

Range corrections to three-body observables near a Feshbach resonance

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A nonrelativistic system of three identical particles will display a rich set of universal features known as *Efimov* physics if the scattering length a is much larger than the range l of the underlying two-body interaction. An appropriate effective theory facilitates the derivation of both results in the $|a| \rightarrow \infty$ limit and finite- l/a corrections to observables of interest. Here we use such an effective-theory treatment to consider the impact of corrections linear in the two-body effective range, r_s , on the three-boson bound-state spectrum and recombination rate for $|a| \gg |r_s|$. We do this by first deriving results appropriate to the strict limit $|a| \rightarrow \infty$ in coordinate space. We then extend these results to finite a using once-subtracted momentum-space integral equations. We take the cutoff on these equations to be large compared to $1/|r_s|$, and find that the first-order effects of $|r_s|$ are independent of the cutoff in this regime. We also discuss the implications of our results for experiments that probe three-body recombination in Bose-Einstein condensates near a Feshbach resonance.

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I. INTRODUCTION

Systems in which the two-body scattering length, a , is much larger than the range, l , of the underlying interaction are said to be close to the “unitary limit,” $|a| \rightarrow \infty$. In that limit the two-body scattering saturates the bound set by the requirement that the S matrix be unitary. The dynamics of such systems for energies $E \sim \hbar^2/(Ma^2)$ (M is the particle mass) is universal, in the sense that it is independent of any details of the underlying two-body interaction. One obvious example of this “universality” is that for $a > 0$ all such systems have a two-body bound state (*dimer*) with binding energy $E_D \approx \hbar^2/(Ma^2)$.

A nonrelativistic system of three identical particles with $|a| \gg l$ displays a rich set of universal features [1]. In the limit $|a| \rightarrow \infty$ the system has an infinite tower of three-body bound states (*trimers*). Their binding energies are

$$B_n = (e^{-2\pi s_0})^{n-n^*} B_{n^*}, \quad (1)$$

where B_{n^*} denotes the energy of an arbitrary state in the spectrum, and the numerical constant $s_0 \approx 1.00624$. The nature of the geometric spectrum (1) is a consequence of the discrete scale invariance of the three-body problem in the joint limit $|a| \rightarrow \infty$, $l \rightarrow 0$. Efimov, who made this discovery, pointed out that this discrete scale invariance is also relevant for finite a and that it should be observable in systems for which $l/|a| \neq 0$, but is still $\ll 1$ [2,3].

Evidence for Efimov physics in cold atoms was recently presented by the Innsbruck group [4]. Using a magnetic field to control the scattering length, they measured the recombination rate of cold ^{133}Cs atoms in the lowest hyperfine state

and found resonant enhancement at $a \sim -850a_0$, which can be attributed to the existence of an Efimov trimer at the three-body threshold. Approaches which employ a zero-range model and therefore account for the separation of scales between a and l by taking the limit $l \rightarrow 0$ are very successful in describing this experimental data [5,6]. But a systematic calculation of the corrections to these results due to finite-range effects is desirable, in order to identify the limitations of these previous calculations and also to extend the range of applicability of approaches based on the scale hierarchy $l \ll |a|$.

In recent years Efimov’s results have been rederived in the framework of an effective field theory (EFT) [7]. This EFT employs the ratio $l/|a|$ as a small expansion parameter and allows for a systematically improvable and model-independent calculation of observables of few-body systems with a large scattering length. The leading order (LO) of this EFT reproduces all universal features previously discovered by Efimov.

In this paper we will exploit effective theory methods to derive the correction linear in the effective range to this universal behavior (1). In the EFT expansion this corresponds to the next-to-leading order (NLO) piece. The value of the two-body effective range, r_s , sets the size of these corrections. We evaluate their impact perturbatively, and so obtain for the three-body bound-state spectrum a result of the form

$$B_n = B_{n^*} \left[F_n \left(\frac{\gamma}{\kappa_*} \right) + \kappa_* r_s G_n \left(\frac{\gamma}{\kappa_*} \right) + O[(\kappa_* r_s)^2] \right], \quad (2)$$

where $\kappa_* = \sqrt{MB_{n^*}}$ is the binding momentum of the n^* th state in the three-body bound-state spectrum, $\gamma = 1/a$ (at LO) is the binding momentum of the two-body bound state that is present for $a > 0$, and the functions F_n obey [1]

$$F_n(e^{m\pi/s_0}x) = e^{-2m\pi/s_0} F_{n-m}(x). \quad (3)$$

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We will show that the functions G_n obey approximate relations similar to Eq. (3), since discrete scale invariance connects the G_n 's for different values of n . Knowledge of the functions F_{n*} and G_n therefore provides information on the behavior of the bound-state spectrum in the vicinity of a narrow Feshbach resonance to a precision given by $(r_s/a)^2$. In this work we will give results for the function G_{n*} from the unitary limit up to the value of a where the three-body bound state under consideration hits the threshold. In particular, we will show that this function is zero at particular values of a .

We do this by formally studying systems in which the effective range, r_s , is significantly larger than the range l of the underlying two-body potential, i.e., the scale hierarchy is $l \ll |r_s| \ll |a|$. It is important to point out that in such a three-scale problem it is possible to be in the unitary limit, but still have range effects present in the result, i.e., to not have achieved what is sometimes called the “scaling limit,” where $r_s \rightarrow 0$ and $l \rightarrow 0$. In fact, in the EFT formulation of the problem the range of the underlying two-body potential, l , provides a limiting distance, because the EFT does not specify any details of the two-body potential, and so it does not accurately describe the three-body dynamics in those regions of configuration space where any two-body distance is less than l . Hence it is natural to place a cutoff Λ on the momentum-space integrals present in the EFT calculations, and choose $\Lambda \sim 1/l$. In our case we are interested in systems with $l \ll |r_s|$, and so we will ultimately take the limit $\Lambda \rightarrow \infty \equiv l \rightarrow 0$ in order to obtain our results. The effective range, r_s , will, however, be kept finite.

The inclusion and effects of range corrections in the EFT with contact interactions have been studied extensively. In [8] range corrections in the three-nucleon system were calculated perturbatively in momentum space. A formalism which employs a partial resummation of range effects was developed in Refs. [9–11] and observables were calculated up to next-to-next-to-leading order (N2LO) in the l/a expansion. Hyperradial coordinates and the Wilsonian renormalization group were used to rederive the leading-order amplitude and develop a power counting for subleading effects in Ref. [12].

In Sec. II we will briefly review the hyperradial formalism and show analytically that in the limit of infinite scattering length no linear range correction exists, i.e., that $G_n(0)=0$ for all n . In Sec. III we will review a formalism that allows us to compute numerical results for this linear correction away from the unitary limit and present numerical results for the three-body bound-state spectrum and the function G_n . In Sec. IV we will lay out how the formalism developed for the analysis of Sec. III can be extended to the computation of range corrections for the three-body recombination rate in a cold gas of identical bosons. We will end with a summary and discussion of our findings.

II. NEAR A FESHBACH RESONANCE: RESULTS IN THE UNITARY LIMIT, WITH NONZERO EFFECTIVE RANGE

In this section we first review the use of hyperradial coordinates for the three-body problem in the unitary limit

(Sec. II A). Our presentation closely follows that of Ref. [1]. In order to fix notation and methodology for the subsequent sections we derive the potential between the single particle and the pair in hyperspherical coordinates, and show that it behaves similar to one over the hyperradius, R , squared for $|r_s| \ll R \ll |a|$. We explain why—in the joint limits $r_s \rightarrow 0$ and $|a| \rightarrow \infty$ —this results in the Efimov spectrum described above, and compute the wave functions in this limit. A somewhat different explanation of these results—one based on Efimov's original work [2,3]—can be found in Appendix A of Ref. [12].

We then show how to compute perturbative changes to this spectrum as we move away from the scaling limit. In Sec. II B, we derive the terms in the hyperspherical potential that scale as r_s/R^3 , and compute the matrix elements of this potential between the wave functions obtained with the pure $1/R^2$ potential. These matrix elements diverge in the ultraviolet, but we show that if the spectrum is renormalized by demanding that short-distance physics at $R \sim r_s$ remains unchanged when we move from LO to NLO then there is no change in the Efimov spectrum at $O(r_s)$.

A. Hyperradial formalism: A review

The material of this section is based on the discussion of the hyperradial formalism in the recent review of Ref. [1]. For three particles of equal mass the Jacobi coordinates are defined as

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j; \quad \mathbf{r}_{k,ij} = \mathbf{r}_k - \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j), \quad (4)$$

where the triple (ijk) is a cyclic permutation of the particle indices (123). The hyperradius R and hyperangle α_k are then defined by

$$R^2 = \frac{1}{3}(\mathbf{r}_{12}^2 + \mathbf{r}_{23}^2 + \mathbf{r}_{31}^2) = \frac{1}{2}\mathbf{r}_{ij}^2 + \frac{2}{3}\mathbf{r}_{k,ij}^2; \quad \alpha_k = \arctan\left(\frac{\sqrt{3}|\mathbf{r}_{ij}|}{2|\mathbf{r}_{k,ij}|}\right). \quad (5)$$

We will assume that the potential V depends only on the magnitude of the interparticle separation and write

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V(r_{12}) + V(r_{23}) + V(r_{31}). \quad (6)$$

We now employ the usual Faddeev decomposition of ψ for three identical bosons and neglect subsystem angular momentum

$$\Psi(R, \alpha, \Omega) = \psi(R, \alpha_1) + \psi(R, \alpha_2) + \psi(R, \alpha_3). \quad (7)$$

We exploit the fact that $\psi(R, \alpha_1)$ is independent of $\hat{r}_{23}$ and $\hat{r}_{1,23}$ to obtain the simplified Faddeev equation

$$\begin{aligned}
& \left[\frac{\hbar^2}{2M} R^{-5/2} \left(-\frac{\partial}{\partial R^2} + \frac{15}{4R^2} \right) R^{5/2} \right. \\
& \quad \left. + \frac{\hbar^2}{2MR^2} \frac{1}{\sin 2\alpha} \left(-\frac{\partial^2}{\partial \alpha^2} - 4 \right) \sin 2\alpha - E \right] \psi(R, \alpha) \\
& = -V(\sqrt{2}R \sin \alpha) \left(\psi(R, \alpha) \right. \\
& \quad \left. + \frac{4}{\sqrt{3}} \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \frac{\sin 2\alpha'}{\sin 2\alpha} \psi(R, \alpha') d\alpha' \right). \quad (8)
\end{aligned}$$

The solution of this equation can be expanded in a set of eigenfunctions of the hyperangular operator, i.e.,

$$\psi(R, \alpha) = \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha), \quad (9)$$

where the functions ϕ_n satisfy

$$\begin{aligned}
& \left[-\frac{\partial^2}{\partial \alpha^2} - \lambda_n(R) \right] \phi_n(R, \alpha) \\
& = -\frac{2MR^2}{\hbar^2} V(\sqrt{2}R \sin \alpha) \left[\phi_n(R, \alpha) \right. \\
& \quad \left. + \frac{4}{\sqrt{3}} \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \phi_n(R, \alpha') d\alpha' \right], \quad (10)
\end{aligned}$$

with boundary conditions $\phi_n(R, 0) = \phi_n(R, \frac{\pi}{2}) = 0$. Meanwhile the hyperradial functions $f_n(R)$ satisfy

$$\begin{aligned}
& \left[\frac{\hbar^2}{2M} \left(-\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\
& + \sum_m \left[2P_{nm}(R) \frac{\partial}{\partial R} + Q_{nm}(R) \right] f_m(R) = E f_n(R), \quad (11)
\end{aligned}$$

with the hyperradial potential $V_n(R)$ defined by

$$V_n(R) = [\lambda_n(R) - 4] \frac{\hbar^2}{2MR^2}, \quad (12)$$

and $P_{nm}(R)$ and $Q_{nm}(R)$ (defined in [1]) potentials that induce coupling between different hyperradial channels. These potentials vanish exactly if the ϕ_n are independent of R . As we shall now show, if the potential V is short ranged, then the ϕ_n 's are independent of R . It turns out that in this situation the hyperradial coupling potentials P_{nm} and Q_{nm} vanish exactly.

For hyperradii R which are much larger than the range over which V is nonzero, say l , we can consider α large enough that $V(\sqrt{2}R \sin \alpha)$ is zero. The solution of Eq. (10) in this region is then

$$\phi_n^{(\text{high})}(\alpha) \approx \sin \left[\sqrt{\lambda_n} \left(\frac{\pi}{2} - \alpha \right) \right]. \quad (13)$$

Note that we have implicitly assumed that λ_n becomes independent of R , an assumption which will be justified *a posteriori* in the limit $R \ll |a|$.

On the other hand, as long as we choose $R \gg l$, there is always a region $\alpha \leq l/(\sqrt{2}R) \ll 1$. Here $\hbar^2 \lambda_n(R)/(2MR^2)$ is much less than $V(\sqrt{2}R \sin \alpha)$, which typically will be of order $\hbar^2/(2Ml^2)$. Consequently for this α domain the differential equation for $\phi_n(\alpha)$ becomes

$$\begin{aligned}
& -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \phi_n^{(\text{low})}}{\partial \alpha^2} + V(\sqrt{2}R \alpha) \phi_n^{(\text{low})}(\alpha) \\
& = -\frac{8\alpha}{\sqrt{3}} V(\sqrt{2}R \alpha) \phi_n^{(\text{high})} \left(\frac{\pi}{3} \right). \quad (14)
\end{aligned}$$

Here we have used the fact that $\alpha \ll 1$ in this region to approximate the integral in Eq. (10), and to make the replacement $\sin \alpha \rightarrow \alpha$. If the derivatives are all of natural size, then the corrections to this equation are suppressed by α^2 .

The homogenous version of Eq. (14) has the solution

$$\phi_n^{(\text{low})}(\alpha) = A \psi_{k=0}(\sqrt{2}R \alpha), \quad (15)$$

where $\psi_k(r)$ is the wave function of the two-body system for two-body energy $\frac{\hbar^2 k^2}{2M}$, and A is a constant. Adding a particular solution of the inhomogeneous equation we find

$$\phi_n^{(\text{low})}(\alpha) = A \psi_0(\sqrt{2}R \alpha) - \frac{8\alpha}{\sqrt{3}} \sin \left(\sqrt{\lambda_n} \frac{\pi}{6} \right). \quad (16)$$

We now consider Eq. (14) for values of α such that $l/R \lesssim \alpha \ll 1$. In this domain the analysis that led to Eq. (16) still applies. But because we are now beyond the range of V , we have

$$\psi_k(r) = \frac{\sin[kr + \delta(k)]}{k} = \frac{\sin \delta(k)}{k} [\cos(kr) + \cot \delta \sin(kr)]. \quad (17)$$

As $k \rightarrow 0$ this yields $\psi_0(r) = r - a$, and we can use this asymptotic two-body wave function in Eq. (16). This gives

$$\phi_n^{(\text{low})}(\alpha) = A(\sqrt{2}R \alpha - a) - \frac{8\alpha}{\sqrt{3}} \sin \left(\sqrt{\lambda_n} \frac{\pi}{6} \right). \quad (18)$$

Because $V=0$ in this region this result must match onto that of Eq. (13). This occurs through a suitable choice of A and if the condition

$$\cos \left(\sqrt{\lambda_n} \frac{\pi}{2} \right) - \frac{8}{\sqrt{3\lambda_n}} \sin \left(\sqrt{\lambda_n} \frac{\pi}{6} \right) = \sqrt{\frac{2}{\lambda_n}} \sin \left(\sqrt{\lambda_n} \frac{\pi}{2} \right) \frac{R}{a} \quad (19)$$

is satisfied. We note that if these equations are satisfied λ_n , and hence ϕ_n , is independent of R for $R \ll |a|$, as promised. Indeed, with the correct choice for A and as long as Eqs. (19) is satisfied the form (13) is the result for ϕ for all α such that $\alpha > l/R$. Solving Eq. (19) in the limit $R \ll |a|$ we find the lowest eigenvalue

$$\lambda_0 = -s_0^2 \left(1 + 1.897 \frac{R}{a} \right), \quad (20)$$

with $s_0 = 1.00624 \dots$. This is the only negative eigenvalue, and hence only this channel potential is attractive. So, if we

now focus on the unitary limit, where $|a| \rightarrow \infty$, we have $\lambda_0 = -s_0^2$. The hyperradial equation (11) in this channel then becomes

$$\frac{\hbar^2}{2M} \left(-\frac{\partial^2}{\partial R^2} - \frac{s_0^2 + \frac{1}{4}}{R^2} \right) f_0(R) = E f_0(R). \quad (21)$$

This equation will hold for $R \gg l$. If we desire a solution for negative E the necessity to have f_0 be normalizable mandates that

$$f_0^{(0)}(R) = \sqrt{R} K_{is_0}(\sqrt{2}\kappa R), \quad (22)$$

where $K_{is_0}(z)$ is a Bessel function with imaginary index and the superscript (0) indicates that we are working in the unitary limit, while the subscript 0 refers to the solution for the hyperchannel corresponding to λ_0 , which is the only one that supports bound states. The binding energy of these bound states is related to the κ of Eq. (22) by

$$|E_3| \equiv B \equiv \frac{\hbar^2 \kappa^2}{M}. \quad (23)$$

Since the attractive $1/R^2$ potential produces a spectrum that is unbounded from below some other short-distance physics is needed in order to stabilize the system. If the two-body potential is known this short-distance physics is provided by the two-body potential V , that becomes operative for $R \sim l$. But an alternative approach is to add an additional term to Eq. (21) that summarizes the impact of the two-body V . Here we take this potential to be a surface δ function at a radius $1/\Lambda$ [13]

$$V_{SR}(R) = H_0(\Lambda) \Lambda^2 \delta\left(R - \frac{1}{\Lambda}\right), \quad (24)$$

with H_0 adjusted as a function of Λ such that the binding energy of a particular state, say B_{n*} [with a corresponding κ_* , given by Eq. (23)], is reproduced. Note that since V_{SR} is operative only at distances $R \sim 1/\Lambda$ it corresponds to a three-body force. (See Ref. [7] for a realization of this in a momentum-space formalism.)

In physical terms we anticipate $l \sim 1/\Lambda$, since we know that once we consider hyperradii of order $1/\Lambda$ the potential V_{SR} starts to affect the solutions. Our goal here is to derive *universal* results, that are independent of details of the original two-body potential. Consequently we will use the procedure described in the previous paragraph to fix the value of H_0 for a given Λ , and then examine the residual Λ dependence of our results. If we can take the limit $\Lambda \rightarrow \infty$ without it significantly impacting our calculations of observables (formally, if all effects scale with negative powers of Λ) then these observables are not sensitive to the physics that takes place in the three-body system for $R \sim l$. Although we know that our theory is not valid in the region $R \sim l$, its predictions that have only small ($\sim 1/\Lambda$) corrections due to that short-distance physics will be “universal” and not significantly affected by the inaccuracies in the short-distance region. As we shall see, the excited-state spectrum is one such

observable—which is not surprising, since these states have very small binding energy with respect to the scale l .

Given that our focus is on predictions of the theory that are independent of details of V we can consider the extreme case and take the limit $l \rightarrow 0$. In this limit the form of K_{is_0} as $R \rightarrow 0$ guarantees that once H_0 is fixed to give a bound state at $|E_3| = B_{n*}$, the other binding energies in this hyperradial eigenchannel form a geometric spectrum. Namely, $B_n = \hbar^2 \kappa_n^2 / M$ with

$$\kappa_n = (e^{-\pi/s_0})^{n-n_*} \kappa_*, \quad (25)$$

with n_* the index of the bound state corresponding to κ_* . Thus Eq. (1) is obtained. Equation (25) will hold for all κ_n such that $\kappa_n \ll \Lambda$. (Note that now the subscript on κ denotes the index of the bound state in adiabatic channel zero.) The continuous scale invariance of the $1/R^2$ potential has been broken down to a discrete scale invariance by the imposition of particular short-distance physics on the problem through the short-distance potential (24) [13].

B. The linear range correction in the unitary limit

The solutions derived in the previous section are valid in the strict limit $|a| \rightarrow \infty$. In this section we consider the corrections to this limit that are order $(r_s \kappa_*)$. In performing this computation we shall assume that the range of the two-body potential l is zero.¹ The calculation we are performing therefore corresponds to the limit where $l \ll |r_s| \ll |a|$. Since the natural scale for the cutoff $\Lambda \sim 1/l$ we will consider $\Lambda |r_s| \gg 1$.

The first-order correction to EFT results in the $|a| \rightarrow \infty$ limit was already discussed in Ref. [8]. However, in that work Hammer and Mehen considered $r_s/|a|$ effects, and so the first-order corrections they discuss go to zero in the limit $|a| \rightarrow \infty$ that we are considering here. As we shall show here, in principle there is a first-order correction proportional to $r_s \kappa_*$ that could survive if one considers $|a| \rightarrow \infty$ but keeps r_s finite. However, it turns out that the discrete scale invariance that relates wave functions for different bound states at $r_s = 0$ guarantees that the particular operator in question has zero matrix element in the limit $\Lambda \rightarrow \infty$.

To prove this we first observe that in the limit $\Lambda |r_s| \gg 1$ the form (13) applies for all values of α , as long as Eq. (19) is obeyed. There are then two possible sources of corrections to the Efimov spectrum (25) that are linear in r_s . First, we must reconsider Eq. (14), and take account of the previously neglected term $\hbar^2 \lambda_n / (2MR^2)$. As we shall see this leads to the conclusion that $\phi_n(\alpha)$ is not independent of R , and so then we must also check that P_{nm} and Q_{nm} remain zero at the level of accuracy we work to here.

Reconsidering Eq. (10) we assume that $V \sim \hbar^2 / (2ml^2)$ and consider the region $l/R \lesssim \alpha \ll 1$. We can then write

¹One might be concerned that this precludes consideration of a positive effective range because of the bound derived in Refs. [14,15]. But that bound applies only to energy-independent potentials. If the underlying two-body potential $V(R)$ is energy dependent positive effective ranges can be generated no matter how small l is.

$$\left[-\frac{\partial^2}{\partial \alpha^2} - \lambda_n + \frac{2MR^2}{\hbar^2} V(\sqrt{2}R \sin \alpha) \right] \phi_n^{(\text{low})}(\alpha) \approx -\frac{2MR^2}{\hbar^2} V(\sqrt{2}R \sin \alpha) \frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{high})}\left(R, \frac{\pi}{3}\right). \quad (26)$$

The approximate form adopted here for the integral on the right-hand side of Eq. (10) is itself accurate to relative order α^2 , meaning that the right-hand side omits only terms of overall order α^3 . Therefore the first correction to Eq. (26) is $O(l^3/R^3)$ relative to leading. This equation thus goes one order in $\alpha \sim l/R$ beyond Eq. (14).

In general the homogeneous form of Eq. (26) is difficult to solve. But, for the specific case of $l \rightarrow 0$, we need only consider the effects of V for $\alpha \rightarrow 0$. There $\sin \alpha = \alpha$ still holds. Hence the solution of the homogenous form of Eq. (26) for any $\alpha > 0$ becomes

$$\phi_n^{(\text{low})}(R, \alpha) = A(R) \psi_{k_n}(\sqrt{2}R\alpha), \quad (27)$$

with $\psi_k(r)$ defined by Eq. (17) and $k_n = \frac{\sqrt{\lambda_n}}{\sqrt{2}R}$. Meanwhile, the particular solution of Eq. (26) is unchanged up to the accuracy to which we work here, and so

$$\phi_n^{(\text{low})}(R, \alpha) = A(R) \psi_{k_n}(\sqrt{2}R\alpha) - \frac{8\alpha}{\sqrt{3}} \sin\left(\sqrt{\lambda_n} \frac{\pi}{6}\right) + O(\alpha^3). \quad (28)$$

Substituting in the form (17) for $\psi_{k_n}(r)$ we see that $\phi_n^{(\text{low})}$ will still match smoothly with the large- α solution (13), provided that

$$\begin{aligned} & \cos\left(\sqrt{\lambda_n} \frac{\pi}{2}\right) - \frac{8}{\sqrt{3\lambda_n}} \sin\left(\sqrt{\lambda_n} \frac{\pi}{6}\right) \\ &= -Rk_n \cot(\delta(k_n)) \sqrt{\frac{2}{\lambda_n}} \sin\left(\sqrt{\lambda_n} \frac{\pi}{2}\right) \\ &= \left[\frac{R}{a} - \frac{\lambda_n r_s}{4R} + O\left(\frac{l^3}{R^3}\right) \right] \sqrt{\frac{2}{\lambda_n}} \sin\left(\sqrt{\lambda_n} \frac{\pi}{2}\right). \end{aligned} \quad (29)$$

In Eq. (30) we have assumed only that the coefficients of all terms in the effective-range expansion after the $O(k^2)$ one scale with l .

In this case Eq. (30) can be solved by noting that $\lambda_n(R)$ has a self-consistent solution in the region $|r_s| \ll R \ll |a|$,

$$\lambda_n(R) = \lambda_n^{(0)}(0) \left(1 + \gamma_n \frac{R}{a} + \xi_n \frac{r_s}{R} + \dots \right), \quad (31)$$

with $\lambda_n^{(0)}(R)$ the solution of Eq. (19), which here we need only at $R=0$. By a perturbative evaluation of Eq. (30) in powers of r_s we find that

$$\xi_n = -\frac{\lambda_n^{(0)}(0) \gamma_n}{4} \quad (32)$$

encodes the $O(r_s)$ correction to λ_n in the limit $|a| \rightarrow \infty$. Hence, the value of the lowest eigenvalue is

$$\lambda_0(R) = -s_0^2 \left(1 + 1.897 \frac{R}{a} + 0.480 \frac{r_s}{R} + \dots \right), \quad (33)$$

where we have omitted corrections of second order in R/a and r_s/R and have not considered r_s/a corrections (in the following section we will include the effects of such corrections numerically). In what follows, we will focus only on the lowest eigenvalue $\lambda_0(R)$.

With $\lambda_0(R)$ in hand we can compute

$$\phi_0(\alpha; R) = \sin \left[\sqrt{\lambda_0(R)} \left(\frac{\pi}{2} - \alpha \right) \right] \quad (34)$$

for $\alpha > l/R$, up to corrections of order $(r_s/R)^2$. The normalization of the ϕ_n is arbitrary, and for our purposes it makes sense to normalize them such that [16], for instance,

$$\int_0^{\pi/2} \phi_n^2(\alpha; R) d\alpha = 1. \quad (35)$$

This results in

$$\phi_0(\alpha; R) = \frac{1}{N(R)} \sin \left[\sqrt{\lambda_0(R)} \left(\frac{\pi}{2} - \alpha \right) \right],$$

$$\text{with } N^2(R) = \frac{\pi}{4} [1 - j_0(\pi \sqrt{\lambda_0(R)})], \quad (36)$$

where j_0 is the spherical Bessel function of zeroth order.

The fact that λ_n now depends on R opens the possibility that the hyperchannel coupling potentials P_{nm} and Q_{nm} will no longer be zero. In the unitary limit, both can be at most $O(\frac{r_s}{R^3})$, since if $r_s=0$ the eigenvalues are independent of R and these potentials vanish. Thus, at leading order in r_s , we need only consider $P_{nn}(R)$ and $Q_{nn}(R)$. Given that our focus here is on the bound-state spectrum this means we only have to examine $P_{00}(R)$ and $Q_{00}(R)$. The use of normalized $\phi_0(\alpha; R)$ can then easily be seen to guarantee that $P_{00}(R)$ is identically zero [16]. A brief computation also shows that

$$Q_{00}(R) = 0 + O\left(\frac{r_s^2}{R^2}\right), \quad (37)$$

and so we need not concern ourselves further with these adiabatic coupling potentials in pursuing our calculation of effects linear in r_s . This extends the analysis of Ref. [17] since, in the specific limit considered here, Q_{00} vanishes as $1/R^4$, i.e., one power faster than the $1/R^3$ proven in Ref. [17]. The results of this paragraph thus show that, in the limit $l \ll |r_s| \ll |a|$, the hyperadiabatic approximation yields the correct hyperradial potential up to corrections of $O(r_s^2/R^4)$.

Consequently, in the unitary ($|a| \rightarrow \infty$), but not scaling ($r_s \neq 0$ and $|r_s| \gg l$) limit the hyperradial equation (11) becomes

$$\frac{\hbar^2}{2M} \left(-\frac{\partial^2}{\partial R^2} - \frac{s_0^2 + \frac{1}{4}}{R^2} - \frac{s_0^2 \xi_0 r_s}{R^3} \right) f^{(1)}(R) = E^{(1)} f^{(1)}(R), \quad (38)$$

where the superscript 1 indicates that we have worked only to first order in r_s .

The form of Eq. (38), but not the specific prefactor of the $1/R^3$ potential, was proposed by Efimov on dimensional grounds [19]. A later calculation by Efimov gave an implicit expression for the prefactor [20]. Meanwhile, the explicit expression

$$-c_E \frac{|r_s|}{R^3} \left(1 + \frac{\pi^2}{48s_0^2} \right) \quad (39)$$

with c_E equal to our numerical coefficient $s_0^2 \xi_0 = 0.48 \dots$ was obtained by Nielsen *et al.* for the correction to the $1/R^2$ hyperradial potential due to the finite (but still $\ll |a|$) range of the two-body interaction [18]. Indeed, in Ref. [18] an explicit analytical form for c_E is presented: it is a derivative of the function on the left-hand side of the leading-order eigenvalue equation (19). But the additional factor on the right-hand side of Eq. (39) as compared to our result occurs because in Ref. [18] the matching between the “high” and “low” hyperangular solutions was done at $\alpha = \arcsin(l/R\sqrt{2})$ and not, as here, at an arbitrarily small α . The sensitivity to short-distance details of the two-body potential that could thereby occur does not affect the formula of Ref. [18] because the square-well potential considered there has $r_s = l$ in the unitary limit. In contrast our result is derived by first taking the limit $l \rightarrow 0$ and then considering the linear-in- r_s correction to the hyperradial potential. In this way we treat the two scales l and r_s as independent, and also show that the $1/R^3$ piece of the hyperradial potential is universal in the limit $|r_s| \gg l$, i.e., it is determined solely by r_s , and does not depend on the particular form of the two-body interaction at short distances.

In more recent EFT-based work, the momentum-space evaluation of first-order (in $r_s/|a|$) effects in Ref. [8] neglected the operator corresponding to the $1/R^3$ potential that appears in Eq. (38). The authors of Ref. [8] argued that the piece of the two-body scattering amplitude that survives in the limit $|a| \rightarrow \infty$ and is proportional to r_s does not have a two-body bound-state pole and so cannot lead to effects in the three-body system. As we shall now see the conclusion that the $1/R^3$ potential does not affect the spectrum in first-order perturbation theory in the unitary limit is correct, although it is *not* justified to neglect this piece of the three-body dynamics entirely, as was done in Ref. [8].

Now—to the extent that the r_s/R^3 term is a perturbation—the shift in the Efimov spectrum near a Feshbach resonance (i.e., near $|a| = \infty$) can be evaluated by computing the matrix elements of this perturbing $1/R^3$ potential between leading-order hyperradial functions (22). In other words, we have, for the first-order shift in the bound-state energy,

$$\frac{2M}{\hbar^2} \Delta B_n^{(1)} = s_0^2 r_s \xi_0 \int dR f_n^{(0)2}(R) \frac{1}{R^3}, \quad (40)$$

with $E_n^{(1)} = -(\frac{\hbar^2 \kappa_n^2}{M} + \Delta B_n^{(1)})$ the energy of the n th bound state up to first order in r_s . The $f_n^{(0)}$ that appears in Eq. (40) is the normalized zeroth-order hyperradial wave function of the corresponding bound state [1],

$$f_n^{(0)}(R) = 2\kappa_n \sqrt{\frac{\sinh(\pi s_0)}{\pi s_0}} R^{1/2} K_{is_0}(\sqrt{2}\kappa_n R). \quad (41)$$

Unfortunately the integral in Eq. (40) is divergent. This divergence can, however, be absorbed by a modification of the short-distance potential (24) to include terms $\sim r_s$. That is, we now add to $V_{SR}(R)$ of Eq. (24) a piece,

$$V_{SR}^{(1)}(R) = H_1(\Lambda) \Lambda^2 \delta\left(R - \frac{1}{\Lambda}\right), \quad (42)$$

with $H_1 \sim r_s$.

We include this potential in Eq. (38), and then treat it in perturbation theory. Perturbation theory is only manifestly valid if $|r_s|\Lambda \ll 1$. But, as we will show below, after renormalization we can take the limit $\Lambda \rightarrow \infty$ in our perturbation-theory expressions. Although the $1/R^3$ potential and $V_{SR}^{(1)}$ cannot individually be treated in perturbation theory in this limit their combined effects remain perturbative for arbitrarily large Λ . Indeed, we will show that the Λ dependence in the perturbation-theory calculation is confined to terms of $O(1/\Lambda)$ and so it is always small. The complete first-order-perturbation-theory result is therefore independent of details of short-distance dynamics, i.e., it is “universal” in the sense we are using the term here.

At finite Λ the total first-order shift in the energy of the n th bound state in the zeroth eigenchannel is then

$$\frac{2M}{\hbar^2} \Delta B_n^{(1)} = s_0^2 r_s \xi_0 \left[\int_{1/\Lambda}^{\infty} dR f_n^{(0)2}(R) \frac{1}{R^3} - \frac{h_1}{2} \Lambda^2 f_n^{(0)2}\left(\frac{1}{\Lambda}\right) \right], \quad (43)$$

where the theory has been regulated at a distance $1/\Lambda$ and we shall seek to remove this cutoff (i.e., take $\Lambda \rightarrow \infty$) at the end of the calculation. We shall, however, calculate throughout in the limit that $\Lambda \gg \kappa_*$ (indeed, that Λ is much larger than the binding momentum of any states we are interested in). The dimensionless h_1 in Eq. (43) is defined as

$$h_1 = \frac{4H_1 M}{\hbar^2 s_0^2 r_s \xi_0}, \quad (44)$$

and the subscripts on the hyperradial wave functions indicate which binding energy B_n they correspond to. From now on we will drop the superscript on these f_n 's, since for the rest of this section all wave functions are to be evaluated at leading order.

Substituting for f_n from Eq. (41), and using the short-distance form of $K_{is_0}(x)$,

$$K_{is_0}(x) = -\sqrt{\frac{\pi}{s_0 \sinh(\pi s_0)}} \sin \left[s_0 \ln \left(\frac{x}{\sqrt{2}} \right) + \alpha_0 \right] + O(x^2), \quad (45)$$

with [1]

$$\alpha_0 = -\frac{1}{2}s_0 \ln 2 - \frac{1}{2} \arg\left(\frac{\Gamma(1+is_0)}{\Gamma(1-is_0)}\right), \quad (46)$$

we obtain

$$\begin{aligned} \frac{2M\Delta B_n^{(1)}}{\hbar^2} = 4r_s \xi_0 \kappa_n^2 \left\{ \frac{s_0 \sinh(\pi s_0)}{\pi} \int_{1/\Lambda}^{\infty} dR \frac{K_{is_0}^2(\sqrt{2}\kappa_n R)}{R^2} \right. \\ \left. - \frac{h_1(\Lambda)\Lambda}{2} \sin^2 \left[s_0 \ln\left(\frac{\kappa_n}{\Lambda}\right) + \alpha_0 \right] \right\}. \quad (47) \end{aligned}$$

The result of the integral inside the square brackets can be expressed in terms of hypergeometric functions. Since we are only interested in the result for $\Lambda \gg \kappa_n$ we expand these functions in powers of κ_n/Λ and keep the first two terms to obtain

$$I(z) \equiv \int_z^{\infty} \frac{dx}{x^2} K_{is_0}^2(x) = \frac{1}{z} I_{-1}(z) + z I_1(z) + O(z^3), \quad (48)$$

with the functions I_{-1} and I_1 encoding nonanalytic dependence of the coefficients on z ,

$$\begin{aligned} I_{-1}(z) = \frac{\pi}{2s_0 \sinh(\pi s_0)} \left(1 - \frac{1}{\sqrt{1+4s_0^2}} \right. \\ \left. \times \cos[2s_0 \ln(z/\sqrt{2}) + 2\alpha_0 + \arctan(2s_0)] \right), \quad (49) \end{aligned}$$

$$\begin{aligned} I_1(z) = -\frac{\pi}{4s_0(1+s_0^2)\sinh(\pi s_0)} \left\{ 1 - \sqrt{\frac{1+s_0^2}{1+4s_0^2}} \right. \\ \left. \times \cos \left[2s_0 \ln(z/\sqrt{2}) + 2\alpha_0 - \arctan\left(\frac{3s_0}{1-2s_0^2}\right) \right] \right\}. \quad (50) \end{aligned}$$

Inserting Eq. (48) into the expression for the energy shift we have

$$\begin{aligned} \frac{2M\Delta B_n^{(1)}}{\hbar^2} = 4r_s \xi_0 \kappa_n^2 \left\{ \frac{s_0 \sinh(\pi s_0)}{\pi} \left[\Lambda I_{-1}\left(\frac{\sqrt{2}\kappa_n}{\Lambda}\right) \right. \right. \\ \left. \left. + \frac{2\kappa_n^2}{\Lambda} I_1\left(\frac{\sqrt{2}\kappa_n}{\Lambda}\right) + \dots \right] \right. \\ \left. - \frac{h_1(\Lambda)\Lambda}{2} \sin^2 \left[s_0 \ln\left(\frac{\kappa_n}{\Lambda}\right) + \alpha_0 \right] \right\}. \quad (51) \end{aligned}$$

It follows from Eqs. (51) and (49) that the coefficient of the linear divergence can be forced to zero if we choose

$$\begin{aligned} h_1(\Lambda) = \frac{1}{\sin^2(s_0 \ln x_0 + \alpha_0)} \left(1 - \frac{1}{\sqrt{1+4s_0^2}} \right. \\ \left. \times \cos[2s_0 \ln(x_0) + 2\alpha_0 + \arctan(2s_0)] \right) \quad (52) \end{aligned}$$

with $x_0 \equiv \kappa_*/\Lambda$. With this choice of h_1 we can write

$$\frac{2M\Delta B_n^{(1)}}{\hbar^2} = 4r_s \xi_0 \kappa_n^2 \left[\Lambda f\left(\frac{\kappa_n}{\Lambda}\right) + \frac{\kappa_n^2}{\Lambda} g\left(\frac{\kappa_n}{\Lambda}\right) \right], \quad (53)$$

where the function g is proportional to $I_1(\sqrt{2}\kappa_n/\Lambda)$ up to small corrections and so is generically of order 1—although it does have logarithm-periodic dependence on κ_n/Λ . Similarly the function f inherits the logarithm periodicity of I_{-1} ,

$$f(ze^{\pi/s_0}) = f(z), \quad (54)$$

while the renormalization condition guarantees

$$f(x_0) = 0. \quad (55)$$

Since, at leading order, $\kappa_n = \kappa_* e^{(n-n_*)\pi/s_0}$, Eqs. (54) and (55) result in

$$f\left(\frac{\kappa_n}{\Lambda}\right) = f\left(\frac{\kappa_*}{\Lambda}\right) = 0. \quad (56)$$

Therefore once the linearly divergent part of the shift is renormalized away for one state, it will—at first order in perturbation theory—be zero for all bound states in the Efimov spectrum. Consequently,

$$\Delta B_n^{(1)} = \frac{4\xi_0 r_s \kappa_n^4}{\Lambda} g\left(\frac{\kappa_n}{\Lambda}\right). \quad (57)$$

Numerical results generated for the first-order shift to the spectrum in the unitary limit conform to this pattern [21]. Equation (57) holds as long as we renormalize in such a way as to keep κ_* fixed.

Therefore a renormalization procedure that leaves one of the Efimov states unperturbed by r_s corrections leaves all states unperturbed in the limit $\Lambda \rightarrow \infty$. This conclusion is a consequence of the discrete scale invariance of the Efimov spectrum. Because the leading-order spectrum has discrete scale invariance, and states are related to one another by a scale transformation of a factor $e^{(n-n_*)\pi/s_0}$, the integrals (40) for different states n are also related to one another by that factor, i.e.,

$$\Delta B_n^{(1)} = e^{-3\pi(n-n_*)/s_0} \Delta B_{n_*}^{(1)}. \quad (58)$$

Thus, if $\Delta B_{n_*}^{(1)}$ is forced—by our renormalization procedure—to equal zero, we must also have

$$\Delta B_n^{(1)} = 0 \quad \text{for all } n. \quad (59)$$

In order for this conclusion to hold it is critical to use a regularization and renormalization prescription which respects discrete scale invariance. The cutoff-plus- δ -function procedure we adopted does this in the limit $\Lambda \rightarrow \infty$. At any finite Λ the violations of discrete scale invariance result in $1/\Lambda$ -suppressed corrections to Eq. (59).

However, even if Λ is kept $\sim 1/l$, the perturbative corrections to the spectrum would only be $\sim r_s l \kappa_n^4$, and so are of higher order than we are considering here. We note also that this conclusion is independent of the prefactor multiplying the r_s/R^3 potential, and so would apply also if the potential obtained in Ref. [18] for the case $r_s = l$ were evaluated in

perturbation theory between the scale-invariant wave functions obtained in the joint unitary and scaling limits.

III. THE LINEAR RANGE CORRECTION FOR ARBITRARY SCATTERING LENGTH

In the previous section we focused on the limit in which $|a| \rightarrow \infty$, i.e., $\kappa_* \gg 1/|a|$. The hyperradial formalism is particularly well suited to the analysis of that limit, as the hyperradial potential is a power law (or, more generally, sum of powers of r_s/R) for all values of R . In this section we use a momentum-space formalism [23] to obtain the corrections to the Efimov spectrum for arbitrary values of $1/(a\kappa_*)$. Numerical calculation of scattering and bound-state observables is straightforward, and it is easy to compute range corrections—regardless of the value of a [10]. We review the relevant formalism in Sec. III A. Results for the range correction are presented in Sec. III B and analyzed for their content in terms of universality in Sec. III C.

A technical point is that to do this we employ the effective-range expansion around the bound state pole

$$k \cot \delta_0 = -\gamma + \frac{\rho}{2}(k^2 + \gamma^2) + \dots, \quad (60)$$

rather than the one around $k=0$, since Eq. (60) facilitates the treatment of the two-body threshold cut generated by the three-body scattering equations. In this way we are able to always work with the dimer binding momentum γ , rather than with $1/a$. The two are equal for $r_s=0$, but for finite r_s , γ receives corrections proportional to r_s/a^2 , which means that both the atom-dimer threshold and the three-body binding energies receive corrections. By working with γ , rather than a , as the two-body physical scale, we circumvent this complication. The parameter ρ in Eq. (60) is equal to the effective range r_s up to corrections $\sim r_s^2/a$, which means that setting $\rho=r_s$ is adequate for our NLO calculation. Therefore in what follows we do not distinguish between ρ and r_s .

A. Subtracted momentum space equations: A review

For zero-range, $l=0$, two-body potentials the usual momentum-space Faddeev equations for the atom-dimer scattering amplitude reduce to the equation derived by Skorniakov and Ter-Martirosian (STM) [22],²

$$\begin{aligned} \mathcal{A}^{(0)}(q, q'; E) &= \mathcal{Z}(q, q'; E) + \int_0^\Lambda dq'' q''^2 \mathcal{Z}(q, q'', E) \\ &\times \frac{S^{(0)}(E; q'')}{E + E_D - \frac{3}{4} \frac{\hbar^2 q''^2}{M} + i\epsilon} \mathcal{A}^{(0)}(q'', q'; E), \end{aligned} \quad (61)$$

where $\mathcal{A}^{(0)}$ denotes the leading-order atom-dimer scattering amplitude, E_D is the dimer binding energy, Λ a momentum cutoff which regularizes the integral equation, ϵ a positive infinitesimal, and

$$\mathcal{Z}(q, q'; E) = -\frac{M}{qq'\hbar^2} \ln \left(\frac{q^2 + q'^2 + qq' - \frac{ME}{\hbar^2}}{q^2 + q'^2 - qq' - \frac{ME}{\hbar^2}} \right). \quad (62)$$

Finally, in Eq. (61), $S^{(0)}(E; q'') \equiv S^{(0)}(E - \frac{3\hbar^2 q''^2}{4m})$ denotes the function

$$S^{(0)}(E) = \frac{2\hbar^4}{\pi M^2} (\gamma - ik) \quad (63)$$

with $E_D = \hbar^2 \gamma^2 / M$ and $E = \hbar^2 k^2 / M$. Note that here we have E_D (equivalently $|a|$) finite, in contrast to the analysis of the previous section. Note also that if $\gamma < 0$ then the pole corresponding to the dimer is on the second sheet of the complex energy plane, and so does not correspond to an actual bound state, but instead to a “quasibound state.” Regardless of the sign of γ Eq. (61) may be made more compact by introducing the dimer propagator

$$\tau^{(0)}(E) \equiv \frac{S^{(0)}(E)}{E + E_D + i\epsilon}. \quad (64)$$

Although now 50 years old the STM equation has recently been rederived as the leading-order result for the atom-dimer scattering amplitude in an effective field theory (EFT) with contact interactions alone. This EFT is the appropriate low-energy theory for nonrelativistic systems with $a \gg l$ and $\kappa l \ll 1$, where κ denotes the momentum scale of observable of interest [7].

The EFT analysis of Ref. [7] showed that Eq. (61) leads to strongly cutoff-dependent results for the bound-state spectrum. In coordinate space this is reflected by the necessity to fix the phase of the hyperradial wave function at short distances before a prediction about the three-body bound-state spectrum can be made. As in the analysis of Sec. II, it transpires that fixing a single three-body observable (in what follows we choose the threshold atom-dimer scattering amplitude) is sufficient to remove the cutoff dependence of the spectrum. We will now show how to achieve this using a once-subtracted version of the integral equation (61). Our presentation follows Ref. [23].

We begin by considering

$$\begin{aligned} \mathcal{A}^{(0)}(q, 0; -E_D) &= \mathcal{Z}(q, 0; -E_D) - \frac{4M}{3} \int_0^\Lambda dq'' \mathcal{Z}(q, q''; -E_D) \\ &\times S^{(0)}(-E_D; q'') \mathcal{A}^{(0)}(q'', 0; -E_D). \end{aligned} \quad (65)$$

We can insert the required three-body input by demanding that the scattering amplitude reproduces the correct particle-dimer scattering length at threshold. The on-shell amplitude at $E = -E_D$ obeys Eq. (65) with $q=0$, but the value of $\mathcal{A}^{(0)}(q, 0; -E_D)$ is fixed if the atom-dimer scattering length a_3 is taken as input. Therefore by subtracting the equation at

²The STM form appears different, but is equivalent [7,23].

$q=0$ from Eq. (65) and relating $\mathcal{A}^{(0)}$ to a_3 we obtain the subtracted equation at threshold:

$$\begin{aligned} \mathcal{A}^{(0)}(q, 0; -E_D) &= \frac{3Ma_3}{8\gamma} + \Delta[\mathcal{Z}](q, 0; -E_D) \\ &\quad - \frac{4M}{3} \int_0^\Lambda dq'' \Delta[\mathcal{Z}](q, q''; -E_D) \\ &\quad \times S^{(0)}(-E_D; q'') \mathcal{A}^{(0)}(q'', 0; -E_D). \end{aligned} \quad (66)$$

Here

$$\Delta[\mathcal{Z}](q, q'; E) = \mathcal{Z}(q, q'; E) - \mathcal{Z}(0, q'; E), \quad (67)$$

and a_3 denotes the atom-dimer scattering length.

Now we can determine the full off-shell amplitude at threshold by exploiting the symmetries of $\mathcal{A}$. Before subtraction the full off-shell amplitude at threshold satisfies

$$\begin{aligned} \mathcal{A}^{(0)}(q, q'; -E_D) &= \mathcal{Z}(q, q'; -E_D) - \frac{4M}{3} \int_0^\Lambda dq'' \mathcal{Z}(q, q''; -E_D) \\ &\quad \times S^{(0)}(-E_D; q'') \mathcal{A}^{(0)}(q'', q'; -E_D). \end{aligned} \quad (68)$$

Since $\mathcal{Z}(q, q'; E) = \mathcal{Z}(q', q; E)$, we also have $\mathcal{A}^{(0)}(0, q; -E_D) = \mathcal{A}^{(0)}(q, 0; -E_D)$. Using this fact as input, together with the solution of Eq. (66), produces the subtracted version of Eq. (68),

$$\begin{aligned} \mathcal{A}^{(0)}(q, q'; -E_D) &= \mathcal{A}^{(0)}(0, q'; -E_D) + \Delta[\mathcal{Z}](q, q'; -E_D) \\ &\quad - \frac{4M}{3} \int_0^\Lambda dq'' \Delta[\mathcal{Z}](q, q''; -E_D) \\ &\quad \times S^{(0)}(-E_D; q'') \mathcal{A}^{(0)}(q'', q'; -E_D), \end{aligned} \quad (69)$$

which determines the fully off-shell amplitude at threshold.

With Eq. (69) in hand resolvent identities may be used to obtain the subtracted amplitude at any energy [23],

$$\begin{aligned} \mathcal{A}^{(0)}(q, k; E) &= \mathcal{A}^{(0)}(q, k; -E_D) + B^{(0)}(q, k; -E_D) \\ &\quad + \int_0^\Lambda dq' q'^2 Y^{(0)}(q, q'; E) \mathcal{A}^{(0)}(q', k; E), \end{aligned} \quad (70)$$

where the second inhomogeneous term is given by

$$\begin{aligned} B^{(0)}(q, k; E) &= \delta[\mathcal{Z}](q, k; E) + \int_0^\Lambda dq'' q''^2 \mathcal{A}^{(0)}(q, q''; -E_D) \\ &\quad \times \tau^{(0)}(-E_D; q'') \delta[\mathcal{Z}](q'', k; E), \end{aligned} \quad (71)$$

with

$$\delta[\mathcal{Z}](q, q'; E) = \mathcal{Z}(q, q'; E) - \mathcal{Z}(q, q'; -E_D), \quad (72)$$

and $\tau(E; q) = \tau(E - \frac{3\hbar^2 q^2}{4M})$. The kernel in Eq. (70) is

$$\begin{aligned} Y^{(0)}(q, q'; E) &= \mathcal{A}^{(0)}(q, q'; -E_D) \delta[\tau^{(0)}](E; q') \\ &\quad + \delta[\mathcal{Z}](q, q'; E) \tau^{(0)}(E; q') \\ &\quad + \int_0^\Lambda dq'' q''^2 \mathcal{A}^{(0)}(q, q''; -E_D) \tau^{(0)}(-E_D; q'') \\ &\quad \times \delta[\mathcal{Z}](q'', q'; E) \tau^{(0)}(E; q') \\ &= \mathcal{A}^{(0)}(q, q'; -E_D) \delta[\tau^{(0)}](E; q') \\ &\quad + B^{(0)}(q, q'; E) \tau^{(0)}(E; q'), \end{aligned} \quad (73)$$

with

$$\delta[\tau^{(0)}](E; q') = \tau^{(0)}(E; q') - \tau^{(0)}(0; q'). \quad (74)$$

The result is a formulation of the three-body problem with short-range forces in which only renormalized quantities appear. Subtracted equations relate the amplitude at different energies (including energies below the atom-dimer scattering threshold). Hence, once a_3 is known, the three-body bound-state spectrum and atom-dimer phase shifts can be computed.

For pure S -wave interactions the only impact of effective-range corrections on the above argument is a modification of the two-body propagator τ [10]. For arbitrary r_s this can be written in the form

$$\tau(E) = -\frac{2\hbar^2}{\pi M^2} \frac{1}{-\gamma - ik + \frac{r_s}{2}(\gamma^2 + k^2)}. \quad (75)$$

This two-body propagator has one additional pole which lies outside the region of validity of the EFT. Therefore, the two-body propagator has to be expanded up to the desired order in r_s/a ,³

$$\tau^{(n)}(E) = \frac{S^{(n)}(E)}{E + E_D + i\epsilon}, \quad (76)$$

with $S^{(n)}$ for $n < 3$,

$$S^{(n)}(E) = \frac{2\hbar^4}{\pi M^2} \sum_{i=0}^n \left(\frac{r_s}{2}\right)^i [\gamma - ik]^{i+1}. \quad (77)$$

This form for $S^{(n)}$ is then inserted into the above equations and the calculation at NLO and N2LO can proceed exactly as with the LO calculation. Below we denote the resulting amplitude by $\mathcal{A}^{(n)}$.

B. Renormalization to κ_*

When the subtracted integral equations are presented as has been done in Sec. III A, it appears that to use them we must choose a_3 as the observable that provides the three-body input to the scattering equation. This constitutes a renormalization scheme that is appropriate for the low-energy properties of a system with a fixed atom-atom scattering length. In the situations of interest to us here though, the two-body scattering length of the atomic species is con-

³Technically, r_s must be replaced by ρ if we wish to consider any order beyond $n=1$.

trolled by an external magnetic field, through the phenomenon of Feshbach resonances. The long-range properties of the atomic system (including the three-body scattering length) can then vary rapidly if the resonance is narrow. But, physically, the short-distance properties of the three-body system—in particular the asymptotic form of the three-body wave function in the high-momentum and short-distance region—should be unaffected by changes in the infrared physics associated with the two-body scattering length. In what follows we will therefore use a renormalization scheme in which the parameter associated with this ultraviolet part of the wave function is independent of the two-body scattering length. The analysis of Sec. II shows that (in the unitary limit) the phase of this part of the wave function is in one-to-one correspondence with the binding momentum of the n -th three-body bound state, κ_* , and so this renormalization scheme amounts to ensuring that the value of κ_* , taken at the point $\gamma=0$, is unchanged when we include corrections that are proportional to r_s .

This means we need to know how to relate observables calculated in the unitary limit, $\gamma=0$, to observables calculated at a finite value of γ . More generally, we would like to develop renormalized equations that encode relations between the amplitude $\mathcal{A}^{(n)}$ at different values of the two-body binding momentum γ . The γ dependence of $\mathcal{A}^{(n)}$ arises in the integral equation through the dependence of τ and $\mathcal{Z}$ on the two-body scattering length.⁴ In what follows we use this fact, together with the subtraction approach, to relate the atom-dimer scattering amplitudes at two different values of γ . Choosing $\gamma=0$ as one point in the resulting equation then allows us to compute $\mathcal{A}^{(n)}$ for any desired value of κ_* .

Consider the operator form of the integral equation for the threshold scattering-amplitude $\mathcal{A}$,

$$\mathcal{A}(\gamma, 0) = \mathcal{Z}(\gamma, 0) + \mathcal{Z}(\gamma, 0) \tau(\gamma, 0) \mathcal{A}(\gamma, 0), \quad (78)$$

where now we have written the γ dependence explicitly, as the first argument in all quantities, and indicated that this is an equation at threshold by including the atom-dimer relative momentum, k as a second argument. This supersedes the $k = \sqrt{ME}/\hbar$ employed above, and the argument E used there is now related to k by

$$E = \frac{3\hbar^2 k^2}{4M} - E_D. \quad (79)$$

At the atom-dimer scattering threshold $k=0$. We have also suppressed the superscript n in Eq. (78) since the argument we develop in the next few paragraphs is independent of the $\tau^{(n)}$ that is employed there.

Equation (78) can be written as

$$\mathcal{Z}(\gamma, 0) = \mathcal{A}(\gamma, 0) [1 + \tau(\gamma, 0) \mathcal{A}(\gamma, 0)]^{-1}. \quad (80)$$

Assuming that we have already calculated the scattering amplitude for a given combination of a_3 and a certain γ , say γ_1 , using Eq. (66) and (69), we can now use Eq. (80) to derive

⁴ $\mathcal{Z}$ is γ dependent at fixed on-shell momentum k , which is how we formulate our scattering problem.

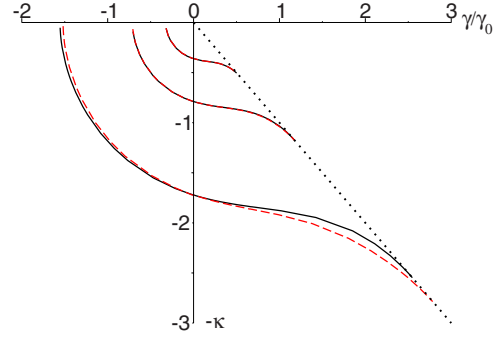


FIG. 1. (Color online) The bound-state spectrum of the three-boson system with short-range interactions. The solid lines denote the leading-order results and the dashed lines give the next-to-leading order result for an effective range of $\gamma_0 r_s = 0.01$. Here the scale has been altered as per Ref. [1], with γ and κ first rewritten in polar form as $\gamma = \gamma_0 H \cos \zeta$ and $\kappa = \gamma_0 H \sin \zeta$, and then the quantities $H^{1/4} \cos \zeta$ and $-H^{1/4} \sin \zeta$ displayed here. The dotted line denotes the point at which trimers become unstable against breakup to an atom and a dimer.

the scattering amplitude for the two-body scattering length that corresponds to γ_2 . Evaluating the expression for $\mathcal{Z}(\gamma_1, 0) - \mathcal{Z}(\gamma_2, 0)$ gives

$$\begin{aligned} \mathcal{Z}(\gamma_1, 0) - \mathcal{Z}(\gamma_2, 0) &= [1 + \mathcal{A}(\gamma_1, 0) \tau(\gamma_1, 0)]^{-1} \mathcal{A}(\gamma_1, 0) \\ &\quad - \mathcal{A}(\gamma_2, 0) [1 + \tau(\gamma_2, 0) \mathcal{A}(\gamma_2, 0)]^{-1}. \end{aligned} \quad (81)$$

Now multiplying with the appropriate expression from the left- and right-hand side, we obtain an integral equation for $\mathcal{A}(\gamma_2, 0)$,

$$\begin{aligned} \mathcal{A}(\gamma_2, 0) &= \mathcal{A}(\gamma_1, 0) + \mathcal{A}(\gamma_1, 0) [\tau(\gamma_2, 0) - \tau(\gamma_1, 0)] \mathcal{A}(\gamma_2, 0) \\ &\quad + [1 + \mathcal{A}(\gamma_1, 0) \tau(\gamma_1, 0)] [\mathcal{Z}(\gamma_2, 0) - \mathcal{Z}(\gamma_1, 0)] \\ &\quad \times [1 + \tau(\gamma_2, 0) \mathcal{A}(\gamma_2, 0)]. \end{aligned} \quad (82)$$

Once the threshold amplitude $\mathcal{A}(\gamma_2, 0)$ is obtained from this equation, it can in turn be used to calculate the scattering amplitude at any energy and for the new value of γ , γ_2 .

The strategy is thus:

- (1) Pick a value of a_3 and set $\gamma_1 = \gamma_0$ (chosen arbitrarily) and compute the threshold $\mathcal{A}$, $\mathcal{A}(\gamma_0, 0)$, using Eqs. (66) and (69).
- (2) Use Eq. (82) to compute $\mathcal{A}(0, 0)$: the threshold amplitude in the unitary limit.
- (3) Use Eq. (70) to obtain $\mathcal{A}(\gamma=0)$ for $E < 0$, thereby determining the bound-state spectrum.
- (4) Adjust $a_3(\gamma_0)$ and repeat steps (1)–(3) until a desired κ_* is obtained.
- (5) Then use Eq. (82) to compute $\mathcal{A}(\gamma, 0)$ for other values of γ .
- (6) And once again use Eq. (70) to determine the bound-state spectrum.

The results of this procedure are shown in Fig. 1. This represents the spectrum of three-body bound states which correspond to different two-body scattering lengths but the same asymptotic behavior in the short-distance region (i.e.,

the same κ_* at $\gamma=0$). The binding energies are renormalized so that there is a state in the unitary limit which has binding momentum $\kappa_* = \sqrt{0.15}\gamma_0$. (From now on we denote this state as the one with index $n=1$.) We also set $M=\hbar=1$. In Fig. 1 the binding momentum $\kappa = \sqrt{MB_3}/\hbar$ is plotted against the inverse of the two-body scattering length. The solid line represents the leading-order results, which reproduce those presented in Ref. [1]. Although the axes are labeled $-\kappa$ and γ , we have actually plotted $-H^{1/4} \sin \zeta$ versus $H^{1/4} \cos \zeta$, with (the dimensionless) H and ζ defined as $\gamma = \gamma_0 H \cos \zeta$ and $\kappa = \gamma_0 H \sin \zeta$. This rescaling of the variables allows us to display a greater range γ and κ as the discrete scaling factor is reduced from 22.7 to $22.7^{1/4} = 2.2$ [1].

We can extend these results to NLO by following steps (1)–(7) above, but using the form (77) for $n=1$ in the integral equations. We reiterate that we renormalize at NLO by demanding that the value of κ_* is unchanged from its LO value. Consequently the value of a_3 obtained at the starting point $\gamma = \gamma_0$ undergoes a shift. In Fig. 1 we show an extended *Efimov* plot which includes the NLO results for a representative value of $r_s \gamma_0 = 0.01$ as the dashed line. The effective range has been chosen such that the deepest three-body bound state shown remains in the range of validity of the EFT for all allowed values of γ . We see that the shift caused by range corrections is only recognizable for this state, and that even the shift in its binding energy vanishes in the unitary limit.

C. Universality in the range correction

We now take the results and methodology of the previous subsection and use them to compute corrections to the bound-state spectrum that are linear in the effective range r_s . This calculation was performed in the limit $|a| \rightarrow \infty$ in Sec. II B, and here we extend it to the case of finite a . In the small r_s regime we can write [c.f. Eq. (2)]

$$B_n = \frac{\hbar^2 \kappa_*^2}{M} \left[F_n \left(\frac{\gamma}{\gamma_0} \right) + \kappa_* r_s G_n \left(\frac{\gamma}{\gamma_0} \right) + \dots \right] \quad (83)$$

where the function F_n gives the leading-order binding energy of the n th state at arbitrary γ [1]. The scale γ_0 is arbitrarily chosen, and for the calculations presented here we have $\kappa_* = \sqrt{0.15}\gamma_0$. In general we will consider values of γ which range from 0 (i.e., the unitary limit) to $\gamma \approx 14.2\kappa_*$, where the $n=1$ trimer state vanishes through the atom-dimer threshold. It needs to be noted, though, that at any nonzero γ there are only a finite number of bound states within the domain of validity of the EFT: the Efimov spectrum of infinitely many shallow three-body bound states is only realized at $\gamma=0$. Indeed, at $\gamma=\gamma_0$ there are only two three-body bound states which are inside the domain of validity of the EFT.

As the linear range correction is the result of an integral over a product of wave functions which display approximate discrete scale invariance, we expect this feature to be reflected in the final result for the functions G_n . We therefore hypothesize that the G_n 's are related to G_{n*} by

$$G_n \left(\frac{\gamma}{\gamma_0} \right) = \exp \left(\frac{(n* - n)3\pi}{s_0} \right) \theta_n \left(\frac{\gamma}{\gamma_0} \right) G_{n*} \left(\frac{\gamma}{\gamma_0} e^{(n-n*)\pi/s_0} \right). \quad (84)$$

The factor out the front comes from assuming κ_n^3 scaling of the first-order correction. If we computed solely in the unitary limit this prefactor would be $e^{(n*-n)3\pi/s_0}$, but a number of effects that modify this scaling for finite γ (when the hyperspherical potential is no longer a sum of power laws, even for small r_s) are accounted for by the function θ_n . This obeys

$$\theta_{n*}(z) = 1 \quad \text{for all } z, \quad (85)$$

$$\theta_n(0) = 1 \quad \text{for all } n, \quad (86)$$

and is ~ 1 everywhere else. The constraint (86) on θ_n incorporates the result already demonstrated analytically in the limit $\gamma \rightarrow 0$: that if the linear-range correction vanishes for one Efimov bound state then it vanishes for all of them. This result is verified by both our results and an independent computation [21].

Here we choose $n*=1$. The extraction of the function G_n from Eq. (83) is numerically quite delicate. It involves a numerical derivative with respect to r_s , but in order to keep consistency with the scales of our EFT we must maintain both $r_s \gamma \ll 1$ and $\Lambda r_s \gg 1$. In practice we compute G for a number of values of r_s that satisfy these constraints and then extrapolate to the $r_s \rightarrow 0$ limit.

The results for G_1 for positive γ are shown as the solid line in the right panel of Fig. 2. Away from $\gamma=0$ the case of G_0 does not give us a wide enough range of γ over which to test Eq. (84) while still remaining inside the domain of validity of the EFT. We can, however, test the prediction (84) for the case $n=2$. Since the binding energies for this case are two orders of magnitude smaller, significantly higher accuracy is needed to compute the necessary numerical derivatives. Results for the function G_2 in the case of positive two-body scattering length are represented by the dashed line in the right panel of Fig. 2, with G_2 having been rescaled according to Eq. (84) with θ_n set to 1. We see that the horizontal scale in the two cases is indeed related by a factor of e^{π/s_0} , exactly as predicted by discrete scale invariance. Also, both functions vanish at the same point, $\gamma \approx 5.5\gamma_0$, once this rescaling is performed ($\gamma \approx 0.24\gamma_0$ for $n=2$ before rescaling). Moreover, the peak in the rescaled G_2 is at approximately the same position ($\gamma \approx 3.5\gamma_0$) as that in G_1 . But, although the factor of 22.7^3 postulated in Eq. (84) allows functions that differ by four orders of magnitude to be plotted on the same vertical scale, the peak of G_2 after rescaling is 30% higher than anticipated based on G_1 and the scaling of binding energies in the unitary limit. Further investigation of these deviations from pure κ_n^3 scaling that are encoded in the functions θ_n will be interesting, but are beyond the scope of this paper.

That the rescaled G_2 and the function G_1 vanish at the same value of γ would seem to be a simple consequence of the fact that the binding momentum of the three-body state goes to zero at this point, and so the first-order correction to the binding energy must be zero. But this straightforward

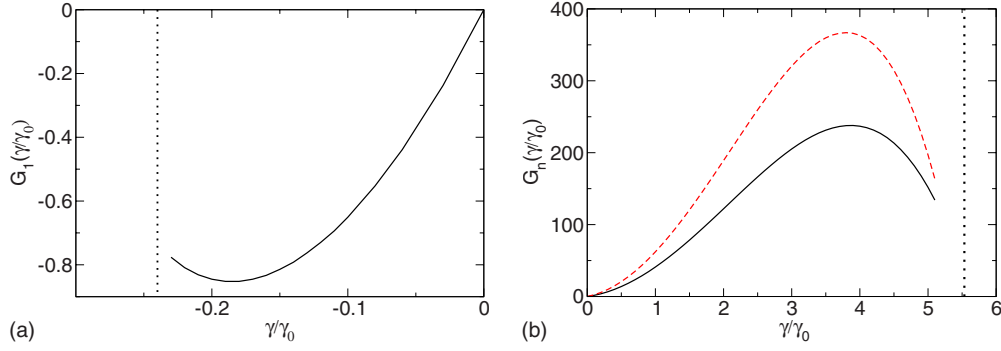


FIG. 2. (Color online) Left panel: The extracted function G_1 for negative scattering length. Right panel: Results for G_1 (solid curve) and rescaled G_2 (dashed curve) for positive scattering length. The vertical lines in both panels denote the thresholds at which the Efimov trimers become unstable.

observation has the important consequence that corrections linear in r_s do not affect the values of $\gamma > 0$ at which states that would be part of the Efimov tower in the $\gamma=0$ limit disappear from the bound-state spectrum.

In the left panel of Fig. 2 we display the result for G_1 at negative scattering length. The rapid decrease of the three-body binding energies in this domain makes the extraction of the functions still more complicated, and in this case we achieve satisfactory numerical accuracy only for G_1 . The state with $n=1$ vanishes above the three-body threshold at $\gamma = -0.26(1)\gamma_0$, so we obtain G_1 over almost the entire range where it is defined. The particularly fast variation of three-body binding energies near this vanishing point means that we could not extract the value of G_1 at the end point with any serious precision.

IV. RANGE CORRECTIONS TO THREE-BODY RECOMBINATION

In this section we will consider effective-range corrections to the three-body recombination rate for positive two-body scattering length. The calculation of the rate at negative scattering length requires the evaluation of the S -matrix element for three-atom-to-three-atom scattering. This is more involved and will be considered in a later publication.

Three-body recombination is a process in which three atoms collide to form a diatomic bound state and the energy that is thereby released causes the particles to leave the trap. The rate of decrease in the number density of atoms in the trap is given by

$$\frac{d}{dt}n = -3\alpha n^3. \quad (87)$$

The factor of 3 in the above equation arises from the assumption that all three particles involved in the recombination process will leave the trap. Atoms with large positive scattering length can recombine into the shallow dimer with $E_D = \hbar^2 \gamma^2 / M$. Recombination can also produce deep dimers with $E_{\text{deep}} \sim \hbar^2 / (Ml^2)$ —if such states are supported by the underlying two-body interaction. For negative scattering length only recombination into such deep dimers is possible.

Range corrections to three-body recombination have previously been considered in an EFT framework [24]. But there only correlations between the atom-dimer scattering length and α were considered. However, the effect that range corrections have on α as the two-body scattering length is varied but the three-body parameter (here denoted by κ_*) remains fixed is particularly relevant to experiment. This case corresponds to the situation in which the two-atom scattering length is tuned using an external magnetic field. Sizeable sensitivity of γ (or equivalently a) to the magnetic field is called a Feshbach resonance and occurs if a change in the magnetic-field strength leads to a threshold crossing of a two-body bound state.

The case of a large effective range was considered by Petrov [34]. Using a two-body propagator with the effective range iterated to all orders as in Eq. (75), he calculated the recombination rate for negative values of r_s . Such an approach might be appropriate if $|a| \sim |r_s| \gg \ell$, however, here we will require that $a \gg r_s$.

Here, for simplicity, we consider three-body recombination at threshold. This corresponds to the zero-temperature limit. We will also neglect three-body recombination into deep dimers. An analytic form for the leading order recombination rate into shallow dimers at threshold has been derived by Macek *et al.* [25] and independently by Petrov [26],

$$\alpha(E=0) = \frac{128\pi^2(4\pi - 3\sqrt{3})\sin^2\left(s_0 \ln \frac{a}{a_{*0}}\right)}{\sinh^2(\pi s_0) + \cos^2\left(s_0 \ln \frac{a}{a_{*0}}\right)} \frac{\hbar}{M\gamma^4}. \quad (88)$$

The recombination rate therefore scales as the product of $\hbar/(M\gamma^4)$ and a logarithm-periodic function. The three-body parameter a_{*0} that determines the phase of this logarithm periodicity defines the scattering length at which the recombination rate has a minimum. At leading order it is related to the three-body parameter κ_* by $a_{*0} = (e^{\pi s_0})^n 0.32 \kappa_*^{-1}$ [1].

The recombination rate can be obtained from the elastic atom-dimer scattering amplitude $\mathcal{A}$ by solving the STM equation and evaluating [24,27]

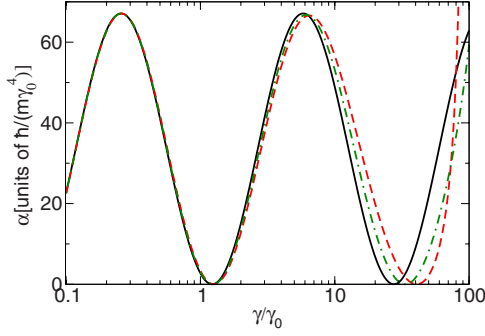


FIG. 3. (Color online) The recombination rate α as a function of γ/γ_0 . The solid line gives the leading order result, while the dashed and dotted-dashed lines give the results for $r_s\gamma_0=0.01$ and 0.005 , respectively.

$$\alpha = \frac{64\pi^2 M}{\sqrt{3}} \frac{1}{\left(1 - \gamma \frac{r_s}{2}\right)^2 (1 - \gamma r)} |\mathcal{A}(0, 2\gamma/\sqrt{3}, 0)|^2. \quad (89)$$

At leading order this reproduces Eq. (88). We now use the subtracted integral equation derived in Sec. III to calculate the amplitude $\mathcal{A}$ in Eq. (89). In any experiment the effective range will vary with the external magnetic field which is used to control the scattering length. However, this variation of r_s will be slow compared to that of γ , and so we assume that the effective range stays constant as γ varies.

Figure 3 shows the resulting dependence of the threshold recombination rate α on γ in units of $\hbar/(\gamma_0^4 M)$. The results display the logarithm-periodic behavior expected from Eq. (89). The solid line gives the three-body recombination rate in the zero-range limit—the LO EFT result—while the dashed and dotted-dashed lines give the NLO results for $r_s\gamma_0=0.01$ and 0.005 , respectively. All calculations are renormalized so that they correspond to the same binding momentum κ_* of the first excited Efimov state at $|a|=\infty$. The renormalization at $|a|=\infty$ makes all curves approach the LO result as $\gamma \rightarrow 0$. Away from $\gamma=0$ it can be seen that the finite effective range leads to a shift of the minima in the recombination rate and also influences the maximum value of α .

Given the recent experiment of Ref. [4] the extension of this computation to the case $\gamma < 0$ will be particularly interesting. In particular, the maxima in the recombination rate are determined by the values of the scattering length at which one of the Efimov states is at threshold. Understanding how corrections $\sim r_s$ affect these maxima is an important subject for further work. More generally for both $\gamma > 0$ and $\gamma < 0$ the inclusion of range corrections into calculations for the recombination of ^{133}Cs atoms will allow for a better description of data and therefore a more precise determination of the relevant three-body parameters. However, in order to obtain truly precise results it will be necessary to account both for the effects of deep dimers and the variation of the effective range as a function of the magnetic field.

V. SUMMARY AND OUTLOOK

Near a Feshbach resonance the scattering length, a , of the two-body system is much larger than all other length scales

in the problem. This leads to an Efimov spectrum in the three-body system, which displays discrete scale invariance. However, corrections due to the effective range, r_s , will in principle always affect observables even in the limit of infinite scattering length. This paper considered the impact of such corrections on three-body system observables.

Using hyperspherical coordinates for the three-body system, we considered the limit $l \ll |r_s| \ll |a|$ (where l is the natural length scale of the two-body interactions) and derived the hyperradial potential that corrects the $1/R^2$ potential governing the unitary limit. Our result for this potential, $V^{(1)}$, extends the potential derived in Refs. [3,17] to the case that $|r_s| \gg l$. In this limit no assumptions whatsoever about the nature of the interparticle interactions are necessary, and the form of $V^{(1)}$ is $\sim r_s/R^3$, as shown in Refs. [3,17]. However, the numerical coefficient of Ref. [17] is modified because effects that scale similar to l^2/r_s^2 are now removed from the hyperspherical potential. In addition, we showed that in the regime $l \ll |r_s| \ll |a|$ corrections to the hyperadiabatic approximation affect the hyperradial potential only at $O(r_s^2/R^4)$. This allows us to derive the full $O(r_s)$ correction to the Efimov spectrum of three-body states using the potential $V^{(1)}$.

Since the potential is singular renormalization is required. If we take as our renormalization condition that the n^{th} state is unperturbed by the $V^{(1)}$ correction, then the shift in the n^{th} state is zero in the limit $\Lambda \rightarrow \infty$, and so the entire spectrum is unperturbed. This is consistent with the discrete scale invariance that is present in that limit, but we note that the conclusion is dependent on the particular renormalization condition chosen.

We then examined how these results change as we move away from the limit $|a| \rightarrow \infty$. We derived subtracted integral equations that allowed us to relate the 1+2 scattering amplitude at different values of the scattering length a . This led us to an extended version of the Efimov plot, where we were able to display how three-body bound state energies vary not only with a , but also with r_s . Such a plot could also be generated with an explicit three-body force, but the subtractive approach employed here has certain advantages, in particular when higher-order calculations at large cutoffs are pursued. This approach has been used with success in Refs. [10,11,24,28] in a variety of nuclear and atomic-physics contexts to compute results that are claimed to be valid up to N2LO in the r_s/a expansion. It is important to point out that the disagreement between these works and Refs. [9,12,29,30] regarding whether additional three-body input is needed to renormalize the three-body problem at N2LO does not affect the results presented here. The formalism employed to compute the first-order correction in r_s reproduces the results of Refs. [9,12,29,30], even if it leads to conclusions at N2LO that disagree with these studies.

Thus the only assumption in our work is that $|a| \gg |r_s| \gg l$, where l is the underlying length scale in the two-body potential. Because of this assumption all calculations of Sec. III were performed in the limit $\Lambda \rightarrow \infty$. We extracted, in this limit, a universal function that describes the linear (in r_s) correction to the bound-state energies of the three-body bound-state spectrum. We chose to examine the correction for the state with $n=n^*=1$, with the universal function defined by

$$B_{n*} = \frac{\hbar^2 \kappa_*^2}{M} \left[F_{n*} \left(\frac{\gamma}{\kappa_*} \right) + \kappa_* r_s G_{n*} \left(\frac{\gamma}{\kappa_*} \right) + O[(\kappa_* r_s)^2] \right], \quad (90)$$

where F_n is given in Ref. [1], and defines the γ dependence of the bound state in the $r_s=0$ (scaling) limit. The result we obtained for G_{n*} is displayed in Fig. 2. We have furthermore showed that the next-to-leading order corrections to different states in the Efimov spectrum are approximately related to each other by the scale transformation defined in Eq. (84). It is an interesting question whether this symmetry of the leading-order wave function impacts other observables too.

Being able to use subtracted equations to relate physical observables at different scattering lengths to each other led us to consider three-body recombination into the shallow dimer. We calculated the recombination rate for different values of the effective range r_s , while renormalizing to the binding momentum κ_* of an Efimov state in the unitary limit. Consequently our results for different values of the two-body effective range approach one another as the two-body scattering length increases.

In order to examine the behavior of the three-body system near a Feshbach resonance we have used an EFT that exploits the hierarchy of scales $|r_s| \ll |a|$. We have focused on effects of $O(r_s/a)$. This enables us to probe the pattern of convergence of the EFT, thereby facilitating reliable error estimates for observables calculated in this framework. It also provides a clear differentiation between effects that are universal (i.e., arise solely as a result of $|a| \gg |r_s|$) and effects that depend on details of the underlying two-body interaction. It is interesting to speculate that features such as the Phillips and Tjon lines in nuclear physics are well reproduced at leading order [31,32], in spite of the large expansion parameter there ($r_s/a \approx 0.3$) because the discrete scale invariance suppresses the NLO correction below the natural expectation.

However, in order to confirm this speculation an analysis of the four-body system in, and near, the unitary limit must be carried out. Such a conjecture might also be rendered

more plausible if the discrete scale invariance that is present at leading order suppressed some of the next-to-next-to-leading order corrections to the Efimov spectrum. In order to examine this possibility we would like to extend the above analysis to $O(r_s^2/a^2)$. While the computation of such corrections to the bound-state spectrum appears straightforward, it is numerically somewhat delicate. However, a computation of the $O(r_s^2)$ corrections to the unitary-limit spectrum can be carried out using the hyperradial formalism in a manner similar to that employed here.

Note added. Recently a paper by Thogerson *et al.* [33] appeared that addresses similar issues. The numerical study of finite-range effects in Ref. [33] found qualitatively similar results to those shown in our Fig. 1. In their work, the authors of Ref. [33] used a Gaussian potential with a fixed range to study the impact of finite range corrections on the Efimov features in the three-body system. The value of r_s was fixed in Ref. [33], and the authors were therefore not able to compute the function G_n . They furthermore presented some results where r_s/a was not a small number. However, Thogerson *et al.* also find that $G_n=0$ at the critical two-body a for which the three-body bound state vanishes through the particle-dimer threshold. Their numerical results for the limit $|a| \rightarrow \infty$ also seem to agree with the conclusion reached by analytic methods in Sec. II, i.e., that the bound state spectrum receives no correction $\sim r_s$ in the limit $|a| \rightarrow \infty$.

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- [1] E. Braaten and H. W. Hammer, Phys. Rep. **428**, 259 (2006).
 - [2] V. N. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971).
 - [3] V. N. Efimov, Sov. J. Nucl. Phys. **29**, 546 (1979).
 - [4] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, Nature (London) **440**, 315 (2006).
 - [5] E. Braaten and H. W. Hammer, Phys. Rev. A **70**, 042706 (2004).
 - [6] E. Braaten, H. W. Hammer, D. Kang, and L. Platter, Phys. Rev. A **78**, 043605 (2008).
 - [7] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999); Nucl. Phys. A **646**, 444 (1999).
 - [8] H.-W. Hammer and T. Mehen, Phys. Lett. B **516**, 353 (2001).
 - [9] P. F. Bedaque, G. Rupak, H. W. Grißhammer, and H.-W. Hammer, Nucl. Phys. A **714**, 589 (2003).
 - [10] L. Platter and D. R. Phillips, Few-Body Syst. **40**, 35 (2006).
 - [11] L. Platter, Phys. Rev. C **74**, 037001 (2006).
 - [12] T. Barford and M. C. Birse, J. Phys. A **38**, 697 (2005).
 - [13] E. Braaten and D. Phillips, Phys. Rev. A **70**, 052111 (2004).
 - [14] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
 - [15] D. R. Phillips and T. D. Cohen, Phys. Lett. B **390**, 7 (1997).
 - [16] Z. Zhen and J. Macek, Phys. Rev. A **38**, 1193 (1988).
 - [17] A. S. Jensen, E. Garrido, and D. V. Fedorov, Few-Body Syst. **22**, 193 (1997).
 - [18] E. Nielsen, D. V. Fedorov, and A. S. Jensen, J. Phys. B **31**, 4085 (1998).
 - [19] V. Efimov, Phys. Rev. C **44**, 2303 (1991).
 - [20] V. Efimov, Phys. Rev. C **47**, 1876 (1993).
 - [21] Daekyoung Kang (private communication).
 - [22] G. V. Skorniakov and K. A. Ter-Martirosian, Zh. Eksp. Teor.

- Fiz. **31**, 775 (1956) [Sov. Phys. JETP **4**, 648 (1957)].
- [23] I. R. Afnan and D. R. Phillips, Phys. Rev. C **69**, 034010 (2004).
- [24] H. W. Hammer, T. A. Lähde, and L. Platter, Phys. Rev. A **75**, 032715 (2007).
- [25] J. H. Macek, S. Ovchinnikov, and G. Gasaneo, Phys. Rev. A **72**, 032709 (2005).
- [26] D. Petrov, talk at the Workshop on Strongly Interacting Quantum Gases, Ohio State University, 2005 (unpublished).
- [27] P. F. Bedaque, E. Braaten, and H. W. Hammer, Phys. Rev. Lett. **85**, 908 (2000).
- [28] H. W. Hammer, D. R. Phillips, and L. Platter, Eur. Phys. J. A **32**, 335 (2007).
- [29] H. W. Griesshammer, Nucl. Phys. A **744**, 192 (2004).
- [30] H. W. Griesshammer, Nucl. Phys. A **760**, 110 (2005).
- [31] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A **676**, 357 (2000).
- [32] L. Platter, H. W. Hammer, and U. G. Meissner, Phys. Lett. B **607**, 254 (2005).
- [33] M. Thøgersen, D. V. Fedorov, and A. S. Jensen, Phys. Rev. A **78**, 020501(R) (2008).
- [34] D. S. Petrov, Phys. Rev. Lett. **93**, 143201 (2004).