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Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extended negativity

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We propose replacing concurrence by convex-roof extended negativity (CREN) for studying monogamy of entanglement (MOE). We show that all proven MOE relations using concurrence can be rephrased in terms of CREN. Furthermore, we show that higher-dimensional (qudit) extensions of MOE in terms of CREN are not disproven by any of the counterexamples used to disprove qudit extensions of MOE in terms of concurrence. We further test the CREN version of MOE for qudits by considering fully or partially coherent mixtures of a qudit W-class state with the vacuum and show that the CREN version of MOE for qudits is satisfied in this case as well. The CREN version of MOE for qudits is thus a strong conjecture with no obvious counterexamples.

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I. INTRODUCTION

Quantum entanglement is a resource with various applications such as quantum teleportation and quantum key distribution in the field of quantum information and quantum computation [1–3]. Whereas entanglement in bipartite quantum systems has been intensively studied with rich understanding, the situation becomes far more difficult for the case of multipartite quantum systems, and very few are known for its characterization and quantification. One important property to characterize multipartite entanglement is known as monogamy of entanglement (MOE) [4], which says that multipartite entanglements cannot be freely shared among the parties.

MOE is a key ingredient to make quantum cryptography secure [5], and it also plays an important role in condensed-matter physics such as the N-representability problem for fermions [6].

Thus, it is an important and necessary task to characterize MOE to understand the whole picture of quantum entanglement in multipartite systems, as well as its possible applications in quantum-information theory.

Although MOE is a typical property of multipartite quantum entanglement, it is however about the relation of bipartite entanglements among the parties in multipartite systems. Thus, the following criteria for an entanglement measure must be satisfied to have a good description of the monogamy nature of entanglement in multipartite quantum systems.

(i) Monotonicity: The property that ensures entanglement cannot be increased under local operations and classical communications (LOCC).

(ii) Separability: Capability of distinguishing entanglement from separability.

(iii) Monogamy: Upper bound on a sum of bipartite entanglement measures thereby showing that bipartite sharing of entanglement is bounded.

There are several possibilities for such a measure, including definitively answering whether the state is entangled or separable, indicating definitively that the state is entangled but inconclusive when the result is separable as well as the reverse case, and stating whether the state is entangled and/or separable with bounded error.

However, there are only a few measures known so far which can show the monogamy property of entanglement in multipartite systems, and their results are restricted to multiqubit systems [4,7]. In other words, there exist quantum states in higher-dimensional systems [8,9] which violate the monogamy properties in terms of the proposed entanglement measures, and this exposes the importance of choosing a proper entanglement measure.

Here we propose the convex-roof extended negativity (CREN) [10] as a powerful candidate for the criteria above. Besides its monotonicity and separability criteria, we claim that CREN is a good alternative for MOE without any known example violating its monogamy property even in higher-dimensional systems. We show that any monogamy inequality of entanglement for multiqubit systems using concurrence [11] can be rephrased by CREN, and this CREN MOE is also true for the counterexamples of concurrence in higher-dimensional systems [8,9].

As the first step toward general CREN MOE studies in higher-dimensional quantum systems, we propose a class of quantum states in n-qudit systems consisting of partially coherent superpositions of a generalized W-class state [9] and the vacuum, $|0\rangle^{\otimes n}$, and show that this class saturates CREN MOE for any arbitrary partition of the set of subsystems. We also show that the CREN value of the proposed class and its dual, CREN of assistance (CRENOA) coincide, and they are not affected by the degree of coherency in the superposition. This is particularly important because the saturation of monogamy relation implies that this class of multipartite higher-dimensional entanglement can have a complete characterization by means of its partial entanglements, and the characterization is not even affected by its decoherency.

The paper is organized as follows. In Sec. II, we review the definition of concurrence, CREN, and their overlap for the case of pure states with Schmidt rank 2, as well as 2-qubit mixed states. In Sec. III A, we rephrase all the monogamy inequalities of entanglement for n-qubit systems in terms of CREN. In Sec. III B, we show that the counterexamples in higher-dimensional quantum systems to the monogamy inequality using concurrence still have a monogamy relation in terms of CREN. In Sec. IV, a class of quantum states in n-qudit systems consisting of partially coherent superpositions of a generalized W-class state and $|0\rangle^{\otimes n}$ is proposed with its CREN monogamy relation of entanglement. In Sec. V, we summarize our results.

II. CONCURRENCE AND CONVEX-ROOF EXTENDED NEGATIVITY

For any bipartite pure state $|\phi\rangle_{AB}$ in a $d\otimes d'$ $(d\leqslant d')$ quantum system, its concurrence, $\mathcal{C}(|\phi\rangle_{AB})$ is defined as [11]

$$\mathcal{C}(|\phi\rangle_{AB}) = \sqrt{2(1-\text{tr }\rho_A^2)}, \tag{1}$$

where $\rho_A=\text{tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For any mixed state ρ_{AB}, it is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \tag{2}$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB}=\Sigma_k p_k|\phi_k\rangle_{AB}\langle\phi|_k$.

Concurrence of assistance (COA) [12], which can be considered to be dual to concurrence, is defined as

$$\mathcal{C}^a(\rho_{AB}) = \max \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \tag{3}$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB}.

Another well-known quantification of bipartite entanglement is the negativity [10,13], which is based on the positive partial transposition (PPT) criterion [14,15]. For a bipartite pure state $|\phi\rangle_{AB}$ in a $d\otimes d'$ $(d\leqslant d')$ quantum system with the Schmidt decomposition,

$$|\phi\rangle_{AB} = \sum_{i=0}^{d-1}\sqrt{\lambda_i}|ii\rangle, \quad \lambda_i \geqslant 0, \quad \sum_{i=0}^{d-1}\lambda_i = 1, \tag{4}$$

(without loss of generality, the Schmidt basis is taken to be the standard basis), the partial transposition of $|\phi\rangle_{AB}$ is

$$\begin{aligned}|\phi\rangle\langle\phi|^{T_B} &= \sum_{i,j=0}^{d-1}\sqrt{\lambda_i\lambda_j}|ij\rangle\langle ji| \\ &= \sum_{i=0}^{d-1}\lambda_i|ii\rangle\langle ii| + \sum_{i<j}\sqrt{\lambda_i\lambda_j}(|ij\rangle\langle ji| + |ji\rangle\langle ij|).\end{aligned} \tag{5}$$

Thus, the negative eigenvalues can be $-\sqrt{\lambda_i\lambda_j}$ for $i<j$ with corresponding eigenvectors $|\psi_{ij}\rangle=\frac{1}{\sqrt{2}}(|ij\rangle-|ji\rangle)$, and the negativity $\mathcal{N}$ of $|\phi\rangle_{AB}$ is defined as [16]

$$\mathcal{N}(|\phi\rangle) = \||\phi\rangle\langle\phi|^{T_B}\|_1 - 1 = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j}, \tag{6}$$

where $\|\cdots\|_1$ is the trace norm.

Based on the reduced density matrix of $|\phi\rangle_{AB}$, we can have an alternative definition of negativity,

$$\mathcal{N}(|\phi\rangle) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} = (\text{tr}\sqrt{\rho_A})^2 - 1, \tag{7}$$

where $\rho_A=\text{tr}_B|\phi\rangle_{AB}\langle\phi|$.

We note that $\mathcal{N}(|\phi\rangle)=0$ if and only if $|\psi\rangle$ is separable, and it can attain its maximal value, $d-1$, for a $d\otimes d$ maximally entangled state,

$$|\phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle. \tag{8}$$

(One can easily check this by the Lagrange multiplier.)

For a mixed state ρ_{AB}, its negativity is defined as

$$\mathcal{N}(\rho_{AB}) = \|\rho_{AB}^{T_B}\|_1 - 1, \tag{9}$$

where ρ^{T_B} is the partial transpose of ρ_{AB}.

It is known that PPT gives a separability criterion for two-qubit systems, and it is also a necessary and sufficient condition for nondistillability in $2\otimes n$ quantum system [17,18]. However, in higher-dimensional quantum systems rather than $2\otimes 2$ and $2\otimes 3$ quantum systems, there exist mixed entangled states with PPT, so-called “bound entangled states” [17,19]. For this case, negativity cannot distinguish PPT bound entangled states from separable states, and thus, negativity itself is not sufficient to be a good measure of entanglement even in a $2\otimes n$ quantum system.

One modification of negativity to overcome its lack of separability criterion is CREN [20], which gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system.

For a bipartite mixed state mixed state ρ_{AB}, CREN is defined as

$$\mathcal{N}_c(\rho) \equiv \min \sum_k p_k \mathcal{N}(|\phi\rangle_k), \tag{10}$$

where the minimum is taken over all possible pure state decompositions of $\rho=\Sigma_k p_k|\phi_k\rangle\langle\phi_k|$.

Whereas a normalized version of the negativity depending on the dimension of the quantum systems was used to show its monotonicity [10], it can be analogously shown with the definitions in Eqs. (9) and (10).

Now, let us consider the relation between CREN and concurrence. For any bipartite pure state $|\phi\rangle_{AB}$ in a $d\otimes d'$ quantum system with Schmidt rank 2,

$$|\phi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle, \tag{11}$$

we have

$$\mathcal{N}(|\phi\rangle) = \||\phi\rangle\langle\phi|^{T_B}\|_1 - 1 = 2\sqrt{\lambda_0\lambda_1} = \sqrt{2(1-\text{tr }\rho_A^2)} = \mathcal{C}(|\phi\rangle), \tag{12}$$

where $\rho_A=\text{tr}_B(|\phi\rangle\langle\phi|)$. In other words, negativity is equivalent to concurrence for any pure state with Schmidt rank 2, and consequently it follows that for any 2-qubit mixed state $\rho_{AB}=\Sigma_i p_i|\phi_i\rangle\langle\phi_i|$,

$$\mathcal{N}_c(\rho_{AB}) = \min \sum_i p_i \mathcal{N}(|\phi_i\rangle) = \min \sum_i p_i \mathcal{C}(|\phi_i\rangle) = \mathcal{C}(\rho_{AB}), \tag{13}$$

where the minima are taken over all pure state decompositions of ρ_{AB}.

Similar to the duality between concurrence and COA, we can also define a dual to CREN, namely CRENOA, by taking the maximum value of average negativity over all possible pure state decomposition. Furthermore, for a 2-qubit system, we have

$$\mathcal{N}_c^a(\rho_{AB}) = \max \sum_i p_i \mathcal{N}(|\phi_i\rangle) = \max \sum_i p_i \mathcal{C}(|\phi_i\rangle) = \mathcal{C}^a(\rho_{AB}), \tag{14}$$

where maxima are taken over all pure state decompositions of ρ_{AB} and $\mathcal{N}_c^a(\rho_{AB})$ is the CRENOA of ρ_{AB}.

From the analysis of CREN and CRENOA, we can see that CREN can be considered as a generalized version of concurrence from 2-qubit systems. Thus, having the monotonicity and separability criteria of CREN, it is natural to investigate MOE in terms of CREN for multiqubit systems and possible higher-dimensional quantum systems.

III. CREN MONOGAMY OF ENTANGLEMENT

In 3-qubit systems, Coffman, Kundu, and Wootters (CKW) [4] first introduced a monogamy inequality in terms of concurrence, as

$$\mathcal{C}_{A(BC)}^2 \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2, \tag{15}$$

where $\mathcal{C}_{A(BC)}=\mathcal{C}(|\psi\rangle_{A(BC)})$ is the concurrence of a 3-qubit state $|\psi\rangle_{A(BC)}$ for a bipartite cut of subsystems between A and BC and $\mathcal{C}_{AB}=\mathcal{C}(\rho_{AB})$. Similarly, its dual inequality in terms of COA,

$$\mathcal{C}_{A(BC)}^2 \leq (\mathcal{C}_{AB}^a)^2 + (\mathcal{C}_{AC}^a)^2, \tag{16}$$

has been shown in [21]. Later, the CKW inequality has been generalized into n-qubit systems [7], and its dual inequality for n-qubit systems has also been introduced [22].

However, a quantum state in a $3\otimes 3\otimes 3$ quantum system was found that violates the CKW inequality [8], and recently another counterexample was found in a $3\otimes 2\otimes 2$ quantum system [9]; therefore, the CKW inequality only holds for multiqubit systems, and even a tiny extension in any of the subsystems leads to a violation.

In this section, we show that all the monogamy inequalities for qubits using concurrence can be rephrased by CREN, and this CREN monogamy inequality is still true for the counterexamples in [8,9].

A. Monogamy inequalities for *n*-qubit systems in terms of CREN

For any pure state $|\psi\rangle_{A_1\cdots A_n}$ in an n-qubit system $A_1 \otimes \cdots \otimes A_n$ where $A_i \cong \mathbb{C}^2$ for $i=1,\ldots,n$, a generalization of the CKW inequality,

$$\mathcal{C}_{A_1(A_2\cdots A_n)}^2 \geq \mathcal{C}_{A_1A_2}^2 + \cdots + \mathcal{C}_{A_1A_n}^2, \tag{17}$$

was conjectured [4] and proved [7]. Another inequality, which can be considered to be dual to Eq. (17) was also introduced in [22],

$$\mathcal{C}_{A_1(A_2\cdots A_n)}^2 \leq (\mathcal{C}_{A_1A_2}^a)^2 + \cdots + (\mathcal{C}_{A_1A_n}^a)^2. \tag{18}$$

Now, let us consider these inequalities in terms of CREN. First, note that any n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$ can have a Schmidt decomposition with at most two nonzero Schmidt coefficients with respect to the bipartite cut between A_1 and the others. Thus, by Eq. (12), we have

$$\mathcal{C}_{A_1(A_2\cdots A_n)} = \mathcal{N}_{cA_1(A_2\cdots A_n)}. \tag{19}$$

Furthermore, for any reduced density matrix $\rho_{A_iA_j}$ of $|\psi\rangle_{A_1\cdots A_n}$ onto two-qubit subsystems $A_i\otimes A_j$, it is a two-qubit mixed state; therefore, by Eqs. (13) and (14), we have

$$\mathcal{C}_{A_iA_j} = \mathcal{N}_{cA_iA_j}, \quad \mathcal{C}_{A_iA_j}^a = \mathcal{N}_{cA_iA_j}^a, \tag{20}$$

for $i, j \in \{1,\ldots,n\}$, $i\neq j$.

Thus, we have the following theorem.

Theorem 1. For any n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$,

$$\mathcal{N}_{cA_1(A_2\cdots A_n)}^2 \geq \mathcal{N}_{cA_1A_2}^2 + \cdots + \mathcal{N}_{cA_1A_n}^2, \tag{21}$$

and

$$\mathcal{N}_{cA_1(A_2\cdots A_n)}^2 \leq (\mathcal{N}_{cA_1A_2}^a)^2 + \cdots + (\mathcal{N}_{cA_1A_n}^a)^2, \tag{22}$$

where $\mathcal{N}_{cA_1(A_2\cdots A_n)}=\mathcal{N}(|\psi\rangle_{A_1(A_2\cdots A_n)})$, $\mathcal{N}_{cA_1A_i}=\mathcal{N}_c(\rho_{A_1A_i})$, and $\mathcal{N}_{cA_1A_i}^a=\mathcal{N}_c^a(\rho_{A_1A_i})$ for $i=2,\ldots,n$.

Proof. It is a direct consequence from the overlap of CREN and concurrence in Eqs. (19) and (20), as well as the monogamy inequalities in Eqs. (17) and (18) by concurrence. ∎

In [23], another monogamy inequality of entanglement for 3-qubit systems in terms of the original negativity [13] was proposed. For a 3-qubit state $|\psi\rangle_{ABC}$, it was shown that

$$\mathcal{N}_{A(BC)}^2 \geq \mathcal{N}_{AB}^2 + \mathcal{N}_{AC}^2, \tag{23}$$

where $\mathcal{N}_{AB}^2=\|\rho_{AB}^{T_B}\|_1-1$ and $\mathcal{N}_{AC}^2=\|\rho_{AC}^{T_C}\|_1-1$ are the original negativities of ρ_{AB} and ρ_{AC}, respectively.

Due to the convexity of the original negativity, we can easily see that CREN is always an upper bound of the original negativity. In other words, for any bipartite mixed state ρ_{AB},

$$\mathcal{N}_c(\rho_{AB}) \geq \mathcal{N}(\rho_{AB}). \tag{24}$$

From Theorem 1 together with Eq. (24), we have the following corollary which encapsulates the result of Eq. (23).

Corollary 1. For any n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$,

$$\mathcal{N}_{A_1(A_2\cdots A_n)}^2 \geq \mathcal{N}_{A_1A_2}^2 + \cdots + \mathcal{N}_{A_1A_n}^2. \tag{25}$$

Thus, besides concurrence, CREN is another good entanglement measure in multiqubit systems for MOE.

B. CREN vs concurrence-based monogamy relations

Two counterexamples in [8,9] are, in fact, all known counterexamples showing the violation of the CKW inequality in higher-dimensional quantum systems. Here we show that they still have a monogamy relation in terms of CREN.

Counterexample 1 (Ou [8]). Let us consider a pure state $|\psi\rangle$ in $3\otimes 3\otimes 3$ quantum systems such that

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle). \tag{26}$$

Since $|\psi\rangle_{ABC}$ is pure, it is easy to check $C^2_{A(BC)}=\frac{4}{3}$. For mixed states ρ_{AB} and ρ_{AC}, it was shown that any pure state in any pure state ensemble has the same constant value, 1, as its concurrence, which implies $C^2_{AB}=C^2_{AC}=1$. Therefore, we have

$$C^2_{AB}+C^2_{AC}=2 \geqslant \frac{4}{3}=C^2_{A(BC)}, \tag{27}$$

which is a violation of the CKW inequality in higher-dimensional quantum systems.

Now, let us consider the case of using CREN as the entanglement measure for the state in Eq. (26).

Since $|\psi\rangle_{ABC}$ is pure, it can be easily checked that

$$\mathcal{N}_{A(BC)}=\mathcal{N}_{cA(BC)}=(\mathrm{tr}\sqrt{\rho_A})^2-1=2. \tag{28}$$

For $\mathcal{N}_{cAB}$, let us consider ρ_{AB} whose spectral decomposition is

$$\rho_{AB}=\tfrac{1}{3}(|x\rangle_{AB}\langle x|+|y\rangle_{AB}\langle y|+|z\rangle_{AB}\langle z|), \tag{29}$$

where

$$|x\rangle_{AB}=\frac{1}{\sqrt{2}}(|23\rangle-|32\rangle),$$

$$|y\rangle_{AB}=\frac{1}{\sqrt{2}}(|31\rangle-|13\rangle),$$

$$|z\rangle_{AB}=\frac{1}{\sqrt{2}}(|12\rangle-|21\rangle). \tag{30}$$

By a straightforward calculation, it can be shown that for arbitrary pure states $|\phi\rangle_{AB}=c_1|x\rangle_{AB}+c_2|y\rangle_{AB}+c_3|z\rangle_{AB}$ with $|c_1|^2+|c_2|^2+|c_3|^2=1$, their reduced density matrix $\rho_A=\mathrm{tr}_B|\phi\rangle_{AB}\langle\phi|$ has the same spectrum $\{\frac{1}{2},\frac{1}{2},0\}$ [24]. Thus, we have

$$\mathcal{N}(|\phi\rangle_{AB})=(\mathrm{tr}\sqrt{\rho_A})^2-1=1, \tag{31}$$

for any $|\phi\rangle_{AB}$ that is a superposition of $|x\rangle_{AB}$, $|y\rangle_{AB}$, and $|z\rangle_{AB}$. By the Hughston-Jozsa-Wootters (HJW) theorem [25], any pure state in any pure state ensemble of ρ_{AB} can be realized as a superposition of $|x\rangle_{AB}$, $|y\rangle_{AB}$, and $|z\rangle_{AB}$ thus we have

$$\begin{aligned}\mathcal{N}_c(\rho_{AB})&=\min_{\sum_k p_k|\phi\rangle_k\langle\phi_k|=\rho_{AB}}\sum_k p_k\mathcal{N}(|\phi\rangle_k)\\&=\frac{1}{3}[\mathcal{N}(|x\rangle_{AB})+\mathcal{N}(|y\rangle_{AB})+\mathcal{N}(|z\rangle_{AB})]=1.\end{aligned} \tag{32}$$

Since Eq. (26) is asymmetric, we also have a similar result for ρ_{AC}, which is

$$\begin{aligned}\mathcal{N}_c(\rho_{AC})&=\min_{\sum_k p_k|\phi\rangle_k\langle\phi_k|=\rho_{AC}}\sum_k p_k\mathcal{N}(|\phi\rangle_k)\\&=\frac{1}{3}[\mathcal{N}(|x\rangle_{AC})+\mathcal{N}(|y\rangle_{AC})+\mathcal{N}(|z\rangle_{AC})]=1.\end{aligned} \tag{33}$$

Now, from Eq. (28) together with Eqs. (32) and (33), we have

$$\mathcal{N}^2_{cA(BC)}=4 \geqslant 1+1=\mathcal{N}^2_{cAB}+\mathcal{N}^2_{cAC}. \tag{34}$$

In other words, even though the state $|\psi\rangle$ in Eq. (26) is a counterexample of the CKW inequality in 3-qutrit systems in terms of concurrence, it still shows a monogamy property in terms of CREN.

Counterexample 2 (Kim and Sanders [9]). Let us consider a pure state $|\psi\rangle$ in $3\otimes 2\otimes 2$ quantum systems such that

$$|\psi\rangle_{ABC}=\frac{1}{\sqrt{6}}(\sqrt{2}|010\rangle+\sqrt{2}|101\rangle+|200\rangle+|211\rangle). \tag{35}$$

It can be easily seen that $\mathcal{C}^2_{A(BC)}=\frac{12}{9}$ whereas $\mathcal{C}^2_{AB}=\mathcal{C}^2_{AC}=\frac{8}{9}$, which implies the violation of the CKW inequality. However, by using a similar method to the previous example, we can have $\mathcal{N}^2_{cA(BC)}=4$ whereas $\mathcal{N}^2_{cAB}=\mathcal{N}^2_{cAB}=\frac{8}{9}$, which implies the example in Eq. (35) also shows a monogamy property in terms of CREN.

Thus, CREN is a powerful alternative for MOE in multipartite higher-dimensional quantum systems without any trivial counterexample so far.

IV. PARTIALLY COHERENT SUPERPOSITION OF AN *n*-QUDIT GENERALIZED *W*-CLASS STATE AND $|0\rangle^{\otimes n}$

Three-qubit systems can have two inequivalent classes of genuine tripartite entangled states by the CKW inequality [26]. One of them is the Greenberger-Horne-Zeilinger (GHZ) class [27] and the other one is the *W* class [26]. These two classes show extreme differences in terms of the CKW and its dual inequalities: The CKW and its dual inequalities are saturated by *W*-class states, whereas the terms for reduced density matrices in the inequalities always vanish for GHZ-class states. Since the saturation of the CKW inequality by *W*-class states can be interpreted as a genuine tripartite entanglement with a complete characterization by means of its partial entanglements, *W*-class states here are especially interesting.

It was shown that there also exists a class of states in *n*-qudit systems which saturate a monogamy relation [9]. By using concurrence as the entanglement measure, the monogamy inequalities are shown to be saturated by incoherent superpositions of a generalized *n*-qudit *W*-class state [9] and the vacuum, $|0\rangle^{\otimes n}$.

In this section, we propose a class of quantum states in *n*-qudit systems consisting of partially coherent superpositions of a generalized *W*-class state and the vacuum, and show that they have the saturation of the monogamy relations in terms of CREN and CRENOA. This saturation is also true for an arbitrary partition of the set of subsystems, and it is not even affected by the degree of coherency.

Let us review the definition of an *n*-qudit generalized *W*-class state [9],

$$\begin{aligned}|W_n^d\rangle_{A_1\cdots A_n}=\sum_{i=1}^{d-1}(&a_{1i}|i0\cdots0\rangle+a_{2i}|0i\cdots0\rangle+\cdots\\&+a_{ni}|00\cdots0i\rangle),\end{aligned}$$

$$\sum_{i=1}^{d-1}(|a_{1i}|^2+|a_{2i}|^2+\cdots+|a_{ni}|^2)=1, \tag{36}$$

which is a coherent superposition of all n-qudit product states with Hamming weight 1.

A partially coherent superposition of a generalized W-class state and $|0\rangle^{\otimes n}$ is given as

$$\rho_{A_1\cdots A_n}=p|W_n^d\rangle\langle W_n^d|+(1-p)|0\rangle^{\otimes n}\langle 0|^{\otimes n}+\lambda\sqrt{p(1-p)}(|W_n^d\rangle \times\langle 0|^{\otimes n}+|0\rangle^{\otimes n}\langle W_n^d|), \tag{37}$$

where λ is the degree of coherence with $0\leqslant\lambda\leqslant 1$. For the case that $\lambda=1$, Eq. (37) becomes a coherent superposition of a generalized W-class state and $|0\rangle^{\otimes n}$, and it is an incoherent superposition, or a mixture when $\lambda=0$. In other words, Eq. (37) is an n-qudit state where the product state of Hamming weight zero is in a partially coherent superposition with all the states of Hamming weight 1.

The state in Eq. (37) can also be interpreted by means of decoherence. In other words, Eq. (37) can be considered as the resulting state from a coherent superposition of a generalized W-class state and $|0\rangle^{\otimes n}$,

$$|\psi\rangle_{A_1,\cdots A_n}=\sqrt{p}|W_n^d\rangle+\sqrt{1-p}|0\rangle^{\otimes n}, \tag{38}$$

after some decoherence process so-called phase damping [28], which can be represented as

$$\rho_{A_1\cdots A_n}=\Lambda(|\psi\rangle\langle\psi|)=E_0|\psi\rangle\langle\psi|E_0^\dagger+E_1|\psi\rangle\langle\psi|E_1^\dagger+E_2|\psi\rangle\langle\psi|E_2^\dagger, \tag{39}$$

with Kraus operators $E_0=\sqrt{\lambda}I$, $E_1=\sqrt{1-\lambda}(I-|0\rangle\langle 0|)$, and $E_2=\sqrt{1-\lambda}|0\rangle\langle 0|$.

Now, we will see that the monogamy relations of the state in Eq. (37) in terms of CREN and CRENOA are saturated with respect to any arbitrary partition of the set of subsystems. Furthermore, the entanglement of the state in Eq. (37) measured by CREN is not affected by the degree of coherency λ.

First, let us consider the CREN and CRENOA of $\rho_{A_1\cdots A_n}$ in Eq. (37) with respect to the bipartite cut between A_1 and all others. The state in Eq. (37) has a pure state decomposition as

$$\rho_{A_1\cdots A_n}=(\sqrt{p}|W_n^d\rangle+\lambda\sqrt{1-p}|0\rangle^{\otimes n})(\sqrt{p}\langle W_n^d|+\lambda\sqrt{1-p}\langle 0|^{\otimes n}) +[\sqrt{(1-p)(1-\lambda^2)}\langle 0|^{\otimes n}][\sqrt{(1-p)(1-\lambda^2)}|0\rangle^{\otimes n}]. \tag{40}$$

Now, let

$$|\tilde{\psi}_1\rangle=\sqrt{p}|W_n^d\rangle+\lambda\sqrt{1-p}|0\rangle^{\otimes n},$$

$$|\tilde{\psi}_2\rangle=\sqrt{(1-p)(1-\lambda^2)}|0\rangle^{\otimes n} \tag{41}$$

be two unnnormalized states in an n-qudit system. Then, by the HJW theorem [25], any other pure state decomposition of $\rho_{A_1(A_2\cdots A_n)}=\Sigma_{i=1}^r|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$ of size r can be realized by the choice of an $r\times r$ unitary matrix (u_{ij}) such that $|\tilde{\phi}_i\rangle=u_{i1}|\tilde{\psi}_1\rangle+u_{i2}|\tilde{\psi}_2\rangle$. In other words, with the normalization of $|\tilde{\phi}_i\rangle=\sqrt{p_i}|\phi_i\rangle$, we can consider an arbitrary pure state decomposition of $\rho_{A_1(A_2\cdots A_n)}=\Sigma_{i=1}^r p_i|\phi_i\rangle\langle\phi_i|$ with arbitrary size r.

By using the method introduced in [9], we can directly evaluate the average negativity of the pure states $|\phi_i\rangle$ for an arbitrary pure state decomposition of $\rho_{A_1(A_2\cdots A_n)}$. After tedious but straightforward calculations, it can be shown that the average negativity is independent from the choice of a unitary matrix (u_{ij}), which is

$$\sum_i p_i\mathcal{N}(|\phi_i\rangle)=2p\sqrt{\mathcal{A}(1-\mathcal{A})}, \tag{42}$$

where $\mathcal{A}=1-\Sigma_{j=1}^{d-1}|a_{1j}|^2$.

Thus, by the definition of CREN and CRENOA, we have

$$\mathcal{N}_c(\rho_{A_1(A_2\cdots A_n)})=\min\sum_i p_i\mathcal{N}(|\phi_i\rangle)=2p\sqrt{\mathcal{A}(1-\mathcal{A})} =\max\sum_i p_i\mathcal{N}(|\phi_i\rangle)=\mathcal{N}_c^a(\rho_{A_1(A_2\cdots A_n)}), \tag{43}$$

where the minimum and maximum are taken over all possible pure state decompositions of $\rho_{A_1(A_2\cdots A_n)}=\Sigma_i p_i|\phi_i\rangle\langle\phi_i|$.

Furthermore, it can be seen from Eq. (42) that this average value is also invariant under the degree of coherency λ. In other words, no matter how much amount of decoherence in Eq. (39) happens to the state in Eq. (38), its entanglement is preserved.

Now, for $\mathcal{N}_{cA_1A_i}$ and $\mathcal{N}^a_{cA_1A_i}$ with $i=2,\ldots,n$, let us first consider the case when $i=2$, whereas all the other cases are analogously following. By tracing over all subsystems except A_1 and A_2 from $\rho_{A_1\cdots A_n}$, we get

$$\rho_{A_1A_2}=p\sum_{i,j=1}^{d-1}(a_{1i}a_{1j}^*|i0\rangle\langle j0|+a_{1i}a_{2j}^*|i0\rangle\langle 0j|+a_{2i}a_{1j}^*|0i\rangle \times\langle j0|+a_{2i}a_{2j}^*|0i\rangle\langle 0j|)+(\mathcal{A}_2+1-p)|00\rangle \times\langle 00|+\lambda\sqrt{p(1-p)}\sum_{k=1}^{d-1}[(a_{1k}|k0\rangle+a_{2k}|0k\rangle) \times\langle 00|+a_{1k}^*|00\rangle(\langle k0|+a_{2k}^*\langle 0k|)], \tag{44}$$

with $\mathcal{A}_2=1-\Sigma_{j=1}^{d-1}(|a_{1j}|^2+|a_{2j}|^2)$.

Let us consider two unnormalized states

$$|\tilde{\psi}_1\rangle=\sqrt{p}\sum_{i=1}^{d-1}(a_{1i}|i0\rangle+a_{2i}|0i\rangle)+\lambda\sqrt{1-p}|00\rangle,$$

$$|\tilde{\psi}_2\rangle=\sqrt{\mathcal{A}_2+(1-p)(1-\lambda^2)}|00\rangle, \tag{45}$$

then we have

$$\rho_{A_1A_2}=|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|+|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|. \tag{46}$$

Thus all possible pure states in an arbitrary pure state decomposition of $\rho_{A_1A_2}$ of size r can be realized as a linear combination of $|\tilde{\psi}_1\rangle$ and $|\tilde{\psi}_2\rangle$ by choosing an $r\times r$ unitary matrix. Again, by using a similar method to the case of $\rho_{A_1\cdots A_n}$, it can been shown that the average negativity of $\rho_{A_1A_2}$ is invariant under the choice of pure state decomposition, which is,

$$\mathcal{N}_{cA_1A_2} = 2p\sqrt{(1-\mathcal{A})(\mathcal{A}-\mathcal{A}_2)} = \mathcal{N}^{a}_{cA_1A_2}. \quad (47)$$

Furthermore, rather surprisingly, this average value is also invariant under the degree of coherency. In other words, no matter how much amount of decoherence in Eq. (39) happens, it does not even affect the entanglement between the subsystems A_1 and A_2.

Similarly, we can have

$$\mathcal{N}_{cA_1A_i} = 2p\sqrt{(1-\mathcal{A})(\mathcal{A}-\mathcal{A}_i)} = \mathcal{N}^{a}_{cA_1A_i}, \quad (48)$$

for $i=3,\ldots,n$ with $\mathcal{A}_i = 1-\Sigma_{j=1}^{d-1}(|a_{1j}|^2+|a_{ij}|^2)$, and thus,

$$\sum_{i=2}^{n}\mathcal{N}^2_{cA_1A_i} = \mathcal{N}^2_{cA_1(A_2\cdots A_n)} = (\mathcal{N}^a_{cA_1(A_2\cdots A_n)})^2 = \sum_{i=2}^{n}(\mathcal{N}^a_{cA_1A_i})^2. \quad (49)$$

In other words, we have obtained a saturation of the CREN monogamy relation for an n-qudit state in Eq. (37), and this saturation does not depend on the choice of coherency λ.

For any arbitrary partition $P=\{P_1,\ldots,P_m\}$ of the set of subsystems, it was shown that an n-qudit generalized W-class state can be also considered as an m-partite generalized W-class state [9], that is

$$\begin{aligned}|W^d_n\rangle_{A_1\cdots A_n} &= \sum_{i=1}^{d-1}(a_{1i}|i\cdots 0\rangle + \cdots + a_{ni}|0\cdots i\rangle)\\ &= \sum_{i=1}^{d-1}|\tilde{x}_{1i}\rangle_{P_1}\otimes\cdots\otimes|\vec{0}\rangle_{P_m} + \cdots + |\vec{0}\rangle_{P_1}\\ &\quad\otimes\cdots\otimes|\tilde{x}_{mi}\rangle_{P_m} = \sum_{i=1}^{d-1}\sqrt{q_{1i}}|i\rangle_{P_1}\otimes\cdots\otimes|0\rangle_{P_m}\\ &\quad+\cdots+\sqrt{q_{mi}}|0\rangle_{P_1}\otimes\cdots\otimes|i\rangle_{P_m} = |W^d_m\rangle_{P_1\cdots P_m},\end{aligned} \quad (50)$$

where

$$|\tilde{x}_{si}\rangle_{P_s} = a_{(n_1+\cdots+n_{s-1}+1)i}|i\cdots 0\rangle_{P_s} + \cdots + a_{(n_1+\cdots+n_s)i}|0\cdots i\rangle_{P_s} \quad (51)$$

and

$$\sqrt{q_{si}}|x_{si}\rangle_{P_s} = |\tilde{x}_{si}\rangle_{P_s}, \quad |\vec{0}\rangle_{P_s} = |0\cdots 0\rangle_{P_s} \quad (52)$$

with renaming $|x_{si}\rangle_{P_s} = |i\rangle_{P_s}$ and $|\vec{0}\rangle_{P_s} = |0\rangle_{P_s}$ for $s \in \{1,\ldots,m\}$.

Therefore Eq. (37) can also be considered to be a partially coherent superposition of an m-partite generalized W-class state and the vacuum, $|0\rangle_{P_1\cdots P_m}$, and thus the result in (49) is also true for any arbitrary partition of the set of subsystems.

Not only for the case of multiqubit systems and the counterexamples in Sec. III, CREN also shows a strong monogamy relation of entanglement for a class of n-qudit states in a partially coherent mixture of a generalized W-class state and the vacuum. Thus, the CREN version of MOE is a strong conjecture for qudit systems with no obvious counterexamples.

V. CONCLUSIONS

The study of higher-dimensional quantum systems is, undoubtedly, important and even necessary to quantum-information science for various kind of reasons. First, qudits for $d>2$ are preferred in some physical systems such as in quantum key distribution where the use of qutrits increases coding density and provide stronger security compared to qubits [29]. In fault-tolerant quantum computation as well as on quantum error-correcting codes (QECCs), many studies are concentrated on the case of binary QECCs in a two-dimensional Hilbert space, whereas generalizations of proofs are often nontrivial when $d>2$.

However, as both qubit and qudit systems occur in the natural world, there is no reason to assume that a theoretical result should hold solely for two-dimensional systems. If an important result (e.g., monogamy of entanglement) is shown to be true for the case $d=2$, then this would suggest that a lot of effort should be directed towards qudit systems, as the case for $d>2$ could be fundamentally different from the case $d=2$. For example, a recent result [30] shows that for subsystem stabilizer codes in d-dimensional Hilbert space, a universal set of transversal gates cannot exist for even one encoded qudit, for any dimension d, which is known as no-go theorem for the universal set of transversal gates in QECC.

The extension of the multipartite entanglement analysis, especially the monogamy relation from qubit-to-qudit case is far more than trivial. The entanglement properties in higher-dimensional systems are hardly known so far, and thus any fundamental step of the challenges to the richness of entanglement studies for system of higher-dimensions and multipartite systems would be fruitful and even necessary to understand the whole picture of quantum entanglement.

In this paper, we have proposed CREN as a powerful alternative for MOE in higher-dimensional quantum systems. We have shown that any monogamy inequality of entanglement for multiqubit systems can be rephrased in terms of CREN. Furthermore, we have pointed out the possibility of CREN MOE in higher-dimensional quantum systems by showing that all the counterexamples for the CKW inequality so far in higher-dimensional quantum systems still have a monogamy inequality in terms of CREN, as well as no trivial counterexamples for CREN MOE so far. This task is one of the key challenges in finding a bipartite entanglement measure that meets our three criteria for qubits and for higher-dimensional systems.

For the studies of CREN MOE in higher-dimensional quantum systems, we have proposed a class of quantum states in n-qudit systems that are in a partially coherent superpositions of a generalized W-class state and the vacuum. The CREN monogamy relation for the proposed class has been shown to be true and it also holds with respect to any arbitrary partition of the subsystems.

Thus CREN is a good candidate for the general monogamy relation of multipartite entanglement, and it shows a strong evidence of its possibility even for the case of mixed states in higher-dimensional systems. We believe that the analysis of CREN MOE derived here will give a full and rich reference for the study of MOE in higher-dimensional quantum systems, which is one of the most important and necessary tasks in the study of quantum entanglement.

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