# Geometric quantum computation and dynamical invariant operators

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An entangling quantum gate based entirely on purely geometric operations is proposed in quantum computation for the Jaynes-Cummings model by the invariant theory, where the qubits include information about the states of photons. By controlling some arbitrary parameters in the invariant operators, the phase accumulated in the quantum gate is a pure geometric phase. This way may be extended to other physical systems.

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Geometric (holonomic) quantum computation is a scheme that is potentially intrinsical fault tolerant and therefore resilient to certain types of computational errors. It is known that the holonomies can be generated when a quantum system is driven in a cyclic evolution through adiabatic or nonadiabatic change in the control parameters in the Hamiltonian. Such holonomies can be either Abelian phase factors or non-Abelian unitary operations if the spectrum of the Hamiltonian is degenerate.

A quantum gate based on the geometric phase can be constructed using only adiabatic evolution [1]. However, it is difficult to experimentally realize quantum computation with adiabatic evolution because the long operation time is required, especially for solid-state systems whose decoherence time is very short [2–4]. This is especially true given that the evolution has to be repeated several times in order to cancel the dynamical phase. Decoherence is the most important limiting factor for quantum computation because its effect is that quantum superpositions decay into statistical mixtures [5]. It may be better, therefore, to construct geometric quantum gates by using the nonadiabatic geometric phase [6–9] since this allows for shortening gate times. For a nonadiabatic cyclic evolution, the total phase between the final and initial states is a sum of the geometric and dynamical phases. In some methods of geometric quantum computation, it is necessary to remove the dynamical component, such as by using dark states [10] and by rotating operations in so-called single-loop and multiloop schemes [3,4]. The experimental errors are, obviously, increased because of the operational process. More worryingly, the dynamic phase accumulated in the gate operation is possibly nonzero and cannot be eliminated. Therefore, it may be better to realize nonadiabatic geometric quantum computation by using varying parameters in the Hamiltonian, where the dynamical and geometric phases are implemented separately without the usual operational process [7,8].

Recently, geometric quantum computation has been attracting increasing interest and was proposed by using nuclear-magnetic-resonance systems [8,9], superconducting nanocircuits [11], semiconducting nanostructures [12], trapped ions, and cavity QED [13–18]. In a really closed system, a useful way to remove the adiabatic constraint in quantum computation is the theory of the dynamical invariant to treat time-dependent Hamiltonian [19,20]. Indeed, the dynamically invariant theory was recently used in a proposal of an interferometric experiment to measure the nonadiabatic geometric phase in cavity quantum electrodynamics [21].

For a closed quantum system, a dynamical invariant $I(t)$ satisfies

$$\frac{\partial I(t)}{\partial t} = i[I(t),H(t)], \tag{1}$$

where $H(t)$ is a Hamiltonian of the system and $I(t)$ is a Hermitian invariant operator with a member of a complete set of commuting observables [22]. Therefore, there exists a set of simultaneous eigenfunctions $|\lambda_n,a;t\rangle$ satisfying

$$I(t)|\lambda_n,a;t\rangle = \lambda_n|\lambda_n,a;t\rangle, \tag{2}$$

$$\langle\lambda_m,b;t|\lambda_n,a;t\rangle = \delta_{mn}\delta_{ba}, \tag{3}$$

where $\lambda_n$ is an eigenvalue of the invariant operator $I(t)$ while $a$ and $b$ are degenerate labels. $|\lambda_n,a;t\rangle$ are also eigenstates of the Hamiltonian $H(t)$. Therefore, an exact solution of the Schrödinger equation with the Hamiltonian $H(t)$ can be expressed as

$$|\psi(t)\rangle = \sum_{n,a} c_{n,a}\mathcal{P}(e^{i\chi_{n,a}})|\lambda_n,a,t\rangle, \tag{4}$$

where $c_n$ do not depend on the involving time and $\mathcal{P}$ stands for the time-ordering operator. The phases $\chi_{n,a}$ are determined by [22,23]

$$\chi_{n,a} = \int_0^t dt\langle\lambda_n,a,t|i\frac{\partial}{\partial t} - H(t)|\lambda_n,a,t\rangle, \tag{5}$$

where the first term is the geometric phase and the second term is the dynamical phase.

It is known that the Jaynes-Cummings model (JCM) [24] is a basis for fully quantum descriptions of radiation matter interactions and has had some extensive applications in quantum optics, quantum electronics and two-level atomic system. Therefore, the geometric properties of the JCM have acquired much interest [25]. Since the supersymmetric structure was found to embed in the JCM [26], it is interesting to physically implement a universal set of quantum gates based on the dynamically supersymmetric invariant.

Consider the Hamiltonian of the JCM under the rotating approximation,

 

$$H(t)=\frac{1}{2}\omega_0\sigma_z+\omega\left(a^\dagger a+\frac{1}{2}\right)+g(t)a^\dagger\sigma_-+g^*(t)a\sigma_+, \quad (6)$$

where $a^\dagger$ and $a$ denote the photon creation and annihilation operators satisfying the commutation relation $[a,a^\dagger]=1$, $\sigma_\pm$ and $\sigma_3$ are Pauli matrices acting on the states of the two-level atom system, $\omega$ is a frequency of the field, and $\omega_0$ is a frequency of the atom, while $g(t)=g_0\exp(i\omega_g t)$ is a coupling constant for the interaction between the atom and field and $g^*(t)$ is a complex conjugation of $g(t)$. The intrinsic relationships among fundamental properties, such as supersymmetry, phase invariance of electromagnetic field, and unitarity and energy spectrum peculiarities are revealed by this model.

By defining the tensor operators [26]

$$V=\sigma_+\sigma_-+a^\dagger a,\quad M=\sigma_+\sigma_--a^\dagger a-1, \quad (7)$$

$$Q_+=Q_-^+=\frac{1}{\sqrt{2}}\sigma_-a^\dagger,\quad \Gamma(t)=\sqrt{2}g(t), \quad (8)$$

the Hamiltonian is rewritten as

$$H(t)=\frac{1}{2}(\omega_0+\omega)V+\frac{1}{2}(\omega_0-\omega)M+\Gamma(t)Q_++\Gamma^*(t)Q_-, \quad (9)$$

which can be easily recognized as an element of the superalgebra associated with the unitary supergroup $u(1,1)$. The tensor operators satisfy the commutation relations

$$\{Q_\epsilon,Q_\eta\}=\frac{1}{2}V\delta_{\epsilon,-\eta},\quad [V,M]=0=[V,Q_\epsilon], \quad (10)$$

$$[M,Q_\epsilon]=-2\epsilon Q_\epsilon,\quad (\epsilon,\eta=\pm), \quad (11)$$

where $V$ is a Casimir operator. Equations (10) and (11) imply that the set of operators $\{V,M,Q_+,Q_-\}$ generates a dynamically closed superalgebra.

In order to compute the geometric phases, we require to get the eigenstates of the invariant operator. In accordance with the closed superalgebra theory, the invariant operator $I(t)$ is of the general form, such as

$$I(t)=\frac{1}{2}[\Omega_0(t)+\Omega(t)]V+\frac{1}{2}[\Omega_0(t)-\Omega(t)]M+\xi(t)Q_+ +\xi^*(t)Q_-, \quad (12)$$

where $\Omega_0(t)$, $\Omega(t)$, $\xi(t)$, and $\xi^*(t)$ are different from $\omega_0$, $\omega$, $\Gamma(t)$, and $\Gamma^*(t)$, respectively, and will be determined by Eq. (1) and the closed superalgebra theory.

Inserting Eqs. (6) and (12) into Eq. (1) and using Eqs. (10) and (11), we find that $\Omega_0(t)$ and $\Omega(t)$ are arbitrary constants because of the relations $\dot{\Omega}(t)=0$ and $\dot{\Omega}_0(t)=0$, and $\xi(t)=[\sqrt{2}(\Omega_0-\Omega)g_0/(\omega_g-\omega_0+\omega)]\exp(i\omega_g t)$. The eigenvalues of operator $I(t)$ can be obtained by $\lambda_n^\pm=\Omega(n+1)\pm\frac{1}{2}\sqrt{(\Omega_0-\Omega)^2+2n|\xi|^2}$, where $n$ is the photon number. It is noted that the eigenvalues are independent of time. The corresponding eigenstates are expressed as

$$|\lambda_n^+,t\rangle=\begin{pmatrix}\cos(\theta_n/2)|n\rangle\\ e^{i\beta}\sin(\theta_n/2)|n+1\rangle)\end{pmatrix}, \quad (13)$$

$$|\lambda_n^-,t\rangle=\begin{pmatrix}-\sin(\theta_n/2)|n\rangle\\ e^{i\beta}\cos(\theta_n/2)|n+1\rangle)\end{pmatrix}, \quad (14)$$

where $e^{i\beta}=\xi(t)/|\xi(t)|$, $|n\rangle$ is a photon number state, and $\theta_n=2\tan^{-1}[G_n/(\Delta_n-1)]$ with $\Delta_n=\sqrt{1+G_n^2}$, $G_n=2\sqrt{n}g_0/(\omega_g-\omega_0+\omega)$. The JCM describes generally a nonresonant interaction between a two-level system with lower state $|-\rangle$, upper state $|+\rangle$, and a harmonic oscillator denoted by the photon number state $|n\rangle$.

Using Eq. (5) and Eqs. (13) and (14), we know that the geometric phases are given by

$$\gamma_{n+}^g=-\pi(1-\cos\theta_n),\quad \gamma_{n-}^g=-\pi(1+\cos\theta_n) \quad (15)$$

and the corresponding dynamical phases are expressed as

$$\gamma_{n+}^d=\frac{2\pi}{\omega_g}\left[\frac{1}{2}\omega_0\cos\theta_n+\omega\left(n+\frac{1}{2}\right)+\sqrt{\frac{n}{2}}|\xi|\sin\theta_n\right], \quad (16)$$

$$\gamma_{n-}^d=\frac{2\pi}{\omega_g}\left[-\frac{1}{2}\omega_0\cos\theta_n+\omega\left(n+\frac{1}{2}\right)-\sqrt{\frac{n}{2}}|\xi|\sin\theta_n\right]. \quad (17)$$

It is interesting to note that, for the photon number $n=0$, the phases are different from zero, which means that the vacuum field introduces a correction in the geometric phases and dynamical phases [20].

It is noted that $\Omega_0$ and $\Omega$ are arbitrary constants. Therefore, by setting $\Omega_0-\Omega=(2k\omega_g-\omega_0\cos\theta_n)/(G_n\sin\theta_n)$ $(k=1,2,\ldots)$, we find

$$\omega_0\cos\theta_n+\sqrt{2n}|\xi|\sin\theta_n=2k\omega_g\quad (k=0,1,2,\ldots). \quad (18)$$

Under the condition, according to Eq. (4), the wave function for a single-qubit system may be expressed as

$$\psi(t)=e^{i(2\pi\omega/\omega_g)(n+1/2)}(e^{i\gamma_{n+}^g}|\lambda_n^+,t\rangle+e^{i\gamma_{n-}^g}|\lambda_n^-,t\rangle), \quad (19)$$

where $\exp[i\frac{2\pi\omega}{\omega_g}(n+1/2)]$ is an overall phase factor, which is not important and may be dropped in the quantum computation. Thus, a pair of orthogonal states $|\lambda_n^\pm,t\rangle$ can evolve cyclically and the relations $u(T)|\lambda_n^\pm,t=0\rangle=\exp(i\gamma_{n\pm}^g)|\lambda_n^\pm,t=T\rangle$ are satisfied. Therefore an arbitrary initial state can be expressed as $|\psi_i\rangle=a_+|\lambda_n^+,t=0\rangle+a_-|\lambda_n^-,t=0\rangle$ with $a_\pm=\langle\lambda_n^\pm,t=0|\psi_i\rangle$. According to Eq. (19), the final state at time $T=2\pi/\omega_g$ is calculated as $|\psi_f\rangle=b_+e^{i\gamma_{n+}^g}|\lambda_n^+,t=T\rangle+b_-e^{i\gamma_{n-}^g}|\lambda_n^-,t=T\rangle$. Under the computational basis $\{|0\rangle=\binom{|n\rangle}{0},|1\rangle=\binom{0}{|n+1\rangle}\}$, where the qubits include information about the photon states, the unitary transformation $u(\gamma_{n+}^g,\gamma_{n-}^g,\theta_n)$, between the input and output states, can be written as

$$u(\gamma_{n+}^g,\gamma_{n-}^g,\theta_n)=\begin{pmatrix}a_1 & b\\ b & a_2\end{pmatrix}, \quad (20)$$

where $a_1=e^{i\gamma_{n+}^g}\cos^2\frac{\theta_n}{2}+e^{i\gamma_{n-}^g}\sin^2\frac{\theta_n}{2}$, $a_2=e^{i\gamma_{n+}^g}\sin^2\frac{\theta_n}{2}+e^{i\gamma_{n-}^g}\cos^2\frac{\theta_n}{2}$, and $b=\frac{1}{2}\sin\theta_n(e^{i\gamma_{n+}^g}-e^{i\gamma_{n-}^g})$.

For a two-qubit system in the JCM, in order to simplify our computation but without loss of generality, we only consider the Casimir interaction between the control and target qubits. The total Hamiltonian is

$$H_{12}(t) = \frac{1}{2}(\omega_0 + \omega)V_1 + \frac{1}{2}(\omega_0 - \omega)M_1 + \Gamma(t)Q_{1+} + \Gamma^*(t)Q_{1-} + \lambda V_1 V_2, \tag{21}$$

where $\lambda$ is the strength of the interaction between two qubits. Similarly to the one-qubit system, the invariant operator for the two-qubit system may be expressed as

$$I_{12}(t) = \frac{1}{2}(\Omega_0 + \Omega)V_1 + \frac{1}{2}(\Omega_0 - \Omega)M_1 + \xi(t)Q_{1+} + \xi^*(t)Q_{1-} + \Lambda(t)V_1 V_2. \tag{22}$$

Substituting Eqs. (21) and (22) into Eq. (1), we find that $\Omega_0$, $\Omega$, and $\Lambda$ are constant, while $\xi(t)$ is the same as the one in the single-qubit system.

Under the computational basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, the invariant operator $I_{12}(t)$ may be rewritten as

$$I_{12}(t) = \begin{pmatrix} \omega_1 & 0 & \sqrt{\frac{n}{2}}\xi^* & 0 \\ 0 & \omega_1 & 0 & \sqrt{\frac{n}{2}}\xi^* \\ \sqrt{\frac{n}{2}}\xi & 0 & \omega_2 & 0 \\ 0 & \sqrt{\frac{n}{2}}\xi & 0 & \omega_2 \end{pmatrix}, \tag{23}$$

where $\omega_1=\Omega(n+1/2)+\Omega_0/2+\Lambda(n+1)^2$ and $\omega_2=\Omega(n+3/2)-\Omega_0/2+\Lambda(n+1)^2$. The eigenvalues of operator $I_{12}(t)$ are degenerate with $\lambda_n^{\pm}=\Omega(n+2)\pm\frac{1}{2}\sqrt{(\Omega_0-\Omega)^2+2n|\xi|^2}$ and the corresponding eigenstates are

$$|\lambda_n^+,1,t\rangle = \cos(\theta_n/2)|00\rangle + e^{i\beta}\sin(\theta_n/2)|10\rangle),$$

$$|\lambda_n^+,2,t\rangle = \cos(\theta_n/2)|01\rangle + e^{i\beta}\sin(\theta_n/2)|11\rangle),$$

$$|\lambda_n^-,1,t\rangle = -\sin(\theta_n/2)|00\rangle + e^{i\beta}\cos(\theta_n/2)|10\rangle),$$

and

$$|\lambda_n^-,2,t\rangle = -\sin(\theta_n/2)|01\rangle + e^{i\beta}\cos(\theta_n/2)|11\rangle),$$

respectively. It is noted that all eigenstates are orthogonal to each other. Using these eigenfuctions, we find that the geometric phases are

$$\gamma_{n+}^g(1) = \gamma_{n+}^g(2) = \gamma_{n+}^g, \quad \gamma_{n-}^g(1) = \gamma_{n-}^g(2) = \gamma_{n-}^g \tag{24}$$

and the corresponding dynamical phases are

$$\gamma_{n+}^d(1) = \gamma_{n+}^d(2) = \frac{2\pi}{\omega_g}\left[\omega(n+1) + \lambda(n+1)^2 + \frac{1}{2}(\omega_0 - \omega)\cos\theta_n + \sqrt{\frac{n}{2}}|\xi|\sin\theta_n\right], \tag{25}$$

$$\gamma_{n-}^d(1) = \gamma_{n-}^d(2) = \frac{2\pi}{\omega_g}\left[\omega(n+1) + \lambda(n+1)^2 - \frac{1}{2}(\omega_0 - \omega)\cos\theta_n - \sqrt{\frac{n}{2}}|\xi|\sin\theta_n\right]. \tag{26}$$

Similar to the single-qubit system, when the arbitrary constants

$$\Omega_0 - \Omega = [2k\omega_g - (\omega_0 - \omega)\cos\theta_n]/(G_n\sin\theta_n)(k=1,2,\ldots)$$

for the two-qubit system, we have

$$(\omega_0 - \omega)\cos\theta_n + \sqrt{2n}|\xi|\sin\theta_n = 2k\omega_g \quad (k=0,1,2,\ldots). \tag{27}$$

Under this condition, thus, the wave functions may be expressed by

$$\Psi(t) = e^{i(2\pi/\omega_g)[\omega(n+1)+\lambda(n+1)^2]}(c_1 e^{i\gamma_{n+}^g}|\lambda_n^+,1,t\rangle + c_2 e^{i\gamma_{n+}^g}|\lambda_n^+,2,t\rangle + c_3 e^{i\gamma_{n-}^g}|\lambda_n^-,1,t\rangle + c_4 e^{i\gamma_{n-}^g}|\lambda_n^-,2,t\rangle). \tag{28}$$

We see that the phase factors $e^{i(2\pi/\omega_g)[\omega(n+1)+\lambda(n+1)^2]}$ can be regarded as an overall phase factor, which may be dropped in the quantum computation.

In terms of the computational basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, where the first (second) bit represents the state of the control (target) qubit, the unitary transformation $U(\gamma_{n+}^g,\gamma_{n-}^g,\theta_n)$ up to a relative phase factor, between the input and output states, can be written as

$$U(\gamma_{n+}^g,\gamma_{n-}^g,\theta_n) = \begin{pmatrix} a_1 & 0 & b & 0 \\ 0 & a_1 & 0 & b \\ b & 0 & a_2 & 0 \\ 0 & b & 0 & a_2 \end{pmatrix}. \tag{29}$$

Thus, we achieve the entangling universal quantum gates based entirely on purely geometric operations (holonomies). Geometric quantum computation demands that the logical gate in computing is realized by using geometric phase shifts, so that it may have the built-in fault-tolerant advantage due to the fact that the geometric phases depend only on some global geometric features. As an example, we choose the parameter as $\theta_n=\pi/3$; the unitary transformation matrix (29) may be written as

$$U = \frac{1}{2}\begin{pmatrix} -i & 0 & -\sqrt{3}i & 0 \\ 0 & -i & 0 & -\sqrt{3}i \\ -\sqrt{3}i & 0 & i & 0 \\ 0 & -\sqrt{3}i & 0 & i \end{pmatrix}, \tag{30}$$

which is a nontrivial geometric phase gate.

In conclusion, we have proposed a way to realize the nonadiabatic geometric computation based on the dynamical invariant theory, where the invariant operator is constructed by supersymmetry algebra. By controlling some arbitrary parameters in the invariant operator, the phase accumulated in the quantum gate is a pure geometric phase. The strategy is applied to the Jaynes-Cummings model considering the qubit-photon interaction Hamiltonian. After interacting with a qubit, a photon can carry away information about the state of the qubit, and this is thus a decoherence process.

In comparison with the conventional geometric gates obtained by rotating operations, our approach does not need any such process, which leads to a possible reduction in experimental errors as well as gate timing. In contrast to the unconventional geometric gates by using global geometric features in the rotating frame, our approach distinguishes the total and geometric phases and offers a wide choice of the solution for other physical systems.


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