Decoherence in adiabatic quantum computation

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We have studied the decoherence properties of adiabatic quantum computation (AQC) in the presence of in general non-Markovian, e.g., low-frequency, noise. The developed description of the incoherent Landau-Zener transitions shows that the global AQC maintains its properties even for decoherence larger than the minimum gap at the anticrossing of the two lowest-energy levels. The more efficient local AQC, however, does not improve scaling of the computation time with the number of qubits n as in the decoherence-free case. The scaling improvement requires phase coherence throughout the computation, limiting the computation time and the problem size n.

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The adiabatic ground-state scheme of quantum computation [1,2] represents an important alternative to the gate-model approach. In adiabatic quantum computation (AQC) the Hamiltonian H_S of the qubit register and its wave function $|\psi\rangle$ undergo adiabatic evolution in such a way that, while the transformations of $|\psi\rangle$ represent some meaningful computation, this state also remains close to the instantaneous ground state $|\psi_G\rangle$ of H_S throughout the process. This is achieved by starting the evolution from a sufficiently simple initial Hamiltonian H_i, the ground state of which can be reached directly (e.g., by energy relaxation), and evolving into a final Hamiltonian H_f, whose ground state provides the solution to some complex computational problem: $H_S=[1-s(t)]H_i+s(t)H_f$, where $s(t)$ changes from 0 to 1 between some initial ($t_i=0$) and final (t_f) times.

The advantage of performing a computation this way, besides its insensitivity to gate errors, is that the energy gap between the ground and excited states of the Hamiltonian H_S ensures some measure of protection against decoherence. This protection, as partly demonstrated in this work, is not absolute. Nevertheless, it allows for the ground state to maintain its coherence properties in time far beyond what would be the single-qubit decoherence time in the absence of the ground-state protection. This feature of the AQC remains intact [3] even if the decoherence strength and/or temperature is much larger than the minimum gap.

In general, the performance of an adiabatic algorithm depends on the structure of the energy spectrum of its Hamiltonian H_S. Here we consider a situation, which is typical for complex search and optimization problems [3], where the performance is limited by the anticrossing of the two lowest-energy states. The minimum gap g_m between those states shrinks with an increasing number n of qubits in the algorithm, although the exact scaling relation is not known in general. In an isolated system with no decoherence, the limitation is due to the usual Landau-Zener tunneling at the anticrossing, which drives the system out of the ground state with the probability given by the "adiabatic theorem." Different formulations of the theorem all give the computation time as some power of the minimum gap: $t_f \propto g_m^{-\delta}$ [4,5].

The main assumption behind the adiabatic theorem is that there exists a well-defined energy gap between the two lowest-energy states of the system. In a more realistic case with decoherence, however, the energy levels of the qubit register are broadened by the coupling to environment, as illustrated in Fig. 1. Even the simplest environment, e.g., a two-state system, splits a single anticrossing of the two qubit levels into four anticrossings with smaller gaps [Fig. 1(b)]. An environment with a continuous spectrum turns the anticrossing point into a continuous region of some width W [Fig. 1(c)] within which incoherent tunneling between the two qubit states can take place. Thus, for such typical models of environment, the gap no longer exists in the "qubits +environment" system. The broadening W is directly related to the decoherence time of the qubit states. Any uncertainty W in the energy of an energy eigenstate makes the accumulated phase of this state also uncertain in time $\tau_{\mathrm{decoh}} \sim 1/W$. Since the broadening W typically increases with the number of qubits, while the minimum gap g_m decreases, the realistic large-scale system will eventually fall in the incoherent regime $W \gg g_m$. This means that studies of the adiabatic theorem do not apply to such *realistic* situations and therefore

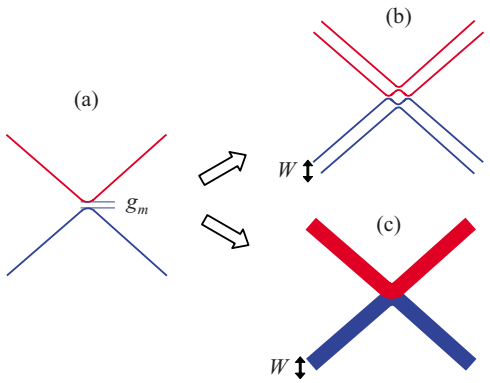

FIG. 1. (Color online) Broadening of the energy levels of a closed system (a) due to coupling to an environment made of (b) a single two-state system or (c) infinitely many degrees of freedom with a continuous energy spectrum. In general, the coupling to an environment splits a single anticrossing into M^2 anticrossings, where M is the number of environment energy eigenstates. For the environment with a continuous spectrum, the anticrossing turns into a continuous transition region of width W.

new ways of understanding AQC performance become necessary. One possible approach toward this goal is to generalize the adiabatic theorem to open quantum systems [6].

In this paper, however, we study the evolution of an adiabatic quantum computer in the "incoherent" regime by developing a corresponding description of Landau-Zener transitions for $W \gg g_m$. We use the model of decoherence appropriate for solid-state circuits, where the AQC approach is particularly promising. One characteristic feature of such a model is that it should allow for non-Markovian, in particular low-frequency environmental noise. Previous studies have mainly considered Markovian environments [3,7–9]. A correct description of the interaction with a low-frequency environment, which has the strongest effect on the AQC algorithms [10], requires a nonperturbative or strong-coupling theory of the environment-qubit interaction.

Another feature of our "solid-state" approach is the assumption that the environment responsible for decoherence is in equilibrium at some temperature T and is sufficiently large to enforce (on some time scale) the equilibration among the qubit states at the same temperature. Even the low-frequency noise that dominates the decoherence of the solid-state qubits (see, e.g., [11,12]) comes usually from equilibrium sources [13]. Previous studies of the AQC decoherence used models that do not account directly for such equilibration [6–8,14,15]. Since the environment temperature cannot be reduced indefinitely, for a sufficiently large system, T will inevitably be larger than the minimum gap g_m. This means that in contrast to closed systems, Landau-Zener transitions in the presence of decoherence are intrinsically linked to thermal excitations out of the ground state, making it necessary to consider the two types of the transitions simultaneously.

Quantitatively, we introduce the decoherence as usual by adding the bath H_B and the interaction Hamiltonian $H_{\rm int}$ to the Hamiltonian H_S of the qubit register: $H_{\rm total}=H_S+H_B+H_{\rm int}$. As discussed above, we use the two-state approximation near the anticrossing, assuming that g_m is much smaller than the energy gaps separating the first two from the other levels [16],

$$H_S = -(\epsilon\sigma_z + g_m\sigma_x)/2, \quad H_{\rm int} = -Q\sigma_z/2, \tag{1}$$

where σ's are the Pauli matrices, Q is an operator of the environmental noise, and $\epsilon \equiv E(s-s_m)$ with $E \gg g_m$ defining the energy scale which characterizes the anticrossing at $s=s_m$. Independent couplings of individual qubits to their environments produce only the σ_z-coupled noise in the two-state model (1) [3]. We assume that the noise is Gaussian so that we do not need to specify H_B explicitly. Then, all averages can be expressed in terms of the spectral density as follows:

$$S(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle Q(t)Q(0)\rangle,$$

where $\langle\cdots\rangle$ denotes averaging over the environment. Gaussian noise is expected if the environment consists of a large number of degrees of freedom all weakly coupled to the system [17].

In the regime of incoherent Landau-Zener transitions considered in this work, both the environment-induced broadening W of the two basis states of the Hamiltonian (1) and temperature T are taken to be much larger than g_m. This means that the time $(\sim 1/W)$, during which the two states lose their relative phase coherence, is much smaller than the typical tunneling time $(\sim 1/g_m)$ which implies that the tunneling between these states will be incoherent. In particular, the off-diagonal elements of the density matrix ρ of system (1) vanish within the time $\tau_{\rm decoh} \sim 1/W$ so that ρ reduces to diagonal elements, i.e., to $\rho_z \equiv p_0 - p_1$, which is governed by the usual kinetic equation,

$$\dot{\rho}_z = -\Gamma(\rho_z - \rho_\infty), \tag{2}$$

where $\Gamma=\Gamma_{01}+\Gamma_{10}$ and $\rho_\infty=[\Gamma_{10}-\Gamma_{01}]/\Gamma$. Here we use the following standard notations: $|0\rangle$ and $|1\rangle$ are the two eigenstates of σ_z with eigenvalues -1 and $+1$, respectively, p_j is the occupation probability of state $|j\rangle$, and Γ_{ij} is the rate of tunneling from state $|i\rangle$ to $|j\rangle$.

The physics behind such an incoherent tunneling is the same as for macroscopic resonant tunneling of flux in superconducting flux qubits which has been studied experimentally [13] and theoretically [18]. In particular, the transition rates have the structure of resonant peaks of width W in the vicinity of the anticrossing point. These rates can be explicitly calculated by a perturbation expansion in g_m and assuming Gaussian noise [18],

$$\Gamma_{01}(\epsilon) = \frac{g_m^2}{4}\int dt\, e^{i\epsilon t} \exp\left\{\int \frac{d\omega}{2\pi} S(\omega)\frac{e^{-i\omega t}-1}{\omega^2}\right\}. \tag{3}$$

The rate of the backward tunneling is determined by the relation $\Gamma_{10}(\epsilon)=\Gamma_{01}(-\epsilon)$. In the case of white noise, $S(\omega)=S(0)$, Eq. (3) gives the tunneling peak in the form of a Lorentzian line shape,

$$\Gamma_{01}(\epsilon) = \frac{1}{2}\frac{g_m^2 W}{\epsilon^2 + W^2}, \quad W = \frac{1}{2}S(0). \tag{4}$$

On the other hand, in the situation characteristic for practical solid-state qubits when the noise is dominated by the low-frequency components, Eq. (3) reduces to a shifted Gaussian [18],

$$\Gamma_{01}(\epsilon) = \sqrt{\frac{\pi}{8}}\frac{g_m^2}{W}\exp\left\{-\frac{(\epsilon-\epsilon_p)^2}{2W^2}\right\},$$

$$W^2 = \int \frac{d\omega}{2\pi} S(\omega), \quad \epsilon_p = \mathcal{P}\int \frac{d\omega}{2\pi}\frac{S(\omega)}{\omega}. \tag{5}$$

For the environment in thermal equilibrium, the width W and the position ϵ_p of the Gaussian are related by [18]

$$W^2 = 2T\epsilon_p. \tag{6}$$

These theoretical results have been experimentally confirmed in flux qubits [13].

Let us first study the kinetic equation (2) in two extreme cases. In the small-T regime $\rho_\infty \simeq \operatorname{sgn}\epsilon$ which implies, with

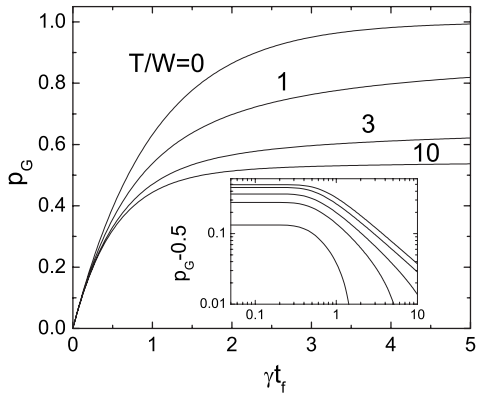

FIG. 2. The occupation probability p_G of the ground state as a function of the dimensionless evolution time γt_f for different temperatures T in the case of the Gaussian tunneling rates (5). The inset shows the dependence of p_G on T/W for γt_f=1, 1.5, 2, 3, and 5 from lower to upper curves, respectively.

the initial condition $\rho_z(0)=1$, that the right-hand side of Eq. (2) is nonzero only for $\epsilon>0$. This leads to the ground-state probability

$$p_G = 1 - e^{-\gamma t_f}, \tag{7}$$

$$\gamma \equiv \frac{1}{t_f}\int_0^\infty \Gamma(\epsilon)\frac{d\epsilon}{\dot{\epsilon}} = \frac{1}{t_f}\int_{-\infty}^\infty \Gamma_{01}(\epsilon)\frac{d\epsilon}{\dot{\epsilon}}. \tag{8}$$

We shall see later that under relatively general conditions $\dot{\epsilon} \propto 1/t_f$ and therefore γ is independent of t_f. These equations assume that the range of ϵ is large enough to effectively cover the whole peak of Γ_{01}, therefore justifying infinite integration limits. In particular, the range of ϵ should be larger than (among other energies) the cutoff energy of the environment excitations. In the opposite large-T regime, one has $|\epsilon| \ll T$ and hence $\rho_\infty=0$ in Eq. (2) for energy ϵ within some relevant interval around the anticrossing point $\epsilon=0$ (this condition is made more precise below). The ground-state probability is then

$$p_G = \frac{1}{2}(1 - e^{-2\gamma t_f}). \tag{9}$$

Because of the thermal excitations, p_G approaches 1/2 in the slow-evolution limit. For the intermediate-T regime, p_G always falls between Eqs. (7) and (9); therefore, these equations give, respectively, upper and lower bounds for the probability of success (see Fig. 2 and discussion below).

An important feature of Eq. (3) is that for uniform evolution, i.e., $\dot{\epsilon}=\text{const}\equiv\nu$, it gives $\gamma t_f = \frac{1}{\nu}\int_{-\infty}^{\infty}\Gamma_{01}(\epsilon)d\epsilon = \pi g_m^2/2\nu$, independent of $S(\omega)$, leading in the small-T regime to the same Landau-Zener probability (7) as in the decoherence-free case. This result extends the recent proofs [19–21] that at $T=0$ the Landau-Zener probability is unaffected by decoherence. The physical reason for this is that the decoherence changes only the profile of the transition region while keeping the total transition probability the same. Therefore, in the two extreme regimes, the ground-state probabilities (7) and (9) are completely independent of the form of the noise spectrum $S(\omega)$.

At intermediate temperatures, on the other hand, the quantitative t_f dependence of the probability p_G is sensitive to the specific form of $S(\omega)$ and therefore to the tunneling rates. For Gaussian rates (5) and uniform evolution, p_G calculated from Eq. (2) is shown in Fig. 2. The curves characterize the transition between the low- [Eq. (7)] and high- [Eq. (9)] temperature limits. At small evolution times when $t_f \ll \gamma^{-1}$ all curves coincide, with $p_G=\gamma t_f$ in the linear approximation, independently of temperature T. The temperature dependence of p_G appears only in the second-order terms in γt_f. For slow evolution, $t_f \geq \gamma^{-1}$, p_G varies from 1 to 1/2 with temperature T (see inset in Fig. 2). If the evolution is infinitely slow, the occupation probabilities of the states $|0\rangle$ and $|1\rangle$ should always reach the local thermal equilibrium. This, however, is not the relevant regime for the present discussion. In the relevant case, the rate ν is comparable to the maximum tunneling rates Γ and therefore becomes much larger than the tunneling rates as the system moves away from the resonance so that the local equilibrium is not maintained. This means that, strictly speaking, the large-T result (9) is valid for any t_f only for $T \gg E$. Asymptotic analysis of the evolution equation for the case of the Gaussian rates (5) shows that in the more interesting regime when $T \gg W$ but $T \ll E$, the ground-state probability is

$$p_G = \frac{1}{2} + \frac{W}{\sqrt{2}T}[\ln\ \gamma t_f]^{1/2}. \tag{10}$$

This equation describes the increase in p_G toward the local equilibrium at sufficiently large evolution time t_f and corresponds to the large-T part of the two curves with larger γt_f in the inset in Fig. 2.

We now use the results presented above to discuss the performance of AQC in the incoherent regime $g_m \ll W, T$. For this, one needs to distinguish global and local adiabatic evolutions. In the *global* scheme, the adiabatic evolution is uniform, $\dot{\epsilon}=\text{const}=E/t_f$, and Eqs. (7) and (9) show that the required computational time $t_f \simeq \gamma^{-1}=2E/\pi g_m^2$ coincides with the decoherence-free case independently of decoherence and temperature T. Even if the large T reduces p_G to $\simeq 1/2$, to find correct solution, one only needs to repeat the computation process on average two times.

Global adiabatic evolution, however, does not yield the optimal performance in coherent AQC. Indeed, for the case of adiabatic Grover search [14], the global adiabatic scheme yields the complexity of the classical exhaustive search, i.e., $t_f=O(N)$, where N $(=2^n)$ is the size of data base. In the more efficient *local* scheme [14], one takes $\dot{\epsilon}(t)=\alpha g(t)^2$ so that the adiabatic condition is satisfied uniformly (the system slows down in the region of small gap) and the computation time is $t_f=\pi/\alpha g_m$ which for the case of adiabatic Grover search yields the optimal $O(\sqrt{N})$ performance. The local evolution plays a crucial role for the scaling analysis of the AQC [14,22,23], although in some cases it is only assumed implicitly. In general, however, finding the gap $g(s)$ is as hard as solving the original problem, and only in some cases, e.g.,

the adiabatic Grover search, $g(s)$ is independent of the final solution and can be found *a priori* analytically.

The enhanced performance of the local scheme comes at a price of its stronger sensitivity to decoherence. A qualitative reason for sensitivity of local AQC is that although decoherence does not change the total integral transition probability, it distributes it over a much larger energy interval $W \gg g_m$, making it necessary to slow down the evolution for a longer period of time. If one uses the same $\epsilon(t)$ as in the decoherence-free case, the average tunneling rate (8) is dominated by the vicinity of the point $\epsilon=0$. Quantitatively, $\dot{\epsilon}=\alpha g^2$ and $t_f=\pi/\alpha g_m$ yield $(t_f\dot{\epsilon})^{-1}=g_m/\pi g^2 \approx \delta(\epsilon)$, which together with Eqs. (3) and (8) give $\gamma \approx \Gamma_{01}(0) \propto g_m^2$. Therefore the computation time is $t_f \simeq \gamma^{-1} \propto g_m^{-2}$, which is similar to the performance of the global scheme with the only possible enhancement compared to the global case being a prefactor. In the case of white noise, Eq. (4) leads to $\gamma=g_m^2/2W$, while for the low-frequency noise, Eq. (5) gives $\gamma=\sqrt{\pi/8}(g_m^2/W)e^{-W^2/8T^2}$. Notice that in the latter case, lowering T with constant width W [13] does not shorten the computation time.

To summarize, we have studied the decoherence effects on AQC due to general non-Markovian environments in the strong-decoherence regime in which the broadening of the energy levels completely smears out the anticrossing region. Our strong-coupling treatment shows that global AQC remains unaffected by strong decoherence $W > g_m$ and is independent of the type of noise, while the local AQC provides only a prefactor improvement of the algorithm running time in this regime and does not change the scaling of this time with g_m as compared to the case without decoherence. Thus, the local AQC can only maintain its properties if $W < g_m$. Since $W \sim 1/\tau_{\mathrm{decoh}}$ and $t_f \sim 1/g_m$ for the local scheme in the weak-decoherence regime, the computation time is limited by the decoherence $t_f < \tau_{\mathrm{decoh}}$ in the same way as in gate-model QC. Therefore, the advantageous scaling of the local AQC requires phase coherence throughout the evolution as in the gate model. Insensitivity of AQC to decoherence only holds for the global scheme and does not apply to local AQC. It should be emphasized that in our treatment we have assumed that the minimum gap is a result of a first-order quantum phase transition for which two-state model holds and the broadening of the energy levels and also thermal excitation do not mix the lowest two states with other excited states. For stronger noise or higher temperatures, one needs to take higher states into consideration.

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