Renormalized coordinate approach to the thermalization process

G. Flores-Hidalgo*
Instituto de Ciências Exatas, Universidade Federal de Itajubá, 37500-903, Itajubá, MG, Brazil

A. P. C. Malbouisson†
Centro Brasileiro de Pesquisas Físicas/MCT, 22290-180, Rio de Janeiro, RJ, Brazil

J. M. C. Malbouisson‡
Instituto de Física, Universidade Federal da Bahia, 40.210-310, Salvador (BA), Brazil

Y. W. Milla§
Instituto de Física Teorica, Universidade Estadual Paulista, 01405-900, São Paulo (SP), Brazil

A. E. Santana‖
Instituto de Física, Universidade de Brasília, 70910-900, Brasília-DF, Brazil

(Received 11 September 2008; revised manuscript received 5 February 2009; published 9 March 2009)

We consider a particle in the harmonic approximation coupled linearly to an environment modeled by an infinite set of harmonic oscillators. The system (particle environment) is considered in a cavity at thermal equilibrium. We employ the recently introduced notion of renormalized coordinates to investigate the time evolution of the particle occupation number. For comparison, we first present this study in bare coordinates. For a long elapsed time, in both approaches, the occupation number of the particle becomes independent of its initial value. The value of the occupation number of the particle is the physically expected one at the given temperature. So we have a Markovian process, describing the particle thermalization with the environment. With renormalized coordinates, no renormalization procedure is required, leading directly to a finite result.

DOI: 10.1103/PhysRevA.79.032105 PACS number(s): 03.65.Yz, 03.65.Ca

I. INTRODUCTION

A thermalization process occurs in some cases for a system of material particles coupled to an environment, in the sense that after an infinitely long time, the matter particles lose the memory of their initial states. This study is, in general, not easy from a theoretical point of view, due to the complex nonlinear character of the interactions between the matter particles and the environment. To get over these difficulties, linearized models have been adopted. An account on the subject of the evolution of quantum systems on general grounds can be found in [1–6]. Besides, the main analytical method used to treat these systems at zero or finite temperature is—except for a few special cases—the perturbation theory. In this framework, the perturbative approach is carried out by means of the introduction of *bare* noninteracting objects (fields, to which are associated bare quanta); the interaction being introduced order by order in powers of the coupling constant.

In spite of the remarkable achievements of the perturbative methods, however, there are situations where they cannot be employed or are of little use. These cases have led to attempts to improve nonperturbative *analytical* methods, in particular, where strong effective couplings are involved. Among these trials, there are methods that perform resummations of perturbative series, even if they are divergent, which amounts in some cases to extending the weak-coupling regime to a strong-coupling domain. One of these methods is the Borel resummation of perturbative series [7–12].

In this paper, we follow a different nonperturbative approach. We investigate a simplified linear version of a particle field or particle-environment system, where the particle—taken in the harmonic approximation—is coupled to the reservoir modeled by independent harmonic oscillators [2,3,5]. We will employ, in particular, *dressed* states and *renormalized* coordinates introduced in [13] and already employed in [14–17]. Using this method, nonperturbative treatments can be considered for both weak and strong couplings. A linear model permits a better understanding of the need for nonperturbative analytical treatments of coupled systems, which is the basic problem underlying the idea of a *dressed* quantum-mechanical system. Of course, the use of such an approach to a realistic nonlinear system is an extremely hard task, while the linear model provides a good compromise between physical reality and mathematical reliability. The whole system is supposed to reside inside a spherical cavity of radius R in thermal equilibrium at temperature $T=\beta^{-1}$. In other words, we consider the spatially regularized theory (finite R) at finite temperature. The free space case is obtained by suppressing the regulator ($R\to\infty$). For a detailed comparison between this procedure and the one considering an *a priori* unbounded space, see [13].

*gflores@ift.unesp.br
†adolfo@cbpf.br
‡jmalboui@ufba.br
§yonym@ift.unesp.br
‖asantana@fis.unb.br

II. MODEL

Let us start by considering a particle approximated by a harmonic oscillator, having *bare* frequency ω_0, linearly coupled to a set of N other harmonic oscillators, with frequencies ω_k, $k=1,2,\ldots,N$. The Hamiltonian for such a system is written in the form,

$$H=\frac{1}{2}\left[p_0^2+\omega_0^2q_0^2+\sum_{k=1}^{N}(p_k^2+\omega_k^2q_k^2)\right]-q_0\sum_{k=1}^{N}c_kq_k, \quad (1)$$

leading to the following equations of motion:

$$\ddot{q}_0+\omega_0^2q_0=\sum_{i=1}^{N}c_iq_i(t), \quad (2)$$

$$\ddot{q}_i+\omega_i^2q_i=c_iq_0(t). \quad (3)$$

In the limit $N\to\infty$, we recover our case of the particle coupled to the environment, after redefining divergent quantities, in a manner analogous to mass renormalization in field theories. A Hamiltonian of the type (1) has been largely used in the literature, in particular, to study the quantum Brownian motion with the path-integral formalism [1,2]. It has also been employed to investigate the linear coupling of a particle to the scalar potential [13–17].

The Hamiltonian (1) is transformed to the principal axis by means of a point transformation,

$$q_\mu=\sum_{r=0}^{N}t_\mu^rQ_r, \quad p_\mu=\sum_{r=0}^{N}t_\mu^rP_r,$$

$$\mu=(0,\{k\}), \quad k=1,2,\ldots,N, \quad r=0,\ldots N, \quad (4)$$

performed by an orthonormal matrix $T=(t_\mu^r)$. The subscripts $\mu=0$ and $\mu=k$ refer, respectively, to the particle and the harmonic modes of the reservoir and r refers to the normal modes. In terms of normal momenta and coordinates, the transformed Hamiltonian reads as

$$H=\frac{1}{2}\sum_{r=0}^{N}(P_r^2+\Omega_r^2Q_r^2), \quad (5)$$

where the Ω_r's are the normal frequencies corresponding to the collective *stable* oscillation modes of the coupled system. Using the coordinate transformation (4) in the equations of motion and explicitly making use of the normalization of the matrix (t_μ^r), $\Sigma_{\mu=0}^{N}(t_\mu^r)^2=1$, we get

$$t_k^r=\frac{c_k}{\omega_k^2-\Omega_r^2}t_0^r, \quad t_0^r=\left[1+\sum_{k=1}^{N}\frac{c_k^2}{(\omega_k^2-\Omega_r^2)^2}\right]^{-1/2}, \quad (6)$$

with the condition

$$\omega_0^2-\Omega_r^2=\sum_{k=1}^{N}\frac{c_k^2}{\omega_k^2-\Omega_r^2}. \quad (7)$$

We take $c_k=\eta(\omega_k)^u$, where η is a constant independent of k. In this case, the environment is classified according to $u>1$, $u=1$, or $u<1$, respectively, as *supraohmic*, *ohmic*, or *subohmic*. This terminology has been used in studies of the quantum Brownian motion and of dissipative systems [2–6]. For a subohmic environment, the sum in Eq. (7) is convergent in the limit $N\to\infty$ and the frequency ω_0 is well defined. For ohmic and supraohmic environments, this sum diverges for $N\to\infty$. This makes the equation meaningless, unless a renormalization procedure is implemented. From now on, we restrict ourselves to an ohmic system. In this case, Eq. (7) is written in the form

$$\omega_0^2-\delta\omega^2-\Omega_r^2=\eta^2\Omega_r^2\sum_{k=1}^{N}\frac{1}{\omega_k^2-\Omega_r^2}, \quad (8)$$

where we have defined the counterterm

$$\delta\omega^2=N\eta^2. \quad (9)$$

There are $N+1$ solutions of Ω_r, corresponding to the $N+1$ normal collective modes. Let us for a moment suppress the index r of Ω_r^2. If $\omega_0^2>\delta\omega^2$, all possible solutions for Ω^2 are positive, physically meaning that the system oscillates harmonically in all its modes. If $\omega_0^2<\delta\omega^2$ then a single negative solution exists. In order to prove this, let us define the function

$$I(\Omega^2)=\omega_0^2-\delta\omega^2-\Omega^2-\eta^2\Omega^2\sum_{k=1}^{N}\frac{1}{\omega_k^2-\Omega^2}, \quad (10)$$

so that Eq. (8) becomes $I(\Omega^2)=0$. We find that

$$I(\Omega^2)\to\infty \quad \text{as} \quad \Omega^2\to-\infty, \quad I(0)=\omega_0^2-\delta\omega^2<0,$$

in the interval $(-\infty,0]$. As $I(\Omega^2)$ is a monotonically decreasing function in this interval, we conclude that $I(\Omega^2)=0$ has a single negative solution in this case. This means that there is a mode whose amplitude grows or decays exponentially, so that no stationary configuration is allowed. Nevertheless, it should be remarked that in a different context, it is precisely this runaway solution that is related to the existence of a bound state in the Lee-Friedrichs model. This solution is considered in the framework of a model to describe qualitatively the existence of bound states in particle physics [18].

Considering the situation where all normal modes are harmonic, which corresponds to the first case above ($\omega_0^2>\delta\omega^2$), we define the *renormalized* frequency

$$\bar{\omega}^2=\omega_0^2-\delta\omega^2=\lim_{N\to\infty}(\omega_0^2-N\eta^2), \quad (11)$$

in terms of which Eq. (8) in the limit $N\to\infty$ becomes

$$\bar{\omega}^2-\Omega^2=\eta^2\sum_{k=1}^{\infty}\frac{\Omega^2}{\omega_k^2-\Omega^2}. \quad (12)$$

In this limit, the above procedure is exactly the analog of the mass renormalization in quantum field theory. The addition of a counterterm $-\delta\omega^2q_0^2$ allows one to compensate the infinity of ω_0^2 in such a way as to leave a finite physically meaningful renormalized frequency $\bar{\omega}$. This simple renormalization scheme has been introduced earlier [19]. Unless explicitly stated, the limit $N\to\infty$ is understood in the following.

Let us define a constant g, with dimension of frequency, by

$$g = \frac{\eta^2}{2\Delta\omega}, \tag{13}$$

where $\Delta\omega = \pi c/R$. The environment frequencies ω_k are given by

$$\omega_k = k\frac{\pi c}{R}, \quad k = 1,2,\ldots, \tag{14}$$

where R is the radius of the cavity that contains the whole system. Then, using the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - u^2} = \frac{1}{2}\left[\frac{1}{u^2} - \frac{\pi}{u}\cot(\pi u)\right], \tag{15}$$

Eq. (12) can be written in a closed form,

$$\cot\left(\frac{R\Omega}{c}\right) = \frac{\Omega}{\pi g} + \frac{c}{R\Omega}\left(1 - \frac{R\bar{\omega}^2}{\pi g c}\right). \tag{16}$$

The solutions of the above equation with respect to Ω give the spectrum of eigenfrequencies Ω_r, corresponding to the collective normal modes.

In terms of the physically meaningful quantities Ω_r and $\bar{\omega}$, the transformation matrix elements turning the particle-field system to the principal axis are obtained. They are

$$t_0^r = \frac{\eta\Omega_r}{\sqrt{(\Omega_r^2 - \bar{\omega}^2)^2 + \frac{\eta^2}{2}(3\Omega_r^2 - \bar{\omega}^2) + \pi^2 g^2 \Omega_r^2}},$$

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r. \tag{17}$$

These matrix elements play a central role in the quantities describing the system.

III. THERMALIZATION PROCESS IN BARE COORDINATES

We now consider the thermalization problem using bare coordinates. For the model described by Eq. (1), this problem was addressed in an alternative way in [20] with the canonical Liouville–von Neumann formalism. We consider the initial state described by the density operator,

$$\rho(t=0) = \rho_0 \otimes \rho_\beta, \tag{18}$$

where ρ_0 is the density operator of the particle that in principle can be in a pure or in a mixed state and ρ_β is the density operator of the thermal bath, at a temperature β^{-1}, that is,

$$\rho_\beta = Z_\beta^{-1} \exp\left[-\beta\sum_{k=1}^{\infty} \omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right)\right], \tag{19}$$

with $Z_\beta = \Pi_{k=1}^N z_\beta^k$ being the partition function of the reservoir, and

$$z_\beta^k = \mathrm{Tr}_k[e^{-\beta\omega_k(a_k^\dagger a_k + 1/2)}] = \frac{1}{2\sinh\left(\frac{\beta_k\omega_k}{2}\right)}. \tag{20}$$

The creation and annihilation operators given by

$$a_\mu = \sqrt{\frac{\bar{\omega}_\mu}{2}} q_\mu + \frac{i}{\sqrt{2\bar{\omega}_\mu}} p_\mu, \tag{21}$$

$$a_\mu^\dagger = \sqrt{\frac{\bar{\omega}_\mu}{2}} q_\mu - \frac{i}{\sqrt{2\bar{\omega}_\mu}} p_\mu, \tag{22}$$

where $\bar{\omega}_\mu = (\bar{\omega}, \omega_k)$. The thermalization problem is addressed by investigating the time evolution of the state $\rho(t)$.

The thermalization problem concerns the time evolution of the initial state to thermal equilibrium. The subsystem corresponding to the particle oscillator is described by an arbitrary density operator ρ_0. As we will show, the expectation value of the number operator corresponding to particles will evolve in time to a value that is independent of the initial density operator ρ_0; the dependence will be exclusively on the mixed density operator corresponding to the thermal bath.

Our aim is to obtain expressions for the time evolution of the expectation values for the occupation number and, in particular, for the one corresponding to particles. We will solve the problem in the framework of the Heisenberg picture. It is to be understood that when a quantity appears without the time argument, it means that such quantity is evaluated at $t=0$. The Heisenberg equation of motion for the annihilation operator $a_\mu(t)$ is given by

$$\frac{\partial}{\partial t} a_\mu(t) = i[\hat{H}, a_\mu(t)]. \tag{23}$$

Due to the linear character of our problem, this equation is solved by writing $a_\mu(t)$ as

$$a_\mu(t) = \sum_{\nu=0}^{\infty} [\dot{B}_{\mu\nu}(t)\hat{q}_\nu + B_{\mu\nu}(t)\hat{p}_\nu], \tag{24}$$

where all the time dependence is in the c-number functions $B_{\mu\nu}(t)$. Then, Eq. (23) reduces to the following coupled equations for $B_{\mu\nu}(t)$:

$$\ddot{B}_{\mu 0}(t) + \bar{\omega}^2 B_{\mu 0}(t) - \sum_{k=1}^{\infty} \eta\omega_k B_{\mu k}(t) = 0, \tag{25}$$

$$\ddot{B}_{\mu k}(t) + \omega_k^2 B_{\mu k}(t) - B_{\mu 0}(t)\sum_{k=1}^{\infty} \eta\omega_k = 0. \tag{26}$$

These equations are formally identical to the classical equations of motion [Eqs. (2) and (3)] for the bare coordinates q_μ. Then we decouple Eqs. (25) and (26) with the same matrix $\{t_\mu^r\}$ that diagonalizes the Hamiltonian (1). In an analogous manner, we write $B_{\mu\nu}(t)$ as

$$B_{\mu\nu}(t) = \sum_{r=0}^{\infty} t_\nu^r C_\mu^r(t), \tag{27}$$

such that from Eqs. (25) and (26), we obtain the following equations for the normal-axis functions $C_\mu^r(t)$,

$$\ddot{C}_\mu^r(t) + \Omega_r^2 C_\mu^r(t) = 0, \tag{28}$$

which gives the solution

$$C_\mu^r(t) = a_\mu^r e^{i\Omega_r t} + b_\mu^r e^{-i\Omega_r t}.$$

Then substituting this expression into Eq. (27), we find

$$B_{\mu\nu}(t) = \sum_{r=0}^{\infty} t_\nu^r (a_\mu^r e^{i\Omega_r t} + b_\mu^r e^{-i\Omega_r t}). \tag{29}$$

The time-independent coefficients a_μ^r and b_μ^r are determined by the initial conditions at $t=0$ for $B_{\mu\nu}(t)$ and $\dot{B}_{\mu\nu}(t)$. From Eqs. (21) and (24), we find that these initial conditions are given by

$$B_{\mu\nu} = \frac{i\delta_{\mu\nu}}{\sqrt{2\bar{\omega}_\mu}},$$

$$\dot{B}_{\mu\nu} = \sqrt{\frac{\bar{\omega}_\mu}{2}}\delta_{\mu\nu}. \tag{30}$$

Using these equations, we obtain for a_μ^r and b_μ^r,

$$a_\mu^r = \frac{it_\mu^r}{\sqrt{8\bar{\omega}_\mu}}\left(1 - \frac{\bar{\omega}_\mu}{\Omega_r}\right), \tag{31}$$

$$b_\mu^r = \frac{it_\mu^r}{\sqrt{8\bar{\omega}_\mu}}\left(1 + \frac{\omega_\mu}{\Omega_r}\right). \tag{32}$$

We write $a_\mu(t)$ and $a_\mu^\dagger(t)$ in terms of a_μ and $a_\mu^\dagger$ using Eqs. (21), (22), and (24),

$$a_\mu(t) = \sum_{\nu=0}^{\infty} [\alpha_{\mu\nu}(t)\hat{a}_\nu + \beta_{\mu\nu}(t)\hat{a}_\nu^\dagger], \tag{33}$$

$$a_\mu^\dagger(t) = \sum_{\nu=0}^{\infty} [\beta_{\mu\nu}^*(t)\hat{a}_\nu + \alpha_{\mu\nu}^*(t)\hat{a}_\nu^\dagger], \tag{34}$$

where $\alpha_{\mu\nu}(t)$ and $\beta_{\mu\nu}(t)$ are the Bogoliubov coefficients given by

$$\alpha_{\mu\nu}(t) = \frac{1}{\sqrt{2\omega_\nu}}\dot{B}_{\mu\nu}(t) - i\sqrt{\frac{\omega_\nu}{2}}B_{\mu\nu}(t) \tag{35}$$

and

$$\beta_{\mu\nu}(t) = \frac{1}{\sqrt{2\omega_\nu}}\dot{B}_{\mu\nu}(t) + i\sqrt{\frac{\omega_\nu}{2}}B_{\mu\nu}(t). \tag{36}$$

Using the definition of $B_{\mu\nu}(t)$, we get

$$\alpha_{\mu\nu}(t) = \sum_{r=0}^{\infty} \sqrt{\frac{\omega_\nu}{\omega_\mu}}\frac{t_\mu^r t_\nu^r}{4\Omega_r}\left\{\frac{\Omega_r}{\omega_\nu}[(\omega_\mu - \Omega_r)e^{i\Omega_r t} + (\omega_\mu + \Omega_r)e^{-i\Omega_r t}] + [(\Omega_r - \omega_\mu)e^{i\Omega_r t} + (\Omega_r + \omega_\mu)e^{-i\Omega_r t}]\right\} \tag{37}$$

and

$$\beta_{\mu\nu}(t) = \sum_{r=0}^{\infty} \sqrt{\frac{\omega_\nu}{\omega_\mu}}\frac{t_\mu^r t_\nu^r}{4\Omega_r}\left\{\frac{\Omega_r}{\omega_\nu}[(\omega_\mu - \Omega_r)e^{i\Omega_r t} + (\omega_\mu + \Omega_r)e^{-i\Omega_r t}] - [(\Omega_r - \omega_\mu)e^{i\Omega_r t} + (\Omega_r + \omega_\mu)e^{-i\Omega_r t}]\right\}. \tag{38}$$

Now we study the time evolution of $n_\mu(t)$, the expectation value of the number operator $N_\mu(t) = a_\mu^\dagger(t)a_\mu(t)$, that is,

$$n_\mu(t) = \mathrm{Tr}[a_\mu^\dagger(t)a_\mu(t)\rho_0 \otimes \rho_\beta]. \tag{39}$$

Using the basis $|n_0, n_1, n_2, \ldots, n_N\rangle$, we obtain

$$n_\mu(t) = \sum_{\nu=0}^{\infty} [|\alpha_{\mu\nu}(t)|^2 + |\beta_{\mu\nu}(t)|^2]n_\nu + \sum_{\nu=0}^{\infty} |\beta_{\mu\nu}(t)|^2, \tag{40}$$

where

$$n_0 = \sum_{n=0}^{\infty} n\langle n|\rho_0|n\rangle \tag{41}$$

is the expectation value of the number operator corresponding to the particle and the set $\{n_k\}$ stands for the thermal expectation values corresponding to the thermal bath oscillators given by the Bose-Einstein distribution,

$$n_k = \frac{1}{e^{\beta\omega_k} - 1}. \tag{42}$$

We are interested in evaluating the expectation value of the number operator corresponding to the particle. Strictly speaking all the modes of the system evolves in time, so that all occupation numbers $n_\mu(t)$ given by Eq. (40) should be considered. However the mode $\mu=0$, corresponding to the particle, is coupled to all of the reservoir modes; while each reservoir mode ($\mu=1,2,3,\ldots$) is under the influence of the particle only, since they are not supposed to interact directly among themselves. Therefore, considering the weak-coupling regime [see comments below Eq. (48)], we work in

the approximation of neglecting the time evolution of the reservoir which remains in thermal equilibrium obeying Eq. (42). Thus taking $\mu=0$ in Eq. (40) and using Eq. (42), we obtain

$$n_0(t) = [|\alpha_{00}(t)|^2 + |\beta_{00}(t)|^2]n_0 + \sum_{k=1}^{\infty} [|\alpha_{0k}(t)|^2 + |\beta_{0k}(t)|^2]\frac{1}{e^{\beta\omega_k}-1} + |\beta_{00}(t)|^2 + \sum_{k=1}^{\infty} |\beta_{0k}(t)|^2, \tag{43}$$

where the coefficients of this expression are [20]

$$\alpha_{00}(t) = \frac{e^{-\pi g t/2}}{16\bar{\omega}\kappa}[(2\bar{\omega}+2\kappa-i\pi g)^2 e^{-i\kappa t} - (2\bar{\omega}-2\kappa-i\pi g)^2 e^{i\kappa t}], \tag{44}$$

$$\beta_{00}(t) = \frac{\pi g e^{-\pi g t/2}}{8\bar{\omega}\kappa}[(\pi g + 2i\kappa)e^{-i\kappa t} - (\pi g - 2i\kappa)e^{i\kappa t}], \tag{45}$$

$$\alpha_{0k}(t) = \sqrt{\frac{\omega_k}{2\bar{\omega}}}\frac{(\bar{\omega}+\omega_k)\sqrt{g\Delta\omega}e^{-i\omega_k t}}{(\omega_k^2 - \bar{\omega}^2 + i\pi g\omega_k)} + \sqrt{\frac{\omega_k}{\bar{\omega}}}\frac{\sqrt{2g\Delta\omega}}{4\kappa}\left[\frac{(2\kappa+2\bar{\omega}-i\pi g)}{(2\kappa-2\omega_k-i\pi g)}e^{-i\kappa t} + \frac{(2\bar{\omega}-2\kappa-i\pi g)}{(2\kappa+2\omega_k+i\pi g)}e^{i\kappa t}\right]e^{-\pi g t/2} \tag{46}$$

and

$$\beta_{0k}(t) = \sqrt{\frac{\omega_k}{2\bar{\omega}}}\frac{(\omega_k-\bar{\omega})\sqrt{g\Delta\omega}e^{i\omega_k t}}{(\omega_k^2 - \bar{\omega}^2 - i\pi g\omega_k)} - \sqrt{\frac{\omega_k}{\bar{\omega}}}\frac{\sqrt{2g\Delta\omega}}{4\kappa}\left[\frac{(2\bar{\omega}+2\kappa-i\pi g)}{(2\kappa+2\omega_k-i\pi g)}e^{-i\kappa t} + \frac{(2\bar{\omega}-2\kappa-i\pi g)}{(2\kappa-2\omega_k+i\pi g)}e^{i\kappa t}\right]e^{-\pi g t/2}, \tag{47}$$

such that

$$\kappa = \sqrt{\bar{\omega}^2 - \pi^2 g^2/4}. \tag{48}$$

The parameter κ measures the intensity of the interaction: if $\kappa^2 \gg 0$, i.e., $g \ll 2\bar{\omega}/\pi$, we are in the *weak*-coupling regime. On the contrary if $\kappa^2 \ll 0$, i.e., $g \gg 2\bar{\omega}/\pi$, the system is in the *strong*-coupling regime. Here we will restrict ourselves to the weak-coupling regime. This case includes the important class of electromagnetic interactions $g=\alpha\bar{\omega}$, with α being the fine-structure constant $\alpha=1/137$ [14].

In the continuum limit $\Delta\omega \to 0$, the sums over k become integrations over a continuous variable ω and we obtain for $n_0(t)$,

$$\begin{aligned} n_0(t) = {} & \frac{e^{-\pi g t}}{\bar{\omega}^2\kappa^2}\left[\bar{\omega}^4 + \frac{\pi^2 g^2}{8}(2\bar{\omega}^2 - \pi^2 g^2)\cos(2\kappa t) - \frac{\pi^3 g^3 \kappa}{4}\sin(2\kappa t)\right]n_0 \\ & + \frac{\pi^2 g^2 e^{-\pi g t}}{16\bar{\omega}^2\kappa^2}[2\bar{\omega}^2 + (2\bar{\omega}^2 - \pi^2 g^2)\cos(2\kappa t) - 2\pi g\kappa\sin(2\kappa t)] + \frac{g}{\bar{\omega}}\int_0^{\infty} d\omega\left[\frac{F(\omega,\bar{\omega},g,t)}{(e^{\beta\omega}-1)} + G(\omega,\bar{\omega},g,t)\right], \end{aligned} \tag{49}$$

where

$$\begin{aligned} F(\omega,\bar{\omega},g,t) = {} & \frac{\omega(\omega^2+\bar{\omega}^2)}{[(\omega^2-\bar{\omega}^2)^2 + \pi^2 g^2\omega^2]}\left\{1 + \frac{e^{-\pi g t}}{4\kappa^2}\left[4\bar{\omega}^2 - \pi^2 g^2\cos(2\kappa t) - 2\pi g\kappa\frac{(\omega^2-\bar{\omega}^2)}{(\omega^2+\bar{\omega}^2)}\sin(2\kappa t)\right]\right. \\ & \left. - \frac{e^{-\pi g t/2}}{\kappa}\left[2\kappa\cos(\omega t)\cos(\kappa t) + \frac{4\omega\bar{\omega}^2}{(\omega^2+\bar{\omega}^2)}\sin(\omega t)\sin(\kappa t) - \pi g\frac{(\omega^2-\bar{\omega}^2)}{(\omega^2+\bar{\omega}^2)}\cos(\omega t)\sin(\kappa t)\right]\right\} \end{aligned} \tag{50}$$

and

$$\begin{aligned} G(\omega,\bar{\omega},g,t) = {} & \frac{\omega(\omega-\bar{\omega})^2}{[(\omega^2-\bar{\omega}^2)^2 + \pi^2 g^2\omega^2]}\left\{1 + \frac{e^{-\pi g t}}{4\kappa^2}\left[4\bar{\omega}^2 + \frac{2\pi^2 g^2\bar{\omega}\omega}{(\omega-\bar{\omega})^2} - \pi^2 g^2\frac{(\omega^2+\bar{\omega}^2)}{(\omega-\bar{\omega})^2}\cos(2\kappa t) - 2\pi g\kappa\frac{(\omega+\bar{\omega})}{(\omega-\bar{\omega})}\sin(2\kappa t)\right]\right. \\ & \left. - \frac{e^{-\pi g t/2}}{\kappa}\left[2\kappa\cos(\omega t)\cos(\kappa t) - 2\bar{\omega}\sin(\omega t)\sin(\kappa t) - \pi g\frac{(\omega+\bar{\omega})}{(\omega-\bar{\omega})}\cos(\omega t)\sin(\kappa t)\right]\right\}. \end{aligned} \tag{51}$$

It is to be noticed that the second and the third lines in Eq. (49) are independent of the initial distribution. Also the integral over $G(\omega,\bar{\omega},g,t)$ is logarithmically divergent. We can understand the origin of these terms in the following way. Suppose that initially, in the absence of the linear interaction, we prepare the system in its ground state, that is, at $t=0$ we have $|0,0,\ldots,0\rangle$. Then, we can compute—in the Heisenberg picture—the time evolution for the expectation value of the number operator corresponding to the particle, that is, $\langle 0,0,\ldots,0|\hat{a}_0^\dagger(t)\hat{a}_0(t)|0,0,\ldots,0\rangle$. We obtain

$$\langle 0,0,\ldots,0|\hat{a}_0^\dagger(t)\hat{a}_0(t)|0,0,\ldots,0\rangle = |\beta_{00}(t)|^2 + \sum_{k=1}^{\infty} |\beta_{0k}(t)|^2, \tag{52}$$

which in the continuum limit gives the second and third lines of Eq. (49). Then, these terms appearing in Eq. (49) are interpreted as the excitations produced from the unstable bare (vacuum) ground state, as a response to the onset of the linear interaction.

The result above is compatible with some results in [5] in the context of quantum dissipative phenomena. In this quoted paper, in the zero-temperature situation, the system is represented by a set of harmonic oscillators. A detailed justification for representing the environment by a set of harmonic oscillators is given in the Appendix C of this reference.

The divergent integral in $G(\omega,\bar{\omega},g,t)$ can be dealt with by a renormalization procedure. The suppression of this term is analogous to the standard Wick ordering in field theory. Thus we write the following *renormalized* expectation value for the particle number operator:

$$\bar{n}_0(t) = K(\bar{\omega},g,t) + \frac{g}{\bar{\omega}}\int_0^\infty d\omega \frac{F(\omega,\bar{\omega},g,t)}{(e^{\beta\omega}-1)}, \tag{53}$$

where

$$K(\bar{\omega},g,t) = \frac{e^{-\pi g t}}{\bar{\omega}^2\kappa^2}\left[\bar{\omega}^4 + \frac{\pi^2 g^2}{8}(2\bar{\omega}^2 - \pi^2 g^2)\cos(2\kappa t) - \frac{\pi^3 g^3 \kappa}{4}\sin(2\kappa t)\right] n_0 + \frac{\pi^2 g^2 e^{-\pi g t}}{16\bar{\omega}^2\kappa^2}[2\bar{\omega}^2 + (2\bar{\omega}^2 - \pi^2 g^2)\cos(2\kappa t) - 2\pi g\kappa \sin(2\kappa t)]. \tag{54}$$

In the limit $t\to\infty$, $\bar{n}_0(t)$ has a well-defined value; that is, the system reaches a final equilibrium state. Also, since $K(\bar{\omega},g,t\to\infty)\to 0$, this final equilibrium state is independent of n_0. The equilibrium expectation value of the number operator corresponding to the particle is independent of its initial value, and the only dependence is on the initial distribution of the thermal bath; that is, the particle thermalizes with the environment. Before the interaction enters into play for $t<0$, $n(t<0)=n_0$, then we have that $K(\bar{\omega},g,t<0)=1$. Taking $t=0$ in Eq. (54), we obtain $K(\bar{\omega},g,t=0)=\frac{1}{\bar{\omega}^2\kappa^2}[\bar{\omega}^4 + \frac{\pi^2 g^2}{8}(2\bar{\omega}^2-\pi^2 g^2)]n_0 + \frac{\pi^2 g^2}{16\bar{\omega}^2\kappa^2}[2\bar{\omega}^2 + (2\bar{\omega}^2-\pi^2 g^2)]$. Thus $K(\bar{\omega},g,t)$ is a discontinuous function of t; the discontinuity appearing just at $t=0$. From the physical standpoint, this discontinuity can be viewed as a response to the sudden onset of the interaction between the particle and the environment.

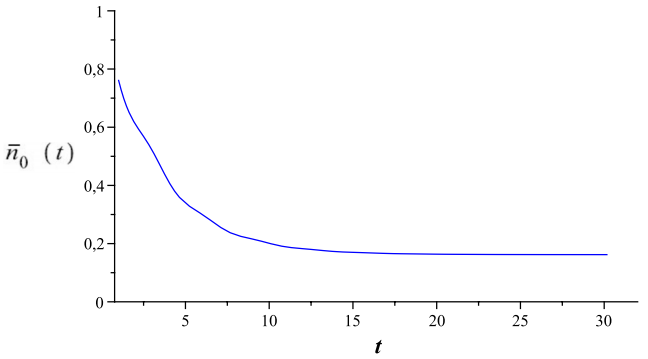

FIG. 1. (Color online) Time behavior for $\bar{n}_0(t)$ given by Eq. (53) for $(t>1)$, $n_0=1$, $\bar{\omega}=1$, $\beta=2$, and $g=0.1$; arbitrary units are used.

It should be mentioned that a very similar problem from the mathematical point of view has been studied in [21]. In this work, the authors studied the damped harmonic oscillator under the optics of a dissipation problem. They apply a method that diagonalizes the Hamiltonian of the system and derive the conditions of validity of the rotating wave approximation.

Although the integral in Eq. (53) cannot be computed analytically, we can perform numerical calculations; for example (in Fig. 1), we display the time behavior for $n_0=1$, $\bar{\omega}=1$, $\beta=2$ and $g=0.1$; $(t>1)$. In Sec. IV, we develop an alternative approach based on the notion of dressed particles. We will find that, in this new realm, no renormalization is needed.

IV. DRESSED COORDINATES AND DRESSED STATES

Let us start with the eigenstates of our system $|n_0,n_1,n_2,\ldots\rangle$ represented by the normalized eigenfunctions in terms of the normal coordinates $\{Q_r\}$,

$$\phi_{n_0 n_1 n_2\ldots}(Q,t) = \prod_s \left[\sqrt{\frac{2^{n_s}}{n_s!}} H_{n_s}\left(\sqrt{\frac{\Omega_s}{\hbar}}Q_s\right)\right] \times \Gamma_0 \exp\left(-i\sum_s n_s\Omega_s t\right), \tag{55}$$

where H_{n_s} stands for the n_sth Hermite polynomial and Γ_0 is the normalized vacuum eigenfunction,

$$\Gamma_0 = \mathcal{N}\exp\left(-\frac{1}{2}\sum_{r=0}^{\infty}\Omega_r^2 Q_r^2\right). \tag{56}$$

We introduce *dressed* or renormalized coordinates q_0' and $\{q_i'\}$ for, respectively, the *dressed* particle and the *dressed* field, defined by

$$\sqrt{\bar{\omega}_\mu}q_\mu' = \sum_r t_\mu^r \sqrt{\Omega_r}Q_r, \tag{57}$$

valid for arbitrary R and where $\bar{\omega}_\mu=\{\bar{\omega},\omega_i\}$. In terms of dressed coordinates, we define for a fixed instant $t=0$ *dressed* states $|\kappa_0\kappa_1\kappa_2\ldots\rangle$ by means of the complete orthonormal set of functions

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \prod_{\mu}\left[\sqrt{\frac{2^{\kappa_\mu}}{\kappa_\mu!}}H_{\kappa_\mu}\left(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu\right)\right]\Gamma_0, \quad (58)$$

where $q'_\mu=\{q'_0,q'_i\}$, $\bar{\omega}_\mu=\{\bar{\omega},\omega_i\}$. Notice that the ground state Γ_0 in the above equation is the same as in Eq. (55). The invariance of the ground state is due to our definition of dressed coordinates given by Eq. (57). Each function $\psi_{\kappa_0,\kappa_1,\ldots}(q')$ describes a state in which the dressed oscillator q'_μ is in its κ_μth excited state.

It is worthwhile to note that our renormalized coordinates are objects different from both the bare coordinates q and the normal coordinates Q. In particular, the renormalized coordinates and dressed states—although both are collective objects—should not be confused with the normal coordinates Q and the eigenstates Eq. (55). While the eigenstates ϕ are stable, the dressed states ψ are all unstable, except for the ground state obtained by setting $\{\kappa_\mu=0\}$ in Eq. (58). The idea is that the dressed states are physically meaningful states. This can be seen as an analog of the wave-function renormalization in quantum field theory, which justifies the denomination of renormalized to the new coordinates q'. Thus, the dressed state given by Eq. (58) describes the particle in its κ_0th excited level and each mode k of the cavity in the κ_kth excited level. It should be noticed that the introduction of the renormalized coordinates guarantees the stability of the dressed vacuum state, since by definition it is identical to the ground state of the system. The fact that the definition given by Eq. (57) assures this requirement can be easily seen by replacing Eq. (57) in Eq. (58). We obtain $\Gamma_0(q')\propto\Gamma_0(Q)$, which shows that the dressed vacuum state given by Eq. (58) is the same ground state of the interacting Hamiltonian given by Eq. (5).

The necessity of introducing renormalized coordinates can be understood by considering what would happen if we write Eq. (58) in terms of the bare coordinates q_μ. In the absence of interaction, the bare states are stable since they are eigenfunctions of the free Hamiltonian. But when we consider the interaction, they all become unstable. The excited states are unstable, since we know this from experiment. On the other hand, we also know from experiment that the particle in its ground state is stable, in contradiction with what our simplified model for the system describes in terms of the bare coordinates. So, if we wish to have a nonperturbative approach in terms of our simplified model, something should be modified in order to remedy this problem. The solution is just the introduction of the renormalized coordinates q'_μ as the physically meaningful ones.

In terms of bare coordinates, the dressed coordinates are expressed as

$$q'_\mu = \sum_\nu \alpha_{\mu\nu}q_\nu, \quad (59)$$

where

$$\alpha_{\mu\nu} = \frac{1}{\sqrt{\bar{\omega}_\mu}}\sum_r t^r_\mu t^r_\nu\sqrt{\Omega_r}. \quad (60)$$

If we consider an arbitrarily large cavity ($R\to\infty$), the dressed coordinates reduce to

$$q'_0 = A_{00}(\bar{\omega},g)q_0, \quad (61)$$

$$q'_i = q_i, \quad (62)$$

with $A_{00}(\bar{\omega},g)$ given by

$$A_{00}(\bar{\omega},g) = \frac{1}{\sqrt{\bar{\omega}}}\int_0^\infty \frac{2g\Omega^2\sqrt{\Omega}d\Omega}{(\Omega^2-\bar{\omega}^2)^2+\pi^2g^2\Omega^2}. \quad (63)$$

In other words, in the limit $R\to\infty$, the particle is still dressed by the field, while for the field there remain bare modes.

Let us consider a particular dressed state $|\Gamma^\mu_1(0)\rangle$ represented by the wave function $\psi_{00\cdots1(\mu)0\ldots}(q')$. It describes the configuration in which only the dressed oscillator q'_μ is in the *first*-excited level. Then the following expression for its time evolution is valid [13]:

$$|\Gamma^\mu_1(t)\rangle = \sum_\nu f^{\mu\nu}(t)|\Gamma^\nu_1(0)\rangle,$$

$$f^{\mu\nu}(t) = \sum_s t^s_\mu t^s_\nu e^{-i\Omega_s t}. \quad (64)$$

Moreover we find that

$$\sum_\nu |f^{\mu\nu}(t)|^2 = 1. \quad (65)$$

Then the coefficients $f^{\mu\nu}(t)$ are simply interpreted as probability amplitudes.

In approaching the thermalization process in this framework, we have to write the initial physical state in terms of dressed coordinates or equivalently in terms of dressed annihilation and creation operators a'_μ and $a'^\dagger_\mu$ instead of a_μ and $a^\dagger_\mu$. This means that the initial dressed density operator corresponding to the thermal bath is given by

$$\rho_\beta = Z_\beta^{-1}\exp\left[-\beta\sum_{k=1}^{\infty}\omega_k\left(a'^\dagger_k a'_k+\frac{1}{2}\right)\right], \quad (66)$$

where we define

$$a'_\mu = \sqrt{\frac{\bar{\omega}_\mu}{2}}q'_\mu + \frac{i}{\sqrt{2\bar{\omega}_\mu}}p'_\mu \quad (67)$$

$$a'^\dagger_\mu = \sqrt{\frac{\bar{\omega}_\mu}{2}}q'_\mu - \frac{i}{\sqrt{2\bar{\omega}_\mu}}p'_\mu. \quad (68)$$

Now we analyze the time evolution of dressed coordinates.

V. THERMAL BEHAVIOR FOR A CAVITY OF ARBITRARY SIZE WITH DRESSED COORDINATES

The solution for the time-dependent annihilation and creation dressed operators follows similar steps as for the bare operators. The time evolution of the annihilation operator is given by

$$\frac{d}{dt}a'_\mu(t) = i[\hat{H},a'_\mu(t)] \quad (69)$$

and a similar equation for $a'^\dagger_\mu(t)$. We solve this equation with the initial condition at $t=0$,

$$a'_\mu(0) = \sqrt{\frac{\omega_\mu}{2}} q'_\mu + \frac{i}{\sqrt{2\omega_\mu}} p'_\mu, \tag{70}$$

which, in terms of bare coordinates, becomes

$$a'_\mu(0) = \sum_{r,\nu=0}^{N} \left(\sqrt{\frac{\Omega_r}{2}} t^r_\mu t^r_\nu \hat{q}_\nu + \frac{i t^r_\mu t^r_\nu}{\sqrt{2\Omega_r}} \hat{p}_\nu \right). \tag{71}$$

We assume a solution for $a'_\mu(t)$ of the type

$$a'_\mu(t) = \sum_{\nu=0}^{\infty} [\dot{B}'_{\mu\nu}(t)\hat{q}_\nu + B'_{\mu\nu}(t)\hat{p}_\nu]. \tag{72}$$

Using Eq. (1) we find

$$B'_{\mu\nu}(t) = \sum_{r=0}^{\infty} t^r_\nu (a'^r_\mu e^{i\Omega_r t} + b'^r_\mu e^{-i\Omega_r t}). \tag{73}$$

In the present case, the time-independent coefficients are different from those in the bare coordinate approach [Eq. (29)]. The initial conditions for $B'_{\mu\nu}(t)$ and $\dot{B}'_{\mu\nu}(t)$ are obtained by setting $t=0$ in Eq. (72) and comparing with Eq. (71). Then

$$B'_{\mu\nu}(0) = i \sum_{r=0}^{\infty} \frac{t^r_\mu t^r_\nu}{\sqrt{2\Omega_r}}, \tag{74}$$

$$\dot{B}'_{\mu\nu}(0) = \sum_{r=0}^{\infty} \sqrt{\frac{\Omega_r}{2}} t^r_\mu t^r_\nu. \tag{75}$$

Using these initial conditions and the orthonormality of the matrix $\{t^r_\mu\}$, we obtain $a'^r_\mu = 0$ and $b'^r_\mu = i t^r_\mu / \sqrt{2\Omega_r}$. Replacing these values for a'^r_μ and b'^r_μ in Eq. (73), we get

$$B'_{\mu\nu}(t) = i \sum_{r=0}^{\infty} \frac{t^r_\mu t^r_\nu}{\sqrt{2\Omega_r}} e^{-i\Omega_r t}. \tag{76}$$

We have

$$\begin{aligned} a'_\mu(t) &= \sum_{r,\nu=0}^{N} t^r_\mu t^r_\nu \left(\sqrt{\frac{\Omega_r}{2}} \hat{q}_\nu + \frac{i}{\sqrt{2\Omega_r}} \hat{p}_\nu \right) e^{-i\Omega_r t} \\ &= \sum_{r,\nu=0}^{N} t^r_\mu t^r_\nu \left(\sqrt{\frac{\omega_\nu}{2}} \hat{q}'_\nu + \frac{i}{\sqrt{2\omega_\nu}} \hat{p}'_\nu \right) e^{-i\Omega_r t} = \sum_{\nu=0}^{\infty} f_{\mu\nu}(t) \hat{a}'_\nu, \end{aligned} \tag{77}$$

where

$$f_{\mu\nu}(t) = \sum_{r=0}^{\infty} t^r_\mu t^r_\nu e^{-i\Omega_r t}. \tag{78}$$

For the occupation number $n'_\mu(t) = \langle a'^\dagger_\mu(t) a'_\mu(t) \rangle$, we get

$$n'_\mu(t) = \mathrm{Tr}[a'^\dagger_\mu(t) a'_\mu(t) \rho'_0 \otimes \rho'_\beta], \tag{79}$$

where ρ'_0 is the density operator for the dressed particle and ρ'_β is the density operator for the thermal bath, which coincides with the corresponding operator for the bare thermal bath if the system is in free space (in the sense of an arbitrarily large cavity) [13,14].

To evaluate $n'_\mu(t)$, we choose the basis $|n_0, n_1, \ldots, n_N\rangle = \Pi_{\mu=0}^{\infty} |n_\mu\rangle$, where $|n_\mu\rangle$ are the eigenvectors of the number operators $a'^\dagger_\mu a'_\mu$. From Eq. (77), we get

$$\begin{aligned} a'^\dagger_\mu(t) a'_\mu(t) &= \sum_{\nu,\rho=0}^{\infty} f^*_{\mu\rho}(t) f_{\mu\nu}(t) \hat{a}'^\dagger_\rho \hat{a}'_\nu \\ &= \sum_{\nu=0}^{\infty} |f_{\mu\nu}(t)|^2 \hat{a}'^\dagger_\nu \hat{a}'_\nu + \sum_{\nu \neq \rho} f^*_{\mu\rho}(t) f_{\mu\nu}(t) \hat{a}'^\dagger_\nu \hat{a}'_\rho. \end{aligned} \tag{80}$$

In the basis $|n_0, n_1, n_2, \ldots\rangle$, we obtain

$$n'_\mu(t) = |f_{\mu 0}(t)|^2 n'_0 + \sum_{k=1}^{\infty} |f_{\mu k}(t)|^2 n'_k, \tag{81}$$

where n'_0 and n'_k are the expectation values of the initial number operators, respectively, for the dressed particle and dressed bath modes. We assume that dressed field modes obey a Bose-Einstein distribution. This can be justified by remembering that in the free space limit $R \to \infty$, dressed field modes are identical to the bare ones, according to Eqs. (61) and (62). Now, no term independent of the temperature appears in the thermal bath. This should be expected since the dressed vacuum is stable; particle production from the vacuum is not possible. Setting $\mu = 0$ in Eq. (81), we obtain the time evolution for the occupation number of the particle,

$$n'_0(t) = |f_{00}(t)|^2 n'_0 + \sum_{k=1}^{\infty} |f_{0k}(t)|^2 n'_k. \tag{82}$$

VI. LIMIT OF ARBITRARILY LARGE CAVITY: UNBOUNDED SPACE

In a large cavity (free space), we must compute the quantities $f_{00}(t)$ and $f_{0k}(t)$ in the continuum limit to study the time evolution of the occupation number for the particle. Remember that in Eq. (17), $\omega_k = k\pi c/R$, $k = 1, 2, \ldots$, and $\eta = \sqrt{2g\Delta\omega}$, with $\Delta\omega = (\omega_{i+1} - \omega_i) = \pi c/R$. When $R \to \infty$, we have $\Delta\omega \to 0$ and $\Delta\Omega \to 0$ and then the sum in Eq. (78) becomes an integral. To calculate the quantities $f_{\mu\nu}(t)$, we first note that, in the continuum limit, Eq. (17) becomes

$$t^r_0 \to t^\Omega_0 \sqrt{\Delta\Omega} \equiv \lim_{\Delta\Omega \to 0} \frac{\Omega \sqrt{2g\Delta\Omega}}{\sqrt{(\Omega^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \Omega^2}}, \tag{83}$$

$$t^r_k \to \frac{\omega \sqrt{2g\Delta\omega}}{\omega^2 - \Omega^2} t^\Omega_0 \sqrt{\Delta\Omega}. \tag{84}$$

In the following, we suppress the labels in the frequencies, since they are continuous quantities.

We start by defining a function $W(z)$,

$$W(z) = z^2 - \bar{\omega}^2 + \sum_{k=1}^{\infty} \frac{\eta^2 z^2}{\omega_k^2 - z^2}. \tag{85}$$

We find that the Ω's are the roots of $W(z)$. Using $\eta^2 = 2g\Delta\omega$, we have in the continuum limit,

$$W(z) = z^2 - \bar{\omega}^2 + 2gz^2 \int_0^\infty \frac{d\omega}{\omega^2 - z^2}. \tag{86}$$

For complex values of z, the above integral is well defined and is evaluated by using Cauchy theorem to be

$$W(z) = \begin{cases} z^2 + ig\pi z - \bar{\omega}^2, & \mathrm{Im}(z) > 0 \\ z^2 - ig\pi z - \bar{\omega}^2, & \mathrm{Im} < 0. \end{cases} \tag{87}$$

We now compute $f_{00}(t) = \Sigma_{r=0}^{\infty} (t_0^r)^2 e^{-i\Omega_r t}$ which, in the continuum limit, is given by

$$f_{00}(t) = \int_0^\infty (t_0^\Omega)^2 e^{-i\Omega t} d\Omega. \tag{88}$$

We find that

$$(t_0^\Omega)^2 = \frac{1}{W(\Omega)}, \tag{89}$$

and since the Ω's are the roots of $W(z)$, we write Eq. (88) as

$$f_{00}(t) = \frac{1}{i\pi} \oint_C \frac{dz e^{-izt}}{W(z)}, \tag{90}$$

where C is a counterclockwise contour in the z plane that encircles the real positive roots of $W(z)$. Choosing a contour infinitesimally close to the positive real axis, that is, $z = \alpha - i\epsilon$ below it and $z = \alpha + i\epsilon$ above it with $\alpha > 0$ and $\epsilon \to 0^+$, we obtain

$$f_{00}(t) = \frac{1}{i\pi} \int_0^\infty d\alpha \alpha e^{-i\alpha t} \left[\frac{1}{W(\alpha - i\epsilon)} - \frac{1}{W(\alpha + i\epsilon)} \right]. \tag{91}$$

In the limit $\epsilon \to 0^+$, Eq. (87) gives $W(\alpha \pm i\epsilon) = \alpha^2 - \bar{\omega}^2 \pm ig\pi\alpha$ which leads to

$$f_{00}(t) = C_1(t;\bar{\omega},g) + iS_1(t;\bar{\omega},g), \tag{92}$$

where

$$C_1(t;\bar{\omega},g) = 2g \int_0^\infty d\alpha \frac{\alpha^2 \cos(\alpha t)}{(\alpha^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \alpha^2}, \tag{93}$$

$$S_1(t;\bar{\omega},g) = -2g \int_0^\infty d\alpha \frac{\alpha^2 \sin(\alpha t)}{(\alpha^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \alpha^2}. \tag{94}$$

Notice that $C_1(t=0;\bar{\omega},g) = 1$ and $S_1(t=0;\bar{\omega},g) = 0$, so that $f_{00}(t=0) = 1$ as expected from the orthonormality of the matrix (t_μ^r). The real part of $f_{00}(t)$ is calculated using the residue theorem. For $\kappa^2 = \bar{\omega}^2 - \pi^2 g^2/4 > 0$, which includes the weak-coupling regime, one finds

$$C_1(t;\bar{\omega},g) = e^{-\pi g t/2} \left[\cos(\kappa t) - \frac{\pi g}{2\kappa} \sin(\kappa t) \right] \quad (\kappa^2 > 0). \tag{95}$$

Although $S_1(t;\bar{\omega},g)$ cannot be analytically evaluated for all t, however, for long times, i.e., $t \gg 1/\bar{\omega}$, we have

$$S_1(t;\bar{\omega},g) \approx \frac{4g}{\bar{\omega}^4 t^3} \quad \left(t \gg \frac{1}{\bar{\omega}} \right). \tag{96}$$

Thus, we get for large t

$$|f_{00}(t)|^2 \approx e^{-\pi g t} \left[\cos(\kappa t) - \frac{\pi g}{2\kappa} \sin(\kappa t) \right]^2 + \frac{16 g^2}{\bar{\omega}^8 t^6}. \tag{97}$$

Next we compute the quantity $f_{0k}(t) = \Sigma_{r=0}^{\infty} t_0^r t_k^r e^{-i\Omega_r t}$ in the continuum limit. It is

$$f_{0\omega}(t) = \eta\omega \int_0^\infty \frac{(t_0^\Omega)^2 e^{-i\Omega t} d\Omega}{(\omega^2 - \Omega^2)} = \frac{\eta\omega}{i\pi} \oint_C \frac{z e^{-izt}}{(\omega^2 - z^2) W(z)}, \tag{98}$$

where $\eta = \sqrt{2g\Delta\omega}$. Taking the same contour as that used to calculate $f_{00}(t)$, we obtain

$$f_{0\omega}(t) = -\frac{\eta\omega}{i\pi} \int_0^\infty d\alpha \left[\frac{\alpha e^{-i\alpha t}}{W(\alpha - i\epsilon)[(\alpha - i\epsilon)^2 - \omega^2]} - \frac{\alpha e^{-i\alpha t}}{W(\alpha + i\epsilon)[(\alpha + i\epsilon)^2 - \omega^2]} \right]. \tag{99}$$

Thus, taking $\epsilon \to 0^+$ $f_{0\omega}(t)$ is written as

$$f_{0\omega}(t) = \omega\sqrt{\Delta\omega} [C_2(\omega,t;\bar{\omega},g) + iS_2(\omega,t;\bar{\omega},g)], \tag{100}$$

where

$$C_2(\omega,t;\bar{\omega},g) = (2g)^{3/2} \int_0^\infty d\alpha \frac{\alpha^2 \cos(\alpha t)}{(\omega^2 - \alpha^2)[(\alpha^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \alpha^2]}, \tag{101}$$

$$S_2(\omega,t;\bar{\omega},g) = -(2g)^{3/2} \int_0^\infty d\alpha \frac{\alpha^2 \sin(\alpha t)}{(\omega^2 - \alpha^2)[(\alpha^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \alpha^2]}. \tag{102}$$

Notice that the integrals defining the functions C_2 and S_2 are actually Cauchy principal values.

The function C_2 is calculated analytically using Cauchy theorem; we find

$$C_2(\omega,t;\bar{\omega},g) = \sqrt{2g} \left[e^{-\pi g t/2} \left\{ \frac{\omega^2 - \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \omega^2} \cos \kappa t - \frac{\pi g}{2\kappa} \frac{\omega^2 + \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \omega^2} \sin \kappa t \right\} + \frac{\pi g \omega}{(\omega^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \omega^2} \sin \omega t \right]. \tag{103}$$

The function S_2 cannot be evaluated analytically for all t; it has to be calculated numerically. For long times, we have

$$S_2(t;\bar{\omega},g) \approx \frac{4\sqrt{2g}\sqrt{g}}{\omega^2 \bar{\omega}^4 t^3} \quad \left(t \gg \frac{1}{\bar{\omega}} \right). \tag{104}$$

In the continuum limit, we get the average of the particle occupation number,

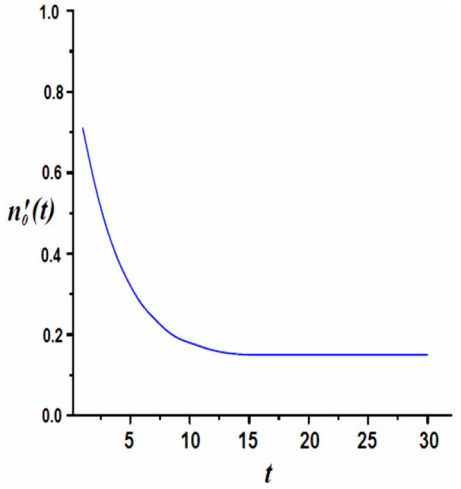

FIG. 2. (Color online) Time behavior for $n_0'(t)$ given by Eq. (105) for $(t>1)$, $n_0=1$, $\bar{\omega}=1$, $\beta=2$, and $g=0.1$; arbitrary units are used.

$$n_0'(t) = [C_1^2(t;\bar{\omega},g) + S_1^2(t;\bar{\omega},g)]n_0' + \int_0^\infty d\omega\omega^2[C_2^2(\omega,t;\bar{\omega},g) + S_2^2(\omega,t;\bar{\omega},g)]n'(\omega), \quad (105)$$

where $n'(\omega)=1/(e^{\beta\omega}-1)$ is the density of occupation of the environment modes, the functions C_1 and C_2 are given by Eqs. (95) and (103) while the functions S_1 and S_2 are given by the integrals (94) and (102), respectively. In Fig. 2, we display the behavior in time for $n_0=1$, $\bar{\omega}=1$, $\beta=2$, and $g=0.1$; $(t>1)$.

The important point that is seen from Figs. 1 and 2 is that, for long times, both the bare and dressed occupation numbers of the particle approach smoothly to asymptotic values which are ≈ 0.160. Moreover, these values are expected on physical grounds for the interacting particle, being slightly higher than the one obtained from the Bose distribution at the equilibrium temperature of the reservoir. In fact, taking $\beta=2$ and $\bar{\omega}=1$, as used in the plots, one has

$$n_\infty(\bar{\omega}) = 1/(e^{\beta\bar{\omega}} - 1) = 0.156.$$

Therefore both methods and, in particular, our dressed state formalism describe correctly the thermalization process.

VII. FINAL REMARKS

We have considered a linearized version of a particle-environment system and we have carried out a nonperturbative treatment of the thermalization process. We have adopted the point of view of renouncing to an approach very close to the real behavior of a nonlinear system to study instead a linear model. As a counterpart, an exact solution has been possible. This realizes a good compromise between physical reality and mathematical reliability. We have presented an ohmic quantum system consisting of a particle, in the larger sense of a material body, an atom, or a Brownian particle coupled to an environment modeled by noninteracting oscillators. We have used the formalism of dressed states to perform a nonperturbative study of the time evolution of the system contained in a cavity or in free space. Distinctly to what happens in the bare coordinate approach, in the dressed coordinate approach, no renormalization procedure is needed. Our renormalized coordinates contain in themselves the renormalization aspects. As far as the thermalization process is concerned from a physical viewpoint, both bare and dressed approaches are in agreement with what we expect for this process. For long times, all the information about the particle occupation numbers depends only on the environment. Both curves in Figs. 1 and 2 approach steadily to asymptotic values of the bare and dressed occupation numbers of the particle, which are physically expected at the given temperature.

ACKNOWLEDGMENTS

We are especially grateful to F. C. Khanna for fruitful discussions and the Theoretical Physics Institute, University of Alberta, for their kind hospitality during the summer 2008. Our thanks also to P. J. Pompeia and R. R. Cuzinatto for help in numerical calculations. This work received partial financial support from CNPq/MCT, FAPEMIG, and FAPERJ.

[1] W. G. Unruh and W. H. Zurek, Phys. Rev. D **40**, 1071 (1989).
[2] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[3] P. Ullersma, Physica **32**, 56 (1966); Physica **32**, 74 (1966); Physica **32**, 90 (1966).
[4] F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).
[5] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[6] H. Grabert, P. Schramm, and G.-L. Ingold, Phys. Rep. **168**, 115 (1988).
[7] V. Rivasseau, *From Perturbative to Constructive Renormalization* (Princeton University Press, Princeton, NJ, 1991), pp. 54–56.
[8] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).
[9] E. J. Weniger, Phys. Rev. Lett. **77**, 2859 (1996).
[10] U. D. Jentschura, Phys. Rev. A **64**, 013403 (2001).
[11] G. Cvetic, C. Dib, T. Lee, and I. Schmidt, Phys. Rev. D **64**, 093016 (2001).
[12] A. P. C. Malbouisson, J. Math. Phys. **35**, 479 (1994).
[13] N. P. Andion, A. P. C. Malbouisson, and A. Mattos Neto, J. Phys. A **34**, 3735 (2001).
[14] G. Flores-Hidalgo, A. P. C. Malbouisson, and Y. W. Milla, Phys. Rev. A **65**, 063414 (2002).
[15] G. Flores-Hidalgo and A. P. C. Malbouisson, Phys. Rev. A **66**, 042118 (2002).
[16] A. P. C. Malbouisson, Ann. Phys. **308**, 373 (2003).
[17] G. Flores-Hidalgo and A. P. C. Malbouisson, Phys. Lett. A

337, 37 (2005); J. Phys. A **34**, 6381 (1996); Phys. Rev. A **66**, 042118 (2002).
[18] A. K. Likhoded and G. P. Pronko, Int. J. Theor. Phys. **36**, 2335 (1997).
[19] W. Thirring and F. Schwabl, Ergeb. Exakten Naturwiss. **36**, 219 (1964).
[20] G. Flores-Hidalgo and R. O. Ramos, Physica A **326**, 159 (2003).
[21] M. Rosenau da Costa, A. O. Caldeira, S. M. Dutra, and H. Westfahl, Jr., Phys. Rev. A **61**, 022107 (2000).