvious that the  $\tilde{S}_j$  for  $j=1, \dots, 2n-1$  represent elementary transpositions. Moreover, it is clear that  $\tilde{S}_{2n}$  can never be constructed solely from the  $\tilde{S}_j$  for  $j=1, \dots, 2n-1$  because of its nontrivial last column. Therefore the operators  $\tilde{S}_1, \dots, \tilde{S}_{2n}$  generate a faithful representation of  $\mathcal{S}_{2n+1}$ .

If  $n=1$  the two  $2 \times 2$  matrices  $\tilde{S}_3$  and  $\tilde{S}_1$  coincide; hence we get a faithful representation of  $\mathcal{S}_3$ . However, for  $n \geq 2$  a

simple argument shows that  $\tilde{S}_{2n+1}$  is independent from all other  $\tilde{S}_j$ . Indeed, suppose there is a way of writing  $\tilde{S}_{2n+1}$  as a product of the  $\tilde{S}_j$  with  $j \leq 2n$ ,

$$\tilde{S}_{2n+1} = \tilde{S}_{\alpha_1} \cdots \tilde{S}_{\alpha_r}, \quad 1 \leq \alpha_i \leq 2n.$$

The right-hand side corresponds to a permutation  $\pi \in \mathcal{S}_{2n+2}$  whose expression in elementary group generators  $\pi_j$  reads

$$\pi \equiv \pi_{\alpha_1} \cdots \pi_{\alpha_r}.$$

Alternatively, the element  $\pi$  can always be written as

$$\pi \equiv \left( \prod_s \pi_{\alpha_s} \right) \pi_{2n} \left( \prod_t \pi_{\beta_t} \right), \quad \alpha_s, \beta_t \leq 2n-1.$$

Therefore we also have the relation

$$\tilde{S}_{2n+1} \equiv \left( \prod_s \tilde{S}_{\alpha_s} \right) \tilde{S}_{2n} \left( \prod_t \tilde{S}_{\beta_t} \right).$$

Now observe that multiplying  $\tilde{S}_{2n}$  with  $\tilde{S}_j$ 's with  $j \leq 2n-1$  simply exchanges rows and columns in  $\tilde{S}_{2n}$ . The number of 1's in  $\tilde{S}_{2n}$  is left unchanged by exchanging rows or columns. Therefore, it is obvious that  $\tilde{S}_{2n+1}$  cannot be constructed like that from  $\tilde{S}_{2n}$  because the number of 1's of both matrices differs. Hence, we have shown that the representation of the permutation group  $\mathcal{S}_{2n+2}$  is faithful for all  $n \geq 2$ .

---

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[2] A. Kitaev, *Ann. Phys. (N.Y.)* **303**, 2 (2003).

[3] R. Alicki, M. Fannes, and M. Horodecki, *Math. Syst. Theory* **40**, 6451 (2007).

[4] Z. Nussinov and G. Ortiz, *Phys. Rev. B* **77**, 064302 (2008).

[5] Z. Nussinov and G. Ortiz, e-print arXiv:cond-mat/0702377.

[6] S. Das Sarma, M. Freedman, and C. Nayak, *Phys. Rev. Lett.* **94**, 166802 (2005).

[7] G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).

[8] C. Nayak and F. Wilczek, *Nucl. Phys. B* **479**, 529 (1996).

[9] N. Read, *J. Math. Phys.* **44**, 558 (2003).

[10] L. S. Georgiev, *Phys. Rev. B* **74**, 235112 (2006).

[11] L. S. Georgiev, *Nucl. Phys. B* **789**, 552 (2008).

[12] L. S. Georgiev, e-print arXiv:0812.2334.

[13] L. S. Georgiev, e-print arXiv:0812.2337.

[14] D. Gottesman, in *Hobart 1998, Group Theoretical Methods in Physics*, pp. 32–43, e-print arXiv:quant-ph/9807006.

[15] S. Aaronson and D. Gottesman, *Phys. Rev. A* **70**, 052328 (2004).

[16] D. Gottesman, Ph.D. thesis, Caltech, 1997.

[17] D. M. Greenberger, M. A. Horne, A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (Kluwer Academic, Dordrecht, 1989), pp. 73–76.

[18] R. Raussendorf, D. E. Browne, and H. J. Briegel, *Phys. Rev. A* **68**, 022312 (2003).

[19] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).

[20] A. Stern, *Ann. Phys.* **323**, 204 (2008).

[21] T. Mansour and S. Severini, e-print arXiv:0806.4809.

[22] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).

[23] P. Furlan, G. Sotkov, and I. Todorov, *Riv. Nuovo Cimento* **12**, 1 (1989).

[24] L. S. Georgiev, *Nucl. Phys. B* **651**, 331 (2003).

[25] D. A. Ivanov, *Phys. Rev. Lett.* **86**, 268 (2001).

[26] F. Wilczek and A. Zee, *Phys. Rev. D* **25**, 553 (1982).

[27] J. K. Slingerland and F. A. Bais, *Nucl. Phys. B* **612**, 229 (2001).

[28] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, *IEEE Trans. Inf. Theory* **44**, 1369 (1998).

[29] J. S. Birman, *Braids, Links and Mapping Class Groups*, 82nd ed., *Annals of Mathematical Studies* (Princeton University Press, Princeton, NJ, 1974).

[30] I. T. Todorov and Y. S. Stanev, *Chiral Current Algebras and Two-Dimensional Conformal Models* (Troisième Cycle de la Physique, En Suisse Romande, Université de Lausanne, Lausanne, 1992).

[31] R. A. Wilson, *The Finite Simple Groups* (Springer, Berlin, 2007).

[32] M. Planat and P. Jorrand, *J. Phys A* **41**, 182001 (2008).

[33] We thank Sergey Bravyi for pointing out this fact to us.

[34] The operators  $R_j$  and  $P, R$  differ only by a global phase factor, which means that the corresponding elements in the projective Clifford group  $\mathcal{PC}_n$  coincide.