Robust strategies for lossy quantum interferometry

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We give a simple multiround strategy that permits us to beat the shot-noise limit when performing interferometric measurements even in the presence of loss. In terms of the average photon number employed, our procedure can achieve twice the sensitivity of conventional interferometric ones in the noiseless case. In addition, it is more precise than the (recently proposed) optimal two-mode strategy even in the presence of loss.

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The shot-noise limit is the minimum noise level that the Heisenberg uncertainty relations permit achieving when classical states are employed in the apparatuses. Many quantum strategies have been proposed to beat the shot noise [1,2] and to achieve the ultimate Heisenberg limit [12], but virtually all of them are very sensitive to noise and loss of photons [3]. Only very recently were some interferometric strategies presented that can beat the shot noise even in the presence of relevant losses of photons [4,5]. These are all instances of parallel strategies [2], where both arms of the interferometer are sampled at the same time using a mode-entangled quantum state of light [see Fig. 1(a)]. In addition to the parallel strategies, in quantum metrology it is also possible to achieve the Heisenberg limit using multiround (or sequential) strategies [6,7], which, in the noiseless case, are equivalent in terms of resources and of achievable precision [2].

Here we detail how multiround strategies can be used to perform interferometry—see Fig. 1(b). An appropriate input state is prepared (we will analyze two examples below). This state is fed into the first interferometer arm. It picks up a phase $\varphi+\vartheta$, where φ is the interferometric phase we want to estimate and ϑ is the absolute phase picked up by the free evolution in the arm (which is equal to the phase which would be picked up also in the reference arm). The main trick of multiround interferometry is the use of the unitary

$$U=\sum_{n=0}^{M}|M-n\rangle\langle n|+\sum_{n=M+1}^{\infty}|n\rangle\langle n|, \tag{1}$$

where $|n\rangle$ is the Fock basis and M is the largest nonzero component of the initial input state (if the input state has infinite nonzero components, one can always choose M sufficiently large so that the errors, due to the fact that not all components of the state are affected by U, can be made arbitrarily small). The purpose of this unitary is to permute the first M components of the Fock-state expansion of a state in such a way that, when the state is sent back through the reference arm, the absolute phase ϑ is removed from the state (only an irrelevant global phase factor $e^{iM\vartheta}$ persists). Thus, at the end of the round-trip of Fig. 1(b) (multiple round trips are also possible), only the relative phase φ is imprinted on the state. A measurement is finally performed to estimate this phase.

Notice that the two boxes of Fig. 1(b) initially need to share a synchronized phase reference. In fact, suppose that a Fock state $|n\rangle$ in the first box corresponds to a state $e^{i\gamma n}|n\rangle$ in the second box, where γ is the phase offset between the two boxes. Then the unitary U would introduce an unwanted phase 2γ in the states traveling through it. [From the point of view of resource accounting, this essentially amounts to having very precise synchronized clocks at the two boxes' locations, so that $\gamma=0$. However, from the practical point of view, this is not a problem: all interferometers have unknown phases between the two arms (because it is in practice impossible to build two macroscopic interferometer arms with length accuracy of a tiny fraction of an optical wavelength). Hence, typically one builds the interferometer and then, fixing the phase φ to zero, moves one of the mirrors in order to cancel all unknown phases, including γ.]

The multiround interferometry employs the same average energy and the same modes as the conventional (parallel) strategies, but it can achieve twice the sensitivity and it is more robust against noise. In fact, we show that, with an appropriate choice of inputs, our protocol permits us to estimate the phase with an error which is smaller than what is achieved by the strategy detailed in Ref. [4], which is claimed to be the optimal two-mode strategy. In the presence of loss, this is not unexpected, as parallel protocols rely on entanglement, which is notoriously fragile to noise. Instead, in the noiseless case, this can be seen easily with a simple example. Recall that the Heisenberg limit [12] is essentially

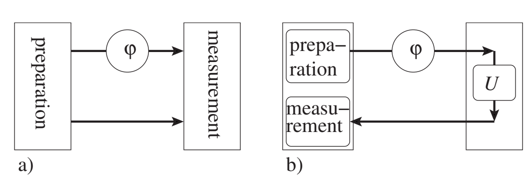

FIG. 1. (a) Parallel strategies for interferometry. Both arms of the interferometer are sampled at the same time using a two-mode entangled state. The phase factor φ is imprinted in the state as a phase difference between the two modes. (b) Sequential strategy proposed here. A loss-resistant single-mode state is sent through the first interferometer arm, sampling the phase φ. Then a unitary transformation is applied and the state is sent *back* through the other interferometer arm. This is needed so that the final phase shift experienced by the state is only the relative phase in the interferometer. The state is measured after one (or more) round-trips.

an application of the time-energy uncertainty, $\Delta\varphi\Delta h \geqslant 1/2$, where h is the generator of the unitary that inserts the phase φ into the system [14]: namely, $h = a^\dagger a$ (a being the annihilation operator of the first arm of the interferometer). Optimal two-mode states, such as the NOON state $(|N0\rangle + |0N\rangle)/\sqrt{2}$, have $\Delta h = N/2$. However, the corresponding single-mode "NO" state of same average number of photons—namely, the state $(|2N\rangle + |0\rangle)/\sqrt{2}$—has $\Delta h = N$. Both states achieve the Heisenberg-limited sensitivity of $\Delta\varphi = 1/(2\Delta h)$, but in terms of the average number of photons N, the NOON state can achieve $\Delta\varphi_{\mathrm{NOON}} = 1/N$, whereas the NO state can achieve $\Delta\varphi_{\mathrm{NO}} = 1/(2N)$—i.e., twice the sensitivity. The NO state is, however, just as sensitive to noise as the NOON state: the loss of a single photon renders both states useless to phase estimation.

The rest of the paper is devoted to presenting two examples of loss-resistant multiround interferometry based on two different input states, the optimal phase state, and the single-mode $|m :: m'\rangle$ state. Both are able to beat the optimal parallel strategy for some values of parameters.

We start by considering the single-mode optimal phase state $|\psi_{opt}\rangle$ introduced in [8,9] (and later extended to the multimode case in [10], building on [11])—i.e., the state

$$|\psi_{opt}\rangle \equiv \sqrt{\frac{2}{M+1}} \sum_{n=0}^{M} \sin\left(\frac{\pi(n+1/2)}{M+1}\right)|n\rangle, \qquad (2)$$

where M is a parameter identifying the average number of photons, $N = M/2$. In the noiseless case, this state achieves a precision $\Delta\varphi \simeq \pi/(N+1)$ [9], which scales as the Heisenberg bound $1/N$—i.e., the ultimate precision [12] in the absolute phase estimation. In addition, this state is highly robust to loss, since the loss tends to deplete primarily the components with large Fock numbers n, which are not very populated in this state. Following the scheme of Fig. 1(b), this state is sent through the first arm of the interferometer, where it is subject to a phase shift of $\varphi + \vartheta$, where φ is the interferometric phase we want to estimate and ϑ is the absolute phase picked up by the free evolution in the arm. During this transit, the state is also evolved by the loss map, described by the Kraus operators $K_i \equiv (\eta^{-1}-1)^{i/2} a^i \eta^{a^\dagger a/2}/\sqrt{i!}$, where a is the annihilation operator of the optical mode and η is its transmissivity or quantum efficiency. (Note that the loss and the phase accumulation commute, so that the order in which we apply these two transformations is irrelevant.) Then, the state is subject to the unitary evolution U of Eq. (1) and it is sent back along the reference arm. Thanks to the permutation of the Fock components that U applies to the state, the free evolution is effectively reversed, so that the absolute phase ϑ that was picked up in the first arm is removed while the radiation travels back through the reference arm. Also in the reference arm the state is typically subject to the loss (although there are interesting cases where the reference may be considered noiseless). Finally, the state is subject to the measurement. For the optimal phase state, a good measurement [9] can be obtained by considering the orthogonal positive operator-valued measure (POVM) composed of the projectors on the Pegg-Barnett states

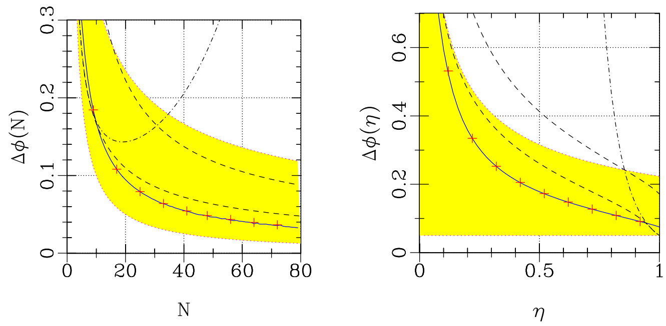

FIG. 2. (Color online) (Left) Solid line: error in the estimation of the phase from the state ρ, i.e., minimum (over φ) of the rms of the probability $p(l)$, as a function of the average number of photons N for $\eta = 0.9$. Lower dashed line: plot of the error from the optimal two-mode lossy interferometry from [4]. Our single-mode method achieves a higher precision for a wide range of parameters. Plus signs: Holevo variance $\Delta\Phi$, which closely tracks the minimum rms. Dot-dashed line: behavior of the two-mode state $(|N0\rangle + |0N\rangle)/\sqrt{2}$, which is optimal only for high η and low N. Upper dashed line: average over φ of the rms. The gray shaded area encloses the "quantum" trajectories—i.e., the ones included between the shot-noise limit $1/\sqrt{N\eta}$ and the Heisenberg limit $1/N$. (Right) Same as the previous, but as a function of η for $N = 20$.

$$|\Phi_l\rangle \equiv \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} e^{i\Phi_l}|n\rangle \quad \text{with } \Phi_l = \frac{2\pi l}{M+1}, \qquad (3)$$

with $l = 0, \ldots, M$. The root mean square (RMS) of the probability distribution obtained from this POVM is a function of the interferometer phase φ. Its minimum value gives the minimum error that our interferometer can achieve, which is plotted as the solid lines in Fig. 2. In addition, we have also directly estimated the error through the Holevo variance [13] $\Delta\Phi = (S_\Phi^{-2} - 1)^{1/2}$, where $S_\Phi = |\langle e^{i\Phi}\rangle|$ is the average value of the function $e^{i\Phi}$ weighted with the probability obtained from the continuous POVM $|\Phi\rangle\langle\Phi| d\Phi$, obtained using the phase states of Eq. (3) for arbitrary Φ. The Holevo variance is more appropriate than the rms for the estimation of the phase error, since the phase is a periodic quantity [13]. However, as is clear from our plots, the Holevo variance is well approximated by the rms when these two quantities are small enough (compared to 2π).

It is tedious but straightforward to calculate that, after a round-trip characterized by the same quantum efficiency η in both arms, the state $|\psi_{opt}\rangle$ evolves into

$$\rho = \frac{2}{M+1} \sum_{i,j=0}^{M} (1-\eta)^{i+j} \eta^{M-j} \sum_{n,m} \omega_n \omega_m e^{i\varphi(m-n)} |n\rangle\langle m|,$$

$$\omega_k \equiv \binom{k+j}{j}^{1/2} \binom{M-k-j+i}{i}^{1/2} \times \sin\left[\frac{\pi\left(M-k-j+i+\frac{1}{2}\right)}{M+1}\right],$$

where the second sum runs between max $(0, i-j)$ and $M-j$.

From this state, one can estimate the probability distribution of the POVM of Eq. (3): namely,

$$p(l) = \frac{2}{(M+1)^2} \sum_{i,j} (1-\eta)^{i+j} \eta^{M-j} \left| \sum_n \omega_n e^{-in(\varphi+\Phi_l)} \right|^2.$$

The error in the estimation of the phase from a measurement on ρ is given by the rms of $p(l)$, plotted in Fig. 2 as a function of the average photon number N and quantum efficiency η. The graphs of Fig. 2 show that our scheme can beat the shot noise also for low values of N, where in addition to the asymptotic behavior, the scaling constant is also important [8,15].

The second input state we consider is a single mode analogous of the $|m::m'\rangle$ state introduced in [5]—i.e., the state

$$(|M\rangle + |M'\rangle)/\sqrt{2}, \tag{4}$$

where $M > M'$ and whose average photon number is $N = (M+M')/2$. Again, it is straightforward to obtain the output state, after the round-trip of Fig. 1(b):

$$\sigma = \sum_{j=-\delta}^{M'} \alpha_j |j+\delta\rangle\langle j+\delta| + \sum_{j=0}^{M} \beta_j |j\rangle\langle j| + \frac{1}{2}\sum_{j=0}^{M'} \gamma_j [e^{-i\delta\varphi}|j+\delta\rangle\langle j| + e^{i\delta\varphi}|j\rangle\langle j+\delta|], \tag{5}$$

$$\alpha_j = \sum_{i=max(0,j)}^{M'} f_{ij} \binom{M'}{i} \binom{i+\delta}{i-j} \bigg/ 2, \tag{6}$$

$$\beta_j = \sum_{i=j}^{M} f_{ij} \binom{M}{i} \binom{i}{j} \bigg/ 2, \tag{7}$$

$$\gamma_j = \sum_{i=j}^{M'} f_{ij} \left[\binom{M'}{i} \binom{M}{i} \binom{i+\delta}{i-j} \binom{i}{j} \right]^{1/2}, \tag{8}$$

with $\delta \equiv M - M'$ and $f_{ij} \equiv (1-\eta)^{2i-j} \eta^{M-i+j}$. To extract the phase from the state σ, in analogy to the two-mode case of [5], we can measure the observable

$$A = \sum_{k=0}^{M'} |M-k\rangle\langle M'-k| + |M'-k\rangle\langle M-k|. \tag{9}$$

Then, the error on the phase can be obtained from the rms of A using error propagation: namely,

$$\Delta\varphi = \Delta A \bigg/ \left| \frac{\partial}{\partial\varphi}\langle A\rangle \right| = \frac{\sqrt{\Theta - \cos^2(\delta\varphi)\Gamma^2}}{\delta|\sin(\delta\varphi)\Gamma|}, \tag{10}$$

where $\Theta = \sum_{k=0}^{M'} \alpha_k + \alpha_{k-\delta} + \beta_k + \beta_{k+\delta}$ and $\Gamma = \sum_{k=0}^{M'} \gamma_k$. This quantity is plotted in Fig. 3 from which it is evident that, also in this case, the multiround protocol can achieve a better sensitivity than the optimal two-mode one. One can employ the techniques suggested in [5,16] (for the two-mode case) to implement also the single-mode $|m::m'\rangle$ and its related observable A presented here, but our scheme is simpler as it does not entail entanglement. It may appear surprising that the state σ permits one to achieve a greater precision than the NOON state $(|N0\rangle + |0N\rangle)/\sqrt{2}$ even for high values of η. As discussed above, this is essentially due to the fact that a single-mode state performs better than a two-mode state in terms of the average number of photons, N.

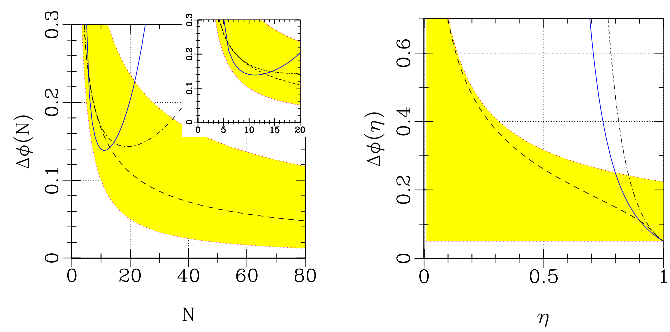

FIG. 3. (Color online) (Left) Solid line: error in the estimation of the phase from the state σ of Eq. (5)—i.e., minimum (over φ) of the function $\Delta\varphi$ of Eq. (10)—as a function of the average number of photons N for $\eta = 0.9$. Here $M' = 3$ and $M = 2N - 3$. Dashed line: plot of the error from the optimal two-mode lossy interferometry from [4]. Again, as in Fig. 2, our single-mode method achieves a better sensitivity in some range. Dot-dashed line: behavior of the state $(|N0\rangle + |0N\rangle/\sqrt{2})$. The inset is an enlargement for small values of N. (Right) Same as the previous, but as a function of η for $N = 20$, $M = 30$, and $M' = 10$.

In conclusion, we have given a strategy for determining the relative phase in an interferometer using single-mode states that are sent through the interferometer in a round-trip, interleaved by the unitary U of Eq. (1). This entails that (a) in the noiseless case a double sensitivity can be reached over the optimal two-mode states (such as the NOON state) in terms of the average number of photons and (b) in the lossy case we can achieve a better phase sensitivity than what is claimed to be the optimal two-mode strategy [4], proving that multiround protocols are preferable in the presence of noise. The robustness in the face of loss stems from two main properties. On one side there is no entanglement between different modes and it is well known that entanglement is very sensitive to noise [3]. On the other side, the fact that we are using a single mode permits a doubling of the phase sensitivity over the two-mode entangled case, since all the photons travel through one mode only. One may object that the increased phase sensitivity arises because we are devoting more resources to the estimation. This objection is unfounded since the average number of photons employed in the two strategies is the same, N. One cannot even say that in the two-mode strategies the phase φ is sampled by fewer photons, because the number of photons that travel through an arm of an interferometer is an undefined quantity (the "which path" information is complementary to the phase information). One can only bound the number of photons traveling through each arm with the total number of photons, N, injected in the interferometer.

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