High-fidelity fast quantum transport with imperfect controls

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Effective transport of quantum information is an essential element of quantum computation. We consider the problem of transporting a quantum state by using a moving potential well while maintaining the encoded quantum information. In particular, we look at a set of cases where the input control defining the position of the potential well is subject to different types of distortion, each of which is motivated by experimental considerations. We show that even under these conditions, we are able to perfectly transfer the quantum information nonadiabatically over any given distance.

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In many current proposed implementations of quantum computers, it is necessary that we can transport computational states between operational sites. This state will often contain information from a previous operation that we want to preserve and use in a further operation. Consequently, we desire that the final quantum state become displaced, but that the initial and final states be equivalent up to a global phase. In this sense, we will have preserved the encoded information. In practice, this transport process is difficult to realize without altering the state: many external sources serve to heat or otherwise decrease the purity of the state while it is transported. In addition, as has been pointed out in [1], transport processes may account for 95% of the operation time of a quantum computation. It is therefore advantageous to minimize the time required for this process while preserving the motional state. In this Rapid Comunication, we present an analytic solution for a one-dimensional system for transporting a quantum state over an arbitrary distance in the nonadiabatic regime using a harmonic potential, subject to a distortion of the input controls. This has application in systems where the control of the transport mechanism is imperfect, as is the case in many realistic experimental implementations, where a well-designed input control becomes distorted, either through a limitation due to the experimental hardware or through the interaction of the apparatus with an uncontrolled environment. Analytic solutions to a driven quantum harmonic oscillator have been known for many years [2,3], but there have been few attempts to utilize the results for the benefit of quantum information transfer [4]. Our approach differs, since it is not focused on any one implementation for transporting quantum states (although it has particular application to trapped particles) and we do not assume perfect control of the system. There are also some early attempts at high-fidelity transport in experiments [1,5–7].

We first analytically solve the Schrödinger equation for a quantum state confined in a driven harmonic oscillator and show that a suitable choice of the transport function $d(t)$ results in an evolution of the wave packet, which constitutes what we shall refer to as "perfect transport:" the final

evolved quantum state is equivalent to the initial state up to an irrelevant global phase which is analogous to a free evolution of the state in the frame of the potential. This motivates us to treat the problem classically in order to derive a particular form of the driving function that satisfies this criterion. We examine the distortion of the ideal transport path analytically by the introduction of a general functional and show that a large class of functionals that describe this broadening have no effect on the success of our transport.

We model the transport of the particle from its initial position over some arbitrary distance by a movement of the potential well according to the functional $\mathcal{D}[d(t)]$ of an input function $d(t)$ (our intended transport path), which defines the position of the center of the potential well along the axis of transport for a given time t. The system can be modeled as a wave packet confined in a static harmonic potential of fixed frequency, subject to the Hamiltonian $\hat{H}(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\{\hat{x} - \mathcal{D}[d(t)]\}^2$, with $\hat{x}$ and $\hat{p}$ being the usual quantum operators corresponding to position and momentum, respectively, and where $\mathcal{D}[d(t)]$ now plays the role of a driving function. Note that we have transformed the variables to make them dimensionless. We prepare the system in a given eigenstate of the harmonic oscillator, $|\psi_n(x,t=0)\rangle$. If we denote the transport distance by Δx, our transport condition for the quantum case is that the fidelity between initial and final states be $\mathcal{F} \equiv |\langle \psi_n(x-\Delta x, 0) | \psi_n(x, T)\rangle|^2 = 1$; in other words, the state should be unchanged in the reference frame moving with the potential well except for a global kinematic phase. To verify this, we must solve the time-dependent Schrödinger equation for our system. The normalized solutions are [2]

$$|\psi_n(x,t)\rangle = e^{-i[E_n t + \phi(t)/2 - F'x]}\hat{T}_F|\psi_n(x,0)\rangle, \tag{1}$$

where E_n is the energy eigenvalue corresponding to the nth eigenstate and $\hat{T}_F$ is the translation operator $\hat{T}_F\psi(x) = \psi(x - F(t))$. The function $F(t)$ is defined as

$$F(t) \equiv \int_0^t \mathcal{D}[d(\tau)]\sin(t-\tau)d\tau, \tag{2}$$

and the phase $\phi(t)$ is given by

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$$\phi(t) = \int_0^t 2F(\tau)F''(\tau) + F''^2(\tau) + F'^2(\tau) \mathrm{d}\tau. \qquad (3)$$

We can satisfy our transport condition if the function $F(t)$ satisfies $F(T)=\Delta x$, $F(0)=F'(0)=F'(T)=0$, such that when $t=0$, the wave function in Eq. (1) reduces to the normalized eigenstates of the harmonic oscillator, and at $t=T$, the wave function reduces to the original wave functions shifted by an amount Δx and with a global phase $\exp[-i(E_n T+\frac{1}{2}\phi(T))]$.

There is a direct correspondence between the wave function in Eq. (1) and the equations of motion for the classical analog of our system, which can be seen by taking the expectation values of the operators $\hat{x}$ and $\hat{p}$ of the state in Eq. (1) and then noting that the time evolution of these quantities obeys the Newtonian equations of motion for a classical particle confined within a harmonic potential well with constant frequency. The equations of motion are given by $\dot{x}(t)=p(t)$ and $\dot{p}(t)=\mathcal{D}[d(t)]-x(t)$, where again all variables are rescaled to make them dimensionless. Here, $x(t)$ and $p(t)$ refer, respectively, to the position and momentum of the classical particle along the axis of transport at time t. These equations have solutions $x(t)=x_c(t)+F(t)$ and $p(t)=p_c(t)+F'(t)$, with $F(t)$ as defined in Eq. (2). The functions $x_c(t)$ and $p_c(t)$ are solutions to the homogeneous equations of motion, which therefore describe the simple harmonic motion undergone by the particle when no transport is undertaken. In the classical picture, the condition for performing perfect transport becomes $x(T)=x_c(T)+\Delta x$ and $p(T)=p_c(T)$, where Δx denotes the displacement of the potential well at the final time. Our perfect transport condition specifies the boundary conditions on $F(t)$ and its first-order derivatives. We can rewrite $F(t)$ in terms of $\mathcal{D}[d(t)]$ by noting that Eq. (2) is a Volterra integral equation of the first kind with a trigonometric kernel [8] and hence has the solution

$$\mathcal{D}[d(t)] = F''(t) + F(t). \qquad (4)$$

We first search for a form of the function $d(t)$ by supposing that $\mathcal{D}[d(t)]=d(t)$. Substituting the transport conditions on $x(t)$ into Eq. (4) and imposing the condition $F''(0)=F''(T)=0$, we find $F(0)=0$ and $F(T)=\Delta x$. If we fix the condition that $d'(t)=0$ when $t=0$ and T, we subsequently place boundary conditions on $F'(t)$ and $F'''(t)$ at the initial and final times. Taking all of these conditions into account for $F(t)$ and its derivatives, we can now construct a function $d(t)$ that transports our particle perfectly. This procedure is as follows: we construct a general $F(t)$ by taking the simplest form of transport function (a linear function) and adding a series of Fourier components. We scale the components so that their period matches the transport time. We then apply the boundary conditions on $F(t)$ to solve for the Fourier coefficients. Due to the periodicity of the components, we have only five independent boundary conditions, and so we may uniquely specify only this many Fourier components. Substituting this into Eq. (4) gives us the solution

$$d_o(t) = \Delta x \left[\frac{t}{T} + \sin\left(\frac{2\pi t}{T}\right)\left(\frac{8\pi}{3T^2} - \frac{2}{3\pi}\right) \right.$$
$$\left. + \sin\left(\frac{4\pi t}{T}\right)\left(\frac{1}{12\pi} - \frac{4\pi}{3T^2}\right) \right], \qquad (5)$$

which we shall call our transport function. This equation depends on the transport time T, so that one must choose the correct transport function for the appropriate transport time. Since we have scaled all the variables in our system, $T=2\pi$ represents transport over one period of the harmonic motion in the potential well (the trap period). We henceforth assume throughout that $d(t) \equiv d_0(t)$. (It should be noted that one can follow a more rigorous derivation by starting from a description in the frame of an optimal control problem, but this was not presented here. See, for instance, [9].)

Through consideration of the above, we may state the following: if the functional $\mathcal{D}[d(t)]$ has a form such that the boundary conditions on $F(t)$ are preserved (and $\mathcal{D}[d(t)]$ is nonsingular for all t), then the functional $\mathcal{D}$ does not affect the transport of the particle. We consider the following three forms for $\mathcal{D}[d(t)]$ and briefly discuss the motives for doing so.

(i) The $\dot{d}(t)$ model: $\mathcal{D}[d(t)]=d(t)+\alpha\dot{d}(t)$, where α is a real constant and $\dot{d}(t)$ represents differentiation with respect to time. A physical interpretation for this model could be that we consistently "overshoot" or "undershoot" our desired potential well position, so that as we move the well more quickly, the deviation from the desired position becomes greater.

(ii) The *piecewise* model: The functional $\mathcal{D}$ casts $d(t)$ into a piecewise form

$$\mathcal{D}[d(t)] = d(t_n) \text{ for } t \in \left(t_n - \frac{T}{2N}, t_n + \frac{T}{2N}\right), \qquad (6)$$

where $t_n = \frac{nT}{N}$ for a given $N \in \mathbb{Z}^+, N>1$. This has the effect that the potential undergoes discrete "jumps" in its position along the transport axis, which could be due to a sampling rate limitation of the equipment used for experiment.

(iii) The *Fourier* model: $\mathcal{D}[d(t)]=d(t)+g(t)$, where $g(t)$ is a discrete Fourier series. This is of relevance since we may decompose any periodic signal (for instance, periodic pulse distortions) into a Fourier series (note that the Fourier representation must converge at all times t to the signal being represented).

We now show that for each of these functionals the conditions for perfect transport are still satisfied. In the case of the $\dot{d}(t)$ model, we begin by writing $d(t)=F''(t)+F(t)$. Hence $\mathcal{D}[d(t)]=\widetilde{F}''(t)+\widetilde{F}(t)$, where $\widetilde{F}(t)=F'(t)+F(t)$. It should be clear that $\widetilde{F}(t)$ satisfies the same boundary conditions as $F(t)$. Hence we conclude that we achieve our desired transport with this functional for any transport time T.

In the case of the piecewise function, we substitute the functional directly into Eq. (2) and solve for $F(t)$. We find in the limit that $T \to T_k = 2k\pi$, $k \in \mathbb{Z}^+$, the boundary conditions for $F(t)$ are satisfied as before. In other words, the piecewise functional can only achieve the boundary conditions when the transport time is an integer multiple of the period of the
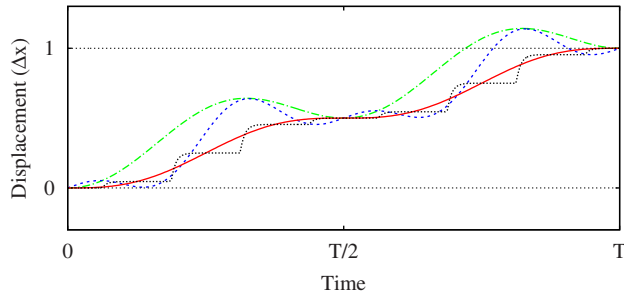
FIG. 1. (Color online) Distortion of the transport function over time period $T=2\pi$. The thick (red) solid line is the transport function from Eq. (5), the dot-dashed (green) line is the $\dot{d}(t)$ model ($\alpha =1$), the dashed (blue) line is the Fourier model (random coefficients in the range $[-1,1]$), and the dotted (black) line is the piecewise model with an exponential-type smoothing ($N=8$).

harmonic trap. It may additionally be the case that the movement of the potential is not exactly stepwise, but that instead the movement is smoothed out (for instance, if we consider a segmented ion trap, this will be due to the charging characteristics of the electrodes). We can model this by writing Eq. (6) as

$$\mathcal{D}[d(t)] = d(t_n) - q\left(t - t_n + \frac{T}{2N}\right)[d(t_n) - d(t_{n-1})], \quad (7)$$

so that $q(t)$ describes the smoothing from the previous value in the stepwise function to the next. Substituting this into Eq. (2), we can calculate that the part of the integral dependent on $q(t)$ evaluates to zero. Hence we may conclude that any smoothing of the transport path due to these terms may be ignored.

Finally, we consider the Fourier model. Again, through direct integration of the functional via Eq. (2), we can obtain the associated function $F(t)$. Here, we see that we satisfy the boundary conditions if the period of the function $g(t)$ is $T_k/2$, with T_k as given above. We can satisfy this by tuning the the frequency of the harmonic potential to accommodate the periodic noise. Figure 1 shows sample transport paths $\mathcal{D}[d(t)]$ over a short transport time $T=2\pi$. One can see that the deviation here is not small; we significantly disturb the motion of the particle. Figure 2 shows the classical trajectories through phase space of the particles transported according to the transport paths $\mathcal{D}[d(t)]$ (or, equivalently, the expectation values of the operators $\hat{x}$ and $\hat{p}$). One sees here that the trajectories begin and end on the constant energy curve given by the free oscillation in the well.

Figure 3 shows snapshots of the evolution of the ground-state wave function subject to distortion; although the probability distributions diverge from each other at intermediate times for the different models for $\mathcal{D}[d(t)]$, at the final time the wave functions converge to the ground-state probability distribution of the displaced potential well. Furthermore, one may verify that the energy expectation at the final time is the predicted value of $\langle E\rangle = E_n = (n+\frac{1}{2})$, which can be immediately found from Eq. (1) (although $\langle E\rangle$ is only uniquely specified for the ground state).
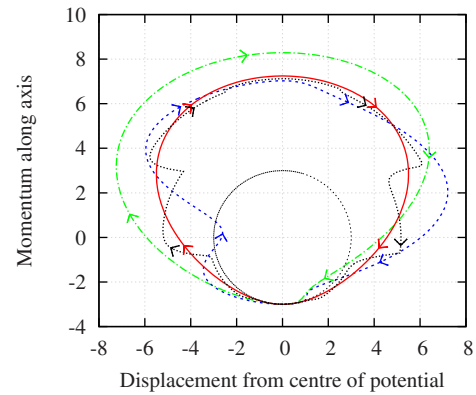

FIG. 2. (Color online) Trajectories through phase space in the frame of the moving potential of a transported particle over time period $T=2\pi$. See caption of Fig. 1 for key. For illustative purposes, the particle was given some initial momentum. The center circle is the trajectory of a freely oscillating particle with constant energy and the same initial momentum in a static potential.

The scheme also allows us to transport far from the adiabatic limit, which can be demonstrated by calculating the fidelity between the actual transported ground state and the instantaneous ground state of the displaced potential well for different values of the transport time. As we transport the state, the fidelity deviates further from unity at intermediate times, only to recover again at the final time (demonstrated in Fig. 4). Although here we have chosen the ground state of the potential, we may well have used any of the other eigenstates of the harmonic oscillator and produced similar results.

We can also begin with superpositions of states. Consider the coherent superposition of eigenstates given by $|\Psi(x,0)\rangle = \Sigma_{n=0}^{M} c_n |\psi_n(x,0)\rangle$, where c_n are normalized coefficients. In order to recover the maximum fidelity, we must preserve the relative phases between initial states. The nth eigenstate acquires a relative phase $\exp[-i(n-m)T]$ with the mth eigenstate during transport, and so in order to preserve its phase relation with the other superposed states, we can choose $T = T_k$. Hence the transport condition is satisfied.

If we instead begin with a mixed state described by the density matrix $\rho(x,0) = \Sigma_{n=0}^{M} \rho_n |\psi_n\rangle\langle\psi_n|$ and take the fidelity [10] $\mathcal{F} = (\mathrm{tr}[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^{1/2})^2$, where ρ_1 and ρ_2 are the initial and final states, respectively, we can show that the fidelity of
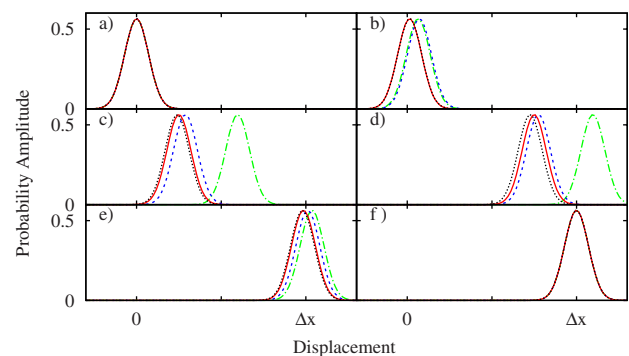

FIG. 3. (Color online) The evolution of the ground-state probability distribution from (a) an initial time $t=0$ to (f) the final time $T=2\pi$. See caption of Fig. 1 for key.
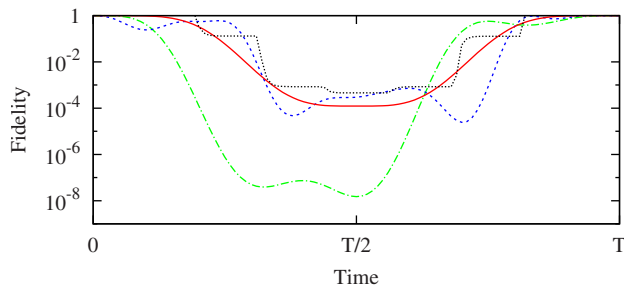
FIG. 4. (Color online) The fidelity between the instantaneous ground state and the transported ground state over a time $T=2\pi$. See caption of Fig. 1 for key.

transport does not depend on the coefficients ρ_n, so that the distribution of states remains constant during transport. We can then infer that the transport is insensitive to temperature.

In this paper, we have derived analytic solutions for the transport of a quantum state via a moving harmonic potential with a constant frequency. We considered the conditions under which a functional that distorts the input control achieves the conditions we have set for perfect transport. In particular, we studied three different models that have a quantitative relevance in experiments dealing with quantum transport and showed that under certain conditions, all three models describe a broadening of the transport path without detriment to the transport success. We briefly review these conditions: (i) The $\dot{d}(t)$ model fulfills the transport condition for any trans-

port time. (ii) The piecewise model fulfills the transport condition when the transport time is an integer multiple of the trap period. (iii) The Fourier model (which models signal distortions) satisfies the boundary conditions if the signal is periodic with half the trap period.

Of course, in a realistic situation beyond our harmonic oscillator model, complete insensitivity to such a broad range of control imperfections is not to be expected: in that case, we not only have distortion of the input controls, but also distortion of the shape of the potential itself. However, our result indicates significant robustness (by which we mean a low sensitivity of the transport fidelity to such distortions) is likely to be obtained at least when potential anharmonicities are small, since the system is well approximated by the harmonic oscillator. This will be the subject of future investigations. Such deviations may also be overcome by application of optimal control methods [11,12]. The consequence of performing robust transport, particularly in the presence of such imperfections, is that we can distribute quantum information to separate elements of a quantum computer with a very high fidelity in short times and over large distances. This is essential for scalability and fault tolerance of quantum systems for future use in computation.

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