Unidirectional decomposition method for obtaining exact localized wave solutions totally free of backward components

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In this paper we use a unidirectional decomposition capable of furnishing localized wave pulses, with luminal and superluminal peak velocities, in exact form and totally free of backward components, which have been a chronic problem for such wave solutions. This decomposition is powerful enough for yielding not only ideal nondiffracting pulses but also their finite energy versions still in exact analytical closed form, avoiding the need for time-consuming numerical simulations. Another advantage of the present approach is that, since the backward spectral components are absent, the frequency spectra of the pulses do not need to possess ultrawidebands, as it is required by the usual localized wave (LW) solutions obtained by other methods. Finally, the present results bring the LW theory nearer to the real experimental possibilities of usual laboratories.

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I. INTRODUCTION

Localized waves (or nondiffracting waves) [1–60] are very special free space solutions of the linear wave equation $(\nabla^2-\partial^2_{ct})\psi=0$, whose main characteristic is that of resisting the diffraction effects for long distances. In their pulse versions, the localized waves (LWs) can possess subluminal, luminal, or supeluminal peak velocities. As it is well known [7,10,17,28,34], these impressive features of the nondiffracting waves are due to the special space-time coupling of their spectra. The possible applications of these waves are many, from secure communications to medicine (and also in theoretical physics).

In recent years, several methods [7,27,28,34] have been developed to yield localized pulses in exact analytical closed form. The most successful approaches are those dealing with bidirectional or unidirectional decomposition [7,28,34], in which the time variable t and the spatial variable z (considered as the propagation direction) are replaced with other two variables, linear combinations of them.

Examples of bidirectional and unidirectional decompositions are (i) $\zeta=z-ct$, $\eta=z+ct$, developed by Besieris *et al.* [7]; (ii) $\zeta=z-Vt$, $\eta=z-c^2t/V$, also introduced by those authors [28]; (iii) $\zeta=z-Vt$, $\eta=z+Vt$, developed by Zamboni-Rached *et al.* [34]; and (iv) $\zeta=z-Vt$, $\eta=z+ct$, proposed by Besieris *et al.* [28].

Even if all those decompositions allow us to get exact analytical LW solutions, all of them suffer with the same problem, that is, the occurrence of backward traveling components in their spectra.[1] This drawback makes it necessary for the use of ultrawideband frequency spectra to minimize [7,28,34] the contribution of the backward components. This fact can suggest the wrong idea that ultrawide frequency bands are a characteristic of the LW pulses. As a matter of fact, it is quite simple to choose spectra that eliminate completely such "noncausal" components and at the same time have narrow frequency bands. The real problem is that no closed form analytical solution is known for those cases, and one has to make recourse to time consuming numerical simulations. This problem was already solved in the case of localized subluminal waves [60] but still persists in the general cases of luminal and superluminal LW pulses.

We are going to show here that, by a very simple decomposition, a unidirectional one, one can overcome the problems cited above, getting ideal and also finite-energy LW pulses[2] with superluminal and luminal peak velocities, constituted by forward[3] traveling components only, and without mandatory recourses to ultrawide frequency bands. In addition, the method is also extended to nonaxially symmetric cases. These results make nondiffracting waves more easily experimentally realizable and applicable.

Finally, we use the method to obtain a kind of functional expression capable of furnishing an infinite family of new totally "forward" LW pulses.

II. METHOD

Section II A forwards a brief overview of the LW theory. The rest of this section is devoted to the method and to its results.

A. Brief overview about LWs

In the case of the linear and homogeneous wave equation in free space, in cylindrical coordinates (ρ,ϕ,z) and using a Fourier-Bessel expansion, we can express a general solution $\psi(\rho,\phi,z,t)$ as

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[1]Only the simplest X-wave pulses, which possess the form $\psi(\rho,\phi,z-Vt)$, does not present this problem.

[2]Ideal LWs present an infinite depth of field, but also infinite energy content. Finite energy LWs resist diffraction effects for long, but finite, distances.

[3]In the following, we shall briefly write "forward components" and even "forward pulses."

$$\Psi(\rho,\phi,z,t)=\sum_{n=-\infty}^{\infty}\left[\int_0^{\infty}dk_\rho\int_{-\infty}^{\infty}dk_z\int_{-\infty}^{\infty}d\omega k_\rho A_n'(k_\rho,k_z,\omega)\times J_n(k_\rho\rho)e^{ik_zz}e^{-i\omega t}e^{in\phi}\right], \quad (1)$$

with

$$A_n'(k_\rho,k_z,\omega)=A_n(k_z,\omega)\,\delta\!\left(k_\rho^2-\left(\frac{\omega^2}{c^2}-k_z^2\right)\right), \quad (2)$$

$A_n(k_z,\omega)$ being an arbitrary function and $\delta(\,)$ the Dirac delta function.

An ideal nondiffracting wave can be defined as a wave capable of maintaining its spatial form indefinitely (except for local variations) while propagating. This property can be expressed in a mathematical way [27] by (when assuming propagation in the z direction)

$$\Psi(\rho,\phi,z,t)=\Psi\!\left(\rho,\phi,z+\Delta z_0,t+\frac{\Delta z_0}{\mathcal{V}}\right), \quad (3)$$

where Δz_0 is a certain length and $\mathcal{V}$ is the pulse-peak velocity, with $0\leqslant\mathcal{V}\leqslant\infty$.

Using Eq. (3) in Eq. (1), and taking into account Eq. (2), we can show [27,1] that any localized wave solution, when eliminating evanescent waves and considering only positive angular frequencies, can be written as

$$\Psi(\rho,\phi,z,t)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\int_0^{\infty}d\omega\int_{-\omega/c}^{\omega/c}dk_zA_{nm}(k_z,\omega)\times J_n\!\left(\rho\sqrt{\frac{\omega^2}{c^2}-k_z^2}\right)e^{ik_zz}e^{-i\omega t}e^{in\phi}\right], \quad (4)$$

with

$$A_{nm}(k_z,\omega)=S_{nm}(\omega)\,\delta(\omega-(\mathcal{V}k_z+b_m)), \quad (5)$$

$b_m=2m\pi\mathcal{V}/\Delta z_0$, and quantity $S_{nm}(\omega)$ being an arbitrary frequency spectrum.

We should note that, due to Eq. (5), each term in the double sum (4), namely, in the expression within square brackets, is a truly nondiffracting wave (beam or pulse), and their sum (4) is just the most general form representing an ideal nondiffracting wave defined by Eq. (3).

We can also notice that Eq. (4) is nothing but a superposition of Bessel beams with a specific space-time coupling in their spectra: more specifically with linear relationships between their angular frequency ω and longitudinal wave number k_z.

Concerning such a superposition, the Bessel beams with $k_z>0$ ($k_z<0$) propagate in the positive (negative) z direction. As we wish to obtain LWs propagating in the positive z direction, the presence of "backward" Bessel beams ($k_z<0$), i.e., of "backward components," is not desirable. This problem can be overcome, however, by appropriate choices of the spectrum (5), which can totally eliminate those components, or minimize their contribution, in superposition (4).

Another important point refers to the energy of the LWs [3,28,33,34]. It is well known that any ideal LW, i.e., any field with the spectrum (5), possesses infinite energy. However finite-energy LWs can be constructed by concentrating the spectrum $A_{nm}(k_z,\omega)$ in the surrounding of a straight line of the type $\omega=\mathcal{V}k_z+b_m$ instead of collapsing it exactly over that line. In such a case, the LWs get a finite energy, but are endowed with finite field depths: i.e., they maintain their spatial forms for long (but not infinite) distances.

Despite the fact that expression (4), with $A_{nm}(k_z,\omega)$ given by Eq. (5), does represent ideal nondiffracting waves, it is difficult to use it for obtaining analytical solutions, especially when having the task of eliminating the backward components. This difficulty becomes even worse in the case of finite-energy LWs.

As an attempt to bypass these problems, many different bidirectional and unidirectional decomposition methods have been proposed in recent years [7,28,34]. Those methods consist essentially in the replacement of the variables z and t in Eq. (4) with new ones $\zeta=z+v_1t$ and $\zeta=z+v_2t$, where v_1 and v_2 are constants possessing *a priori* any value in the range $[-\infty,\infty]$. The names unidirectional and bidirectional decomposition correspond to $v_1/v_2>0$ and $v_1/v_2<0$, respectively.

For instance, in [7] Besieris *et al.* introduced the bidirectional decomposition $\zeta=z-ct$, $\eta=z+ct$ and obtained interesting ideal and finite-energy luminal LWs. They also worked, in [28], with the unidirectional decomposition $\zeta=z-Vt$, $\eta=z-c^2t/V$ (with $V>c$) for obtaining ideal and finite-energy superluminal LWs. In [34] Zamboni-Rached *et al.* introduced the bidirectional decomposition $\zeta=z-Vt$, $\eta=z+Vt$ (with $V>c$), being thus able to provide many other ideal and finite-energy superluminal nondiffracting pulses.

Subluminal LWs have been obtained [28,32,60], for instance, through a decomposition of the type $\zeta=z-vt$, $\eta=z$, with $v<c$.

All such decompositions are very efficient in furnishing LWs in closed forms, but yield solutions that suffer with the problem that backward traveling wave components enter their spectral structure. A way out was found in the case of subluminal waves [60], but the problem still persists for the luminal and superluminal ones. In the latter cases, they succeeded until now only in minimizing the contribution of the backward Bessel beams in Eq. (4) by choosing ultrawideband frequency spectra. This is not the best approach because, in general, the solutions found in this way resulted in being far from the experimental possibilities of the usual laboratories.

As it was said before, the backward components can be totally removed by a proper choice of the spectrum; but none of the previous decompositions are then able to yield analytical solutions for the integral (4).

We are going to introduce, therefore, a unidirectional decomposition that allows one to get ideal and finite-energy LWs, with superluminal and luminal peak velocities, without any occurrence in their spectral structure of backward components.

B. Totally forward LW pulses

Let us start with Eqs. (1) and (2), which describe a general free-space solution (without evanescent waves) of the homogeneous wave equation, and consider in Eq. (2) a spectrum $A_n(k_z,\omega)$ of the type

$$A_n(k_z,\omega) = \delta_{n0}H(\omega)H(k_z)A(k_z,\omega)\,\delta(k_\rho^2-(\omega^2/c^2-k_z^2)), \tag{6}$$

where δ_{n0} is the Kronecker delta function, $H(\,)$ is the Heaviside function, and $\delta(\,)$ is the Dirac delta function, quantity $A(k_z,\omega)$ being an arbitrary function. Spectra of the type (6) restrict the solutions to the axially symmetric case, with only positive values to the angular frequencies and longitudinal wave numbers. With this, the solutions proposed by us get the integral form

$$\psi(\rho,z,t) = \int_0^\infty d\omega \int_0^{\omega/c} dk_z A(k_z,\omega) J_0(\rho\sqrt{\omega^2/c^2-k_z^2})e^{ik_z}e^{-i\omega t}, \tag{7}$$

i.e., result in being general superpositions of zero-order Bessel beams propagating in the positive z direction only. Therefore, any solution obtained from Eq. (7), be they nondiffracting or not, are *completely free* from backward components.

At this point, we introduce the unidirectional decomposition

$$\zeta = z - Vt,$$

$$\eta = z - ct, \tag{8}$$

with $V > c$.

A decomposition of this type was used until now in the context of paraxial approximation only [54]; but we shall show below that it can be much more effective, giving important results in the exact context and in situations that cannot be analyzed in the paraxial approach.

With Eq. (8), we can write the integral solution (7) as

$$\psi(\rho,\zeta,\eta) = (V-c)\int_0^\infty d\sigma \int_{-\infty}^{\sigma} d\alpha A(\alpha,\sigma) \times J_0(\rho\sqrt{\gamma^{-2}\sigma^2-2(\beta-1)\sigma\alpha})e^{-i\alpha\eta}e^{i\sigma\zeta}, \tag{9}$$

where $\gamma=(\beta^2-1)^{-1/2}$, $\beta=V/c$, and where

$$\alpha = \frac{1}{V-c}(\omega - Vk_z),$$

$$\sigma = \frac{1}{V-c}(\omega - ck_z), \tag{10}$$

are the new spectral parameters.

It should be stressed that superposition (9) is not restricted to LWs: It is the choice of the spectrum $A(\alpha,\sigma)$ that will determine the resulting LWs.

1. Totally forward ideal superluminal LW pulses

The X*-type waves*. The most trivial LW solutions are those called X-type waves [13,14]. They are constructed by frequency superpositions of Bessel beams with the same phase velocity $V > c$ and *until now* constitute the only known ideal LW pulses free of backward components. Obviously, it is not necessary to use the approach developed here to obtain such X-type waves, since they can be obtained by using directly the integral representation in the parameters (k_z,ω), i.e., by using Eq. (7). Even so, just as an exercise, let us use the present approach to construct the ordinary X wave.

Consider the spectral function $A(\alpha,\sigma)$ given by

$$A(\alpha,\sigma) = \frac{1}{V-c}\delta(\alpha)e^{-s\sigma}. \tag{11}$$

One can note that the delta function in Eq. (11) implies that $\alpha=0 \rightarrow \omega=Vk_z$, which is just the spectral characteristic of the X-type waves. In this way, the exponential function $\exp(-s\sigma)$ represents a frequency spectrum starting at $\omega=0$, with an exponential decay and frequency bandwidth $\Delta\omega=V/s$.

Using Eq. (11) in Eq. (9), we get

$$\psi(\rho,\zeta) = \frac{1}{\sqrt{(s-i\zeta)^2+\gamma^{-2}\rho^2}} \equiv X, \tag{12}$$

which is the well-known ordinary X-wave solution.

Totally forward superluminal focus wave modes. Focus wave modes (FWMs) [7,28,34] are ideal nondiffracting pulses possessing spectra with a constraint of the type $\omega=\mathcal{V}k_z+b$ (with $b\neq 0$), which links the angular frequency and the longitudinal wave number, and are known for their strong field concentrations.

Until now, all the known FWM solutions possess, however, backward spectral components, a fact that, as we know, forces one to consider large frequency bandwidths to minimize their contribution. However, we are going to obtain solutions of this type completely free of backward components, and able to possess also very narrow frequency bandwidths.

Let us choose a spectral function $A(\alpha,\sigma)$ such as

$$A(\alpha,\sigma) = \frac{1}{V-c}\delta(\alpha+\alpha_0)e^{-s\sigma}, \tag{13}$$

with $\alpha_0 > 0$ a constant. This choice confines the spectral parameters ω, k_z of the Bessel beams to the straight line $\omega=Vk_z-(V-c)\alpha_0$, as it is shown in the Fig. 1 below.

Substituting Eq. (13) in Eq. (9), we have

$$\psi(\rho,\zeta,\eta) = \int_0^\infty d\sigma \int_{-\infty}^{\sigma} d\alpha\,\delta(\alpha+\alpha_0)e^{-s\sigma} \times J_0(\rho\sqrt{\gamma^{-2}\sigma^2-2(\beta-1)\sigma\alpha})e^{-i\alpha\eta}e^{i\sigma\zeta}, \tag{14}$$

which, on using identity 6.616 in Ref. [61], results in

$$\psi(\rho,\zeta,\eta) = Xe^{i\alpha_0\eta}\exp\left[\frac{\alpha_0}{\beta+1}(s-i\zeta-X^{-1})\right], \tag{15}$$

where X is the ordinary X wave given by Eq. (12).

Solution (15) represents an ideal superluminal LW of the type FWM, but is totally free from backward components.

As we already said, the Bessel beams constituting this

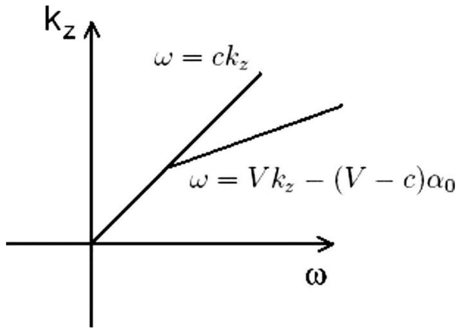

FIG. 1. The Dirac delta function in Eq. (13) confines the spectral parameters ω, k_z of the Bessel beams to the straight line $\omega = Vk_z - (V-c)\alpha_0$, with $\alpha_0 > 0$.

solution have their spectral parameters linked by the relation $\omega = Vk_z - (V-c)\alpha_0$; thus, by using Eqs. (13) and (10), it is easy to see that the frequency spectrum of those Bessel beams starts at $\omega_{\min} = c\alpha_0$ with an exponential decay $\exp(-s\omega/V)$, and so possesses the bandwidth $\Delta\omega = V/s$. It is clear that $\omega_{\min}$ and $\Delta\omega$ can assume any values, so that the resulting FWM, Eq. (15), can range from a quasimonochromatic to an ultrashort pulse. This is a great advantage with respect to the old FWM solutions.

As an example, we plot two situations related with the LW pulse given by Eq. (15).

The first, in Fig. 2, is a quasimonochromatic optical FWM pulse, with $V = 1.5c$, $\alpha_0 = 1.256 \times 10^7\ \mathrm{m}^{-1}$, and $s = 1.194 \times 10^{-4}$ m, which correspond to $\omega_{\min} = 3.77 \times 10^{15}$ Hz, and $\Delta\omega = 3.77 \times 10^{12}$ Hz, i.e., to a picosecond pulse with $\lambda_0 = 0.5\ \mu$m. Figure 2(a) shows the intensity of the complex LW field, while Fig. 2(b) shows the intensity of its real part. Moreover, in Fig. 2(b), in the upper right corner, a zoom is shown of this LW, on the z axis and around the pulse's peak, where the carrier wave of this quasimonochromatic pulse shows up.

The second example, in Fig. 3, corresponds to an ultrashort optical FWM pulse with $V = 1.5c$, $\alpha_0 = 1.256 \times 10^7\ \mathrm{m}^{-1}$, and $s = 2.3873 \times 10^{-7}$ m, which correspond to $\omega_{\min} = 3.77 \times 10^{15}$ Hz, and $\Delta\omega = 1.88 \times 10^{15}$ Hz, i.e., to a femtosecond optical pulse. Figures 3(a) and 3(b) show the intensity of the complex and real part of this LW field, respectively.

Now, we apply the present approach to obtain totally forward finite-energy LW pulses.

2. Totally forward, finite-energy LW pulses

Finite-energy LW pulses are almost nondiffracting, in the sense that they can retain their spatial forms, resistant to the diffraction effects, for long (but not infinite) distances.

There exist many analytical solutions representing finite-energy LWs [7,28,34], but once more, all the known solutions suffer from the presence of backward components. We can overcome this limitation.

Superluminal finite-energy LW pulses, with peak velocity $V > c$, can be obtained by choosing spectral functions in Eq. (7) that are concentrated in the vicinity of the straight line $\omega = Vk_z + b$ instead of lying on it. Similarly, in the case of luminal finite-energy LW pulses the spectral functions in Eq. (7) have to be concentrated in the vicinity of the straight line $\omega = ck_z + b$ (note that in the luminal case, one must have $b \geqslant 0$).

Indeed, from Eq. (10) it is easy to see that, by our approach, finite-energy superluminal LWs can actually be ob-

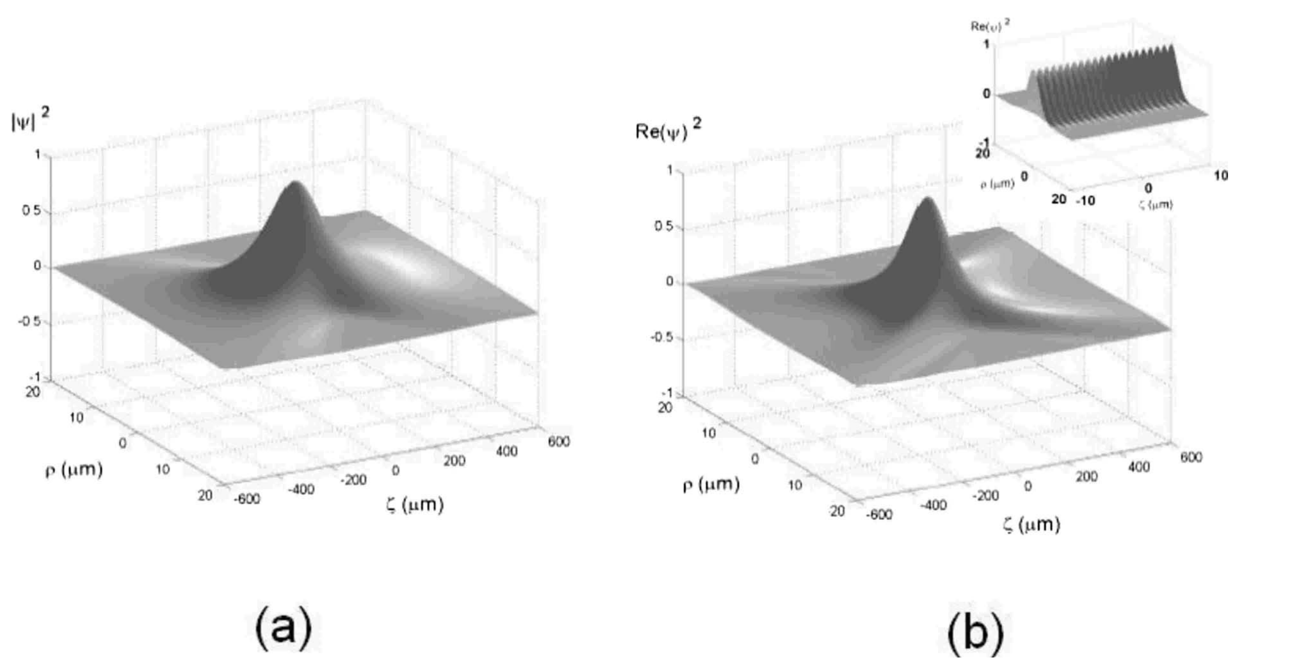

FIG. 2. (a) and (b) show, respectively, the intensity of the complex and real part of a quasimonochromatic, totally forward, superluminal FWM optical pulse, with $V = 1.5c$, $\alpha_0 = 1.256 \times 10^7\ \mathrm{m}^{-1}$, and $s = 1.194 \times 10^{-4}$ m, which correspond to $\omega_{\min} = 3.77 \times 10^{15}$ Hz, and $\Delta\omega = 3.77 \times 10^{12}$ Hz, i.e., to a picosecond pulse with $\lambda_0 = 0.5\ \mu$m.

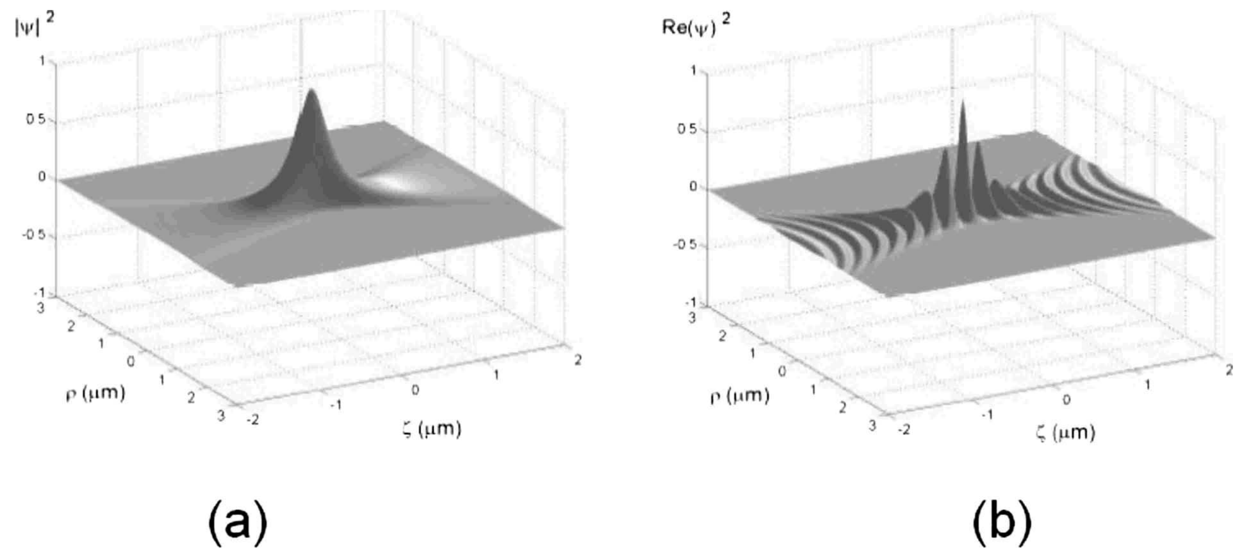

FIG. 3. (a) and (b) show, respectively, the intensity of the complex and real part of an ultrashort, totally forward, superluminal FWM optical pulse, with $V=1.5c$, $\alpha_0=1.256\times10^{7}\ \mathrm{m}^{-1}$, and $s=2.3873\times10^{-7}$ m, which correspond to $\omega_{\min}=3.77\times10^{15}$ Hz, and $\Delta\omega=1.88\times10^{15}$ Hz, i.e., to a femtosecond optical pulse.

tained by concentrating the spectral function $A(\alpha,\sigma)$ entering in Eq. (9), in the vicinity of $\alpha=-\alpha_0$, with α_0 a *positive* constant. And, analogously, the finite-energy luminal case can be obtained with a spectrum $A(\alpha,\sigma)$ concentrated in the vicinity of $\sigma=\sigma_0$, with $\sigma_0\geqslant 0$.

To see this, let us consider the spectrum

$$A(\alpha,\sigma)=\frac{1}{V-c}H(-\alpha-\alpha_0)e^{a\alpha}e^{-s\sigma}, \qquad (16)$$

where $\alpha_0>0$, $a>0$, and $s>0$ are constants, and $H(\ )$ is the Heaviside function.

Due to the presence of the Heaviside function, the spectrum (16), when written in terms of the spectral parameters ω and k_z, has its domain in the region shown in Fig. 4.

We can see that the spectrum $A(\alpha,\sigma)$ given by Eq. (16) is more concentrated on the line $\alpha=-\alpha_0$, i.e., around $\omega=Vk_z-(V-c)\alpha_0$, or on $\sigma=0$ (i.e., around $\omega=ck_z$), depending on the values of a and s: More specifically, the resulting solution will be a superluminal finite-energy LW pulse, with peak velocity $V>c$, if $a\gg s$, or a luminal finite-energy LW pulse if $s\gg a$.

Inserting the spectrum (16) into Eq. (9), we have

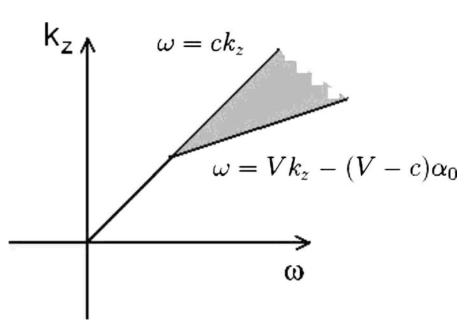

FIG. 4. The spectrum (16), when written in terms of the spectral parameters ω and k_z, has its domain indicated by the shaded region.

$$\psi(\rho,\zeta,\eta)=\int_0^{\infty}d\sigma\int_{-\infty}^{-\alpha_0}d\alpha e^{a\alpha}e^{-s\sigma}\times J_0[\rho\sqrt{\gamma^{-2}\sigma^2-2(\beta-1)\sigma\alpha}]e^{-i\alpha\eta}e^{i\sigma\zeta}, \qquad (17)$$

and, by using identity 6.616 in Ref. [61], we get

$$\psi(\rho,\zeta,\eta)=X\int_{-\infty}^{-\alpha_0}d\alpha e^{a\alpha}e^{-i\alpha\eta}\exp\left[-\frac{\alpha}{\beta+1}(s-i\zeta-X^{-1})\right], \qquad (18)$$

which can be directly integrated to furnish

$$\psi(\rho,\zeta,\eta)=\frac{X\exp\left\{-\alpha_0\left[(a-i\eta)-\dfrac{1}{\beta+1}(s-i\zeta-X^{-1})\right]\right\}}{(a-i\eta)-\dfrac{1}{\beta+1}(s-i\zeta-X^{-1})}. \qquad (19)$$

As far as we know, the new solution (19) is the first one to represent finite-energy LWs completely free of backward components.

Superluminal or luminal peak velocities. The finite-energy LW (19) can be superluminal (peak velocity $V>c$) or luminal (peak velocity c) depending on the relative values of the constants a and s. To see this in a rigorous way, in connection with solution (19), we should calculate how its global maximum of intensity (i.e., its peak), which is located on $\rho=0$, develops in time. One can obtain the peak's motion by considering the field intensity of Eq. (19) on the z axis, i.e., $|\psi(0,\zeta,\eta)|^2$, at a given time t, and finding out the value of z at which the pulse presents a global maximum: We shall call $z_p(t)$ (the peak's position) this value of z. Obviously the peak velocity will be $dz_p(t)/dt$.

The on-axis field intensity of Eq. (19) is

$$|\psi(0,\zeta,\eta)|^2 = \frac{e^{-2a\alpha_0}}{(as-\eta\zeta)^2+(s\eta+a\zeta)^2}. \quad (20)$$

For a given time t, we can find out the z position of the peak by setting $\partial|\psi(0,\zeta,\eta)|^2/\partial z=0$; that is,

$$\frac{e^{-2a\alpha_0}[2(as-\eta\zeta)(\zeta+\eta)-2(s\eta+a\zeta)(a+s)]}{[(as-\eta\zeta)^2+(s\eta+a\zeta)^2]^2}=0, \quad (21)$$

where we have used $\partial/\partial z=\partial/\partial\zeta+\partial/\partial\eta$.

From Eq. (21), we find that

$$\eta\zeta^2+\eta^2\zeta+s^2\eta+a^2\zeta=0. \quad (22)$$

We are interested in the cases where $a \gg s$ or $s \gg a$.

For the case $a \gg s$, we have for the two last terms in the left-hand side of Eq. (22), $s^2\eta+a^2\zeta=s^2(z-ct)+a^2(z-Vt) =(s^2+a^2)z-(s^2c+a^2V)t\approx a^2(z-Vt)$, and so we can approximate Eq. (22) with

$$\eta\zeta^2+\eta^2\zeta+a^2\zeta\simeq 0, \quad (23)$$

which yields three values of z. It is not difficult to show that one of them, $\zeta=0$, i.e., $z=Vt$, furnishes the global maximum of the intensity (20), and therefore also of Eq. (19). We can also show that the other two roots have real values only for $t\geq\sqrt{8}a/(V-c)$, and in any case furnish values of the intensity of Eq. (19) much smaller than the global maximum, already found at $z=Vt\equiv z_p(t)$. So, we can conclude that for $a\gg s$ the peak velocity is $V>c$.

For the case $s^2c\gg a^2V\to s\gg a$, the two last terms in the left-hand side of Eq. (22) are $s^2\eta+a^2\zeta=s^2(z-ct)+a^2(z-Vt) =(s^2+a^2)z-(s^2c+a^2V)t\approx s^2(z-ct)$, and so we can approximate Eq. (22) with

$$\eta\zeta^2+\eta^2\zeta+s^2\eta\simeq 0, \quad (24)$$

and we can show that the root $\eta=0$, i.e., $z=ct\equiv z_p(t)$, furnishes the global maximum of the intensity (20), and therefore also of Eq. (19). The other two roots have real values only for $t\geq\sqrt{8}s/(V-c)$, and furnish, once more, much smaller values of the intensity of Eq. (19). In this way, for $s^2c\gg a^2V\to s\gg a$, the peak velocity is c. Now, let us analyze, in detail, examples of both cases.

Totally forward, finite-energy superluminal LW pulses. As we have seen above, superluminal finite-energy LW pulses can be obtained from Eq. (19) by putting $a\gg s$. In this case, the spectrum $A(\alpha,\sigma)$ is well concentrated around the line $\alpha=-\alpha_0$, and therefore in the plane (k_z,ω) this spectrum starts at $\omega_{min}\approx c\alpha_0$ with an exponential decay, and the bandwidth $\Delta\omega\approx V/s$.

The field depth of the superluminal LW pulse obtained from Eq. (19) can be calculated in a simple way. Let us examine the evolution of its peak intensity by putting $\zeta=0\to\eta=(1-\beta^{-1})z$ in Eq. (20).

$$|\psi(0,\zeta=0,\eta)|^2 = \frac{e^{-2a\alpha_0}}{a^2s^2+s^2(1-\beta^{-1})^2z^2}, \quad (25)$$

where, of course, the coordinate z above is the position of the peak intensity, i.e., $z=Vt\equiv z_p$. It is easy to see that the pulse

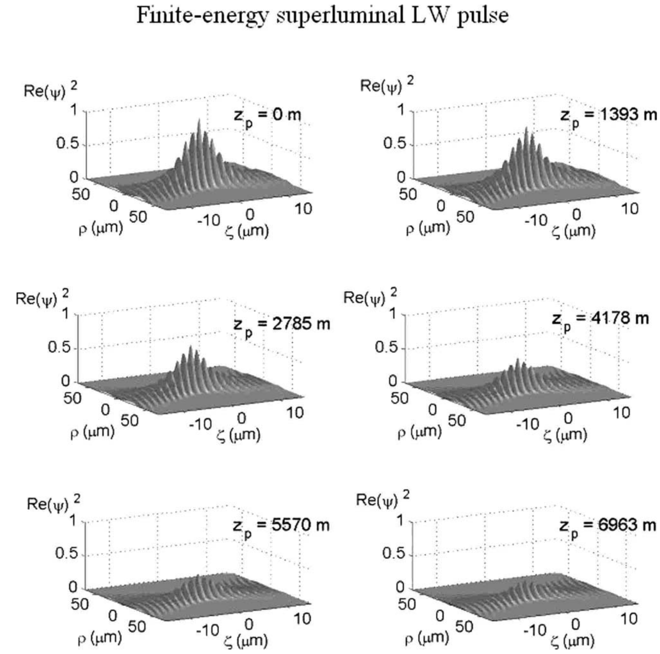

FIG. 5. The space-time evolution, from the pulse's peak at $z_p=0$ to $z_p=Z$, of a totally forward, finite-energy, superluminal LW optical pulse represented by Eq. (19), with the following parameter values: $a=20$ m, $s=3.99\times10^{-6}$ m (note that $a\gg s$), $V=1.005c$, and $\alpha_0=1.26\times10^7$ m^{-1}.

presents its maximum intensity at $z=0$ $(t=0)$, and maintains it for $z\ll a/(1-\beta^{-1})$. Defining the field depth Z as the distance over which the pulse's peak intensity remains at least 25% of its initial value,[4] we can obtain from Eq. (25) the depth of field

$$Z=\frac{\sqrt{3}a}{1-\beta^{-1}}, \quad (26)$$

which depends on a and $\beta=V/c$. Thus, the pulse can get large field depths by suitably adjusting the value of parameter a.

Figure 5 shows the space-time evolution, from the pulse's peak at $z_p=0$ to $z_p=Z$, of a finite-energy superluminal LW pulse represented by Eq. (19) with the following parameter values: $a=20$ m, $s=3.99\times10^{-6}$ m (note that $a\gg s$), $V=1.005c$, and $\alpha_0=1.26\times10^7$ m^{-1}. For such a pulse, we have a frequency spectrum starting at $\omega_{min}\approx3.77\times10^{15}$ Hz (with an exponential decay) and the bandwidth $\Delta\omega\approx7.54\times10^{13}$ Hz. From these values, and since $\Delta\omega/\omega_{min}=0.02$, it is an optical pulse with $\lambda_0=0.5$ μm and a time width of 13 fs. At the distance given by the field depth $Z=\sqrt{3}a/(1-\beta^{-1})=6.96$ km the peak intensity is one-fourth of its initial value. Moreover, it is interesting to note that, in spite of the intensity decrease, the pulse's spot size $\Delta\rho_0=7.5$ μm remains constant during the propagation.

[4] We can expect that, while the pulse-peak intensity is maintained, the same happens for its spatial form.

Totally forward, finite-energy luminal LW pulses. Luminal finite-energy LW pulses can be obtained from Eq. (19) by making $s \gg a$ (more rigorously for $s^2 c \gg a^2 V$). In this case, the spectrum $A(\alpha,\sigma)$ is well concentrated around the line $\sigma=0$, and therefore in the plane (k_z,ω) it starts at $\omega_{\min} \approx c\alpha_0$ with an exponential decay and the bandwidth $\Delta\omega \approx c/a$.

The field depth of the luminal LW pulse obtained from Eq. (19) can be calculated in a simple way. Let us examine its peak intensity evolution by putting $\eta=0 \rightarrow \zeta=(1-\beta)z$ in Eq. (20):

$$|\psi(0,\zeta=0,\eta)|^2 = \frac{e^{-2a\alpha_0}}{a^2 s^2 + a^2(1-\beta)^2 z^2}, \tag{27}$$

where, of course, the coordinate z above is the position of the peak intensity, i.e., $z=ct \equiv z_p$. It is easy to see that the pulse has its maximum intensity at $z=0$ $(t=0)$, and maintains it for $z \ll s/(\beta-1)$. Defining the field depth Z as the distance over which the pulse's peak intensity remains at least 25% of its initial value, we obtain from Eq. (27) the depth of field

$$Z = \frac{\sqrt{3}s}{\beta - 1}, \tag{28}$$

which depends on s and $\beta=V/c$.

Here, we should note that the bigger the value of s, the smaller the transverse field concentration of the luminal pulse. This occurs because for big values of s the spectrum becomes strongly concentrated around $\omega=ck_z$ and, as one knows, in this case, the solution tends to become a plane-wave pulse.[5]

Let us consider, for instance, a finite-energy luminal LW pulse represented by Eq. (19) with $a=1.59\times 10^{-6}$ m, $s=1\times 10^4$ m (note that $s \gg a$), $V=1.5c$, and $\alpha_0=1.26\times 10^7$ m^{-1}. For such a pulse, which has its peak traveling with the light velocity c, the frequency spectrum starts at $\omega_{\min} \approx 3.77\times 10^{15}$ Hz with a bandwidth $\Delta\omega \approx 1.88\times 10^{14}$ Hz. Thus, it is an optical pulse with a time width of 5.3 fs.

The space-time evolution of this pulse, from $z_p=0$ to $z_p=Z$, is shown in Fig. 6. At the distance given by the field depth $Z=\sqrt{3}s/(\beta-1)=23.1$ km the peak intensity is one-fourth of its initial value.

We can see from these two examples, and it can also be shown in a rigorous way, that the superluminal LW pulses obtained from solution (19) are superior to the luminal ones obtained from the same solution, in the sense that the former can possess large field depths and, at same time, present strong transverse field concentrations. To obtain more interesting and efficient luminal LW pulses we should use spectra concentrated around the line $\sigma=\sigma_0>0$.

[5]A possible solution for this limitation would be the use of spectra concentrated around the line $\sigma=\sigma_0>0$.

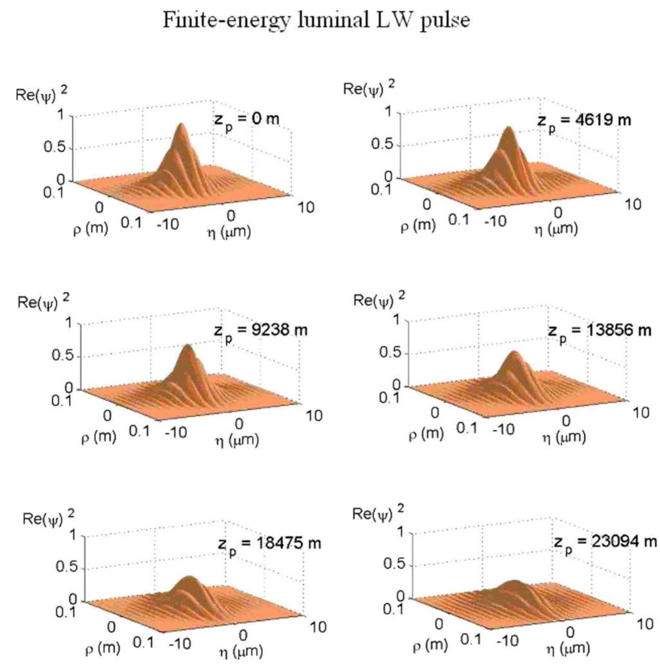

FIG. 6. (Color online) The space-time evolution, from the pulse's peak at $z_p=0$ to $z_p=Z$, of a totally forward, finite-energy, luminal LW optical pulse represented by Eq. (19), with $a=1.59\times 10^{-6}$ m, $s=1\times 10^4$ m (note that $s \gg a$), $V=1.5c$, and $\alpha_0=1.26\times 10^7$ m^{-1}.

3. Nonaxially symmetric, totally forward, LW pulses

So far, we have applied the present method to axially symmetric solutions only.

A simple way to obtain the nonaxially symmetric versions of the previous LW pulses is through the following superposition:

$$\psi(\rho,\phi,\zeta,\eta) = (V-c)\int_0^\infty d\sigma \int_{-\infty}^{\sigma} d\alpha A'(\alpha,\sigma) \times J_\nu(\rho\sqrt{\gamma^{-2}\sigma^2 - 2(\beta-1)\sigma\alpha}) e^{i\nu\phi} e^{-i\alpha\eta} e^{i\sigma\zeta}, \tag{29}$$

where ϕ is the azimuth angle, ν is an integer, and

$$A'(\alpha,\sigma) = \left(\frac{\sigma}{\sigma - 2\alpha/(\beta+1)}\right)^{\nu/2} A(\alpha,\sigma), \tag{30}$$

with $A(\alpha,\sigma)$ being the respective spectra of the axially symmetric LW solutions of the previous sections, which can be recovered from Eq. (29) by making $\nu=0$. Thus, the fundamental superposition (29) is more general than Eq. (9), in the sense that the former can yield both, axially symmetric and nonaxially symmetric LWs totally free of backward components.

Totally forward, nonaxially symmetric LW of the type FWM can be reached from Eq. (29) by inserting Eq. (13) into Eq. (30). After integrating directly over α, the integration over σ can be made by using the identity 3.12.5.6 in Ref. [62], giving

$$\psi(\rho,\phi,\zeta,\eta) = e^{i\nu\phi}\left(\frac{\gamma^{-1}\rho}{s-i\zeta+X^{-1}}\right)^{\nu} X e^{i\alpha_0\eta} \times \exp\left[\frac{\alpha_0}{\beta+1}(s-i\zeta-X^{-1})\right]. \quad (31)$$

Totally forward, finite-energy, nonaxially symmetric LW is obtained from Eq. (29) by inserting Eq. (16) into Eq. (30). By integrating first over σ (with the identity 3.12.5.6 in Ref. [62]), the integration over α can be made directly, and Eq. (29) yields

$$\psi(\rho,\phi,\zeta,\eta) = e^{i\nu\phi}\left(\frac{\gamma^{-1}\rho}{s-i\zeta+X^{-1}}\right)^{\nu} \times \frac{X\exp\left\{-\alpha_0\left[(a-i\eta)-\frac{1}{\beta+1}(s-i\zeta-X^{-1})\right]\right\}}{(a-i\eta)-\frac{1}{\beta+1}(s-i\zeta-X^{-1})}. \quad (32)$$

The solution above can possess superluminal ($a \gg s$) or luminal ($s \gg a$) peak velocity.

4. Functional expression for totally forward LW pulses

In the literature concerning the LWs [36] some interesting approaches appear, capable of yielding functional expressions that describe LWs in closed form. Although interesting, even the LWs obtained from those approaches also possess backward components in their spectral structure.

Now, however, we are able to obtain a kind of functional expression capable of furnishing totally forward LW pulses.

Let us consider, in Eq. (29), spectral functions $A'(\alpha,\sigma)$ of the type

$$A'(\alpha,\sigma) = (V-c)H(-\alpha)\left(\frac{\sigma}{\sigma-2\alpha/(\beta+1)}\right)^{\nu/2}\Lambda'(\alpha)e^{-s\sigma}, \quad (33)$$

where $H(\,)$ is, as before, the Heaviside function, and $\Lambda'(\alpha)$ is a general function of α.

Using Eq. (33) in Eq. (29), performing the integration over σ and making the variable change $\alpha=-u$, we get

$$\psi(\rho,\phi,\zeta,\eta) = e^{i\nu\phi}\left(\frac{\gamma^{-1}\rho}{s-i\zeta+X^{-1}}\right)^{\nu} X\int_0^{\infty} du\Lambda(u)e^{-uS}, \quad (34)$$

quantity X being the ordinary X-wave solution (12) and S being given by

$$S = -i\eta - \frac{1}{\beta+1}(s-i\zeta-X^{-1}). \quad (35)$$

The integral in Eq. (34) is nothing but the Laplace transform of $\Lambda(u)$. In this way, we have that

$$\psi(\rho,\phi,\zeta,\eta) = e^{i\nu\phi}\left(\frac{\gamma^{-1}\rho}{s-i\zeta+X^{-1}}\right)^{\nu} \times XF\left(-i\eta-\frac{1}{\beta+1}(s-i\zeta-X^{-1})\right), \quad (36)$$

with $F(\,)$ an arbitrary function, is an exact solution to the wave equation that can yield ideal, and also finite-energy LW pulses, with superluminal or luminal peak velocities. Besides this, if the chosen function $F(S)$ in Eq. (36) is regular and free of singularities at all space-time points (ρ,ϕ,z,t), we can show that the LW solutions obtained from Eq. (36) will be totally free of backward components.

III. CONCLUSIONS

In conclusion, by using a unidirectional decomposition we were able to get totally forward, ideal and finite-energy LW pulses in exact and closed form, avoiding the need for time-consuming numerical simulations, providing, in this way, a powerful tool for exploring several important properties of the new nondiffracting pulses.

These new solutions are superior to the known LWs already existing in the literature, since the old solutions suffer from the undesirable presence of backward components in their spectra.

By overcoming the problem of these noncausal components, the new LWs here obtained are not obliged to have physical sense only in the cases of ultrawideband frequency spectra; actually, our new LWs can also be quasimonochromatic pulses, and in such a way get closer to a practicable experimental realization.

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