Quantum search in a nonclassical database of trapped ions

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We propose an implementation of the Grover's search algorithm in a nonclassical database using a linear chain of trapped ions. The database comprises all collective states with two ionic excitations; hence the number of database entries scales quadratically with the number of ions N. The system is initialized in an even superposition of all register states, i.e., in the symmetric Dicke state with N ions sharing two excitations. The reflection-about-the-mean operator is produced merely by global addressing of the ion string by an off-resonant pulse with a suitable area, whereas the oracle operator is a control-phase gate. This simplification should allow a demonstration of quantum search in a database of hundreds of elements without needing to synthesize multiqubit quantum gates. The technique does not rely on coherent addition or subtraction of phonons, and hence the ion chain can be sympathetically cooled throughout the process.

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I. INTRODUCTION

Although there are now a growing number of quantum algorithms, much of the interest in quantum-information science is still driven by two early results [1]. One of these, due to Shor, is that a quantum-information processor could factorize large numbers into primes exponentially faster than any classical device [2]. The other is Grover's observation that the superposition principle can furnish us with a quadratic speedup in the solution of the unstructured search problem [3]. Experimental demonstrations of Grover's algorithm have been performed in a range of physical systems, including nuclear magnetic resonance [4–6], linear-optical [7,8], and trapped-ion systems [9] as well as using individual Rydberg atoms [10] and in classical optics [11]. So far, all experiments have been at the proof-of-principle level, and involved databases of very modest dimension. For example, the largest-dimensional quantum search which has been performed using trapped ions involved just two ions and featured a database dimension of just four elements, i.e., $\mathcal{N}=4$ [9]. Of course, in order for quantum search to be of any practical use, it must be demonstrated in much larger systems, but unfortunately, this comes at the cost of complicated highly conditional gates. Using the approach taken in [9], a register comprised of N ions contains $\mathcal{N}=2^N$ elements, but executing the algorithm then requires a conditional gate with respect to all N ions. Recently, a simple physical implementation of Grover's algorithm with trapped ions has been proposed, where the quantum register is prepared in an entangled W state [12]. This register, however, scales linearly with the number of ions and is therefore of classical nature.

In this paper, we propose an approach to increasing the register dimension without the overhead of requiring multiply conditional gates. Indeed, the entire technique involves only a series of laser pulses, each illuminating the whole chain, and also controlled-phase gates between a pair of ions. This simplified approach is made possible by restricting the dynamics to the subspace of states in which exactly two of the ions are in their excited state; hence the register dimension increases quadratically with the number of ions. This type of scaling is superclassical since the number of database elements scales faster than the required physical resources.

II. OVERVIEW OF GROVER'S ALGORITHM

Grover's algorithm for quantum search allows a marked (sought) element $|s\rangle$ to be located within an unstructured database of dimension $\mathcal{N}$ in a number of search steps that scales as $O(\sqrt{\mathcal{N}})$. It provides a quadratic speed up with respect to classical search, which scales as $O(\mathcal{N})$.

The reflection operators. The key element in Grover's search is the Grover operator

$$\hat{G} = \hat{M}_a(\varphi)\hat{M}_s(\varphi), \tag{1}$$

where the operator $\hat{M}_\chi(\varphi)$ is a generalized Householder reflection (HR) with respect to a plane normal to the vector $|\chi\rangle$ supplemented with a phase shift φ [13],

$$\hat{M}_\chi(\varphi) = \hat{I} + (e^{i\varphi} - 1)|\chi\rangle\langle\chi|. \tag{2}$$

The first of these reflections $\hat{M}_s(\varphi)$ is known as an oracle query, since it can be written solely in terms of the oracle function $f_s(x)$: $\hat{M}_s(\varphi)|x\rangle = e^{if_s(x)\varphi}|x\rangle$. Here the oracle function $f_s(x)$ takes integer arguments $(x=1,2,\ldots,\mathcal{N})$ and acts differently on the sought state $|s\rangle$ to all others,

$$f_s(x) = \delta_{sx} = \begin{cases} 1 & (x = s), \\ 0 & (x \neq s). \end{cases} \tag{3}$$

The other component of the Grover operator $\hat{M}_a(\varphi)$ is the inversion-about-average operator, which treats all database elements equally and reflects the state of the system about the symmetric superposition,

*Deceased.

$$|a\rangle = \frac{1}{\sqrt{\mathcal{N}}}\sum_{x=1}^{\mathcal{N}} |x\rangle. \tag{4}$$

The Grover iteration. The Grover algorithm requires the database to be prepared initially in state (4),

$$|\Psi_i\rangle = |a\rangle, \tag{5}$$

and then apply iteratively the Grover operator $\hat{G}$ to this state n_g times, where [3]

$$n_g = \left(\frac{\pi}{4\sin^{-1}\mathcal{N}^{-1/2}}\right) \overset{\mathcal{N}\gg 1}{\sim} \left(\frac{\pi}{4}\sqrt{\mathcal{N}}\right), \tag{6}$$

where $[n]$ stands for the integer part of n. In result, a significant amplification of the marked state amplitude occurs, whose value approaches unity for large $\mathcal{N}$,

$$|\langle s|\hat{G}^{n_g}|a\rangle| \approx 1. \tag{7}$$

Phase matching. In principle, the angles of the two reflections $\hat{M}_s$ and $\hat{M}_a$ could be different. However, it was realized quite early on [14,15] that the maximum amplitude amplification of state $|s\rangle$ arises when the two reflections are phase matched, as in Eq. (1).

Grover's phase. The minimum possible number of iterations for Grover's search is n_g, Eq. (6), and it is obtained when $\varphi=\pi$, as in the original Grover's proposal [3]. However, with this choice of phase the outcome of the algorithm is probabilistic: the marked state is identified with probability (fidelity) $\mathcal{F}\gtrsim 1-1/\mathcal{N}$. If a deterministic search outcome is important, it is always possible to choose an alternative phase φ_d, for which the probability of recovering the marked state is unity [15],

$$\varphi_d = 2\sin^{-1}\left(\sqrt{\mathcal{N}}\sin\frac{\pi}{4n_d+6}\right), \tag{8}$$

where n_d is the corresponding number of iterations. Equation (8) has real solutions for all integer n_d satisfying $n_d > n_g$ and also, Eq. (8) has a real solution corresponding to $n_d=n_g$ (except for certain values of $\mathcal{N}$). We point out that the issue of deterministic search is only relevant for small databases because the uncertainty in the original Grover search decreases as $1/\mathcal{N}$; for any database of practical significance the original Grover search is almost deterministic.

In the context of the present paper, it is useful to rewrite Eq. (8) as

$$n_\varphi = \frac{\pi}{4}\left[\sin^{-1}\left(\frac{\sin(\varphi/2)}{\sqrt{\mathcal{N}}}\right)\right]^{-1} - \frac{3}{2} \overset{\mathcal{N}\gg 1}{\sim} \frac{\pi\sqrt{\mathcal{N}}}{4\sin(\varphi/2)}. \tag{9}$$

This equation provides the number of Grover iterations needed for an arbitrary value of φ. For $\varphi=\pi$ we obtain Grover's number n_g, Eq. (6). For any other value of φ the number of steps increases, $n_\varphi \geqq n_g$; for small φ this number can be very large ($n_\varphi \to \infty$ for $\varphi \to 0$). The important conclusion from Eq. (9) is that there is some leeway in the choice of φ, as long as its value is large enough. For instance, any value of φ in the range

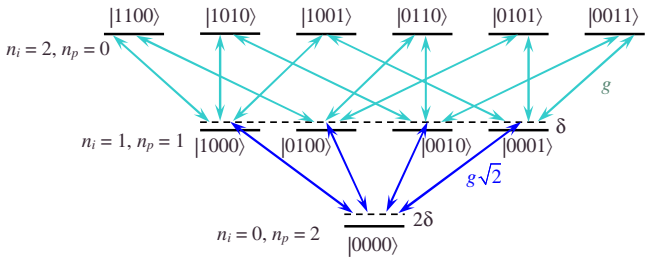

FIG. 1. (Color online) Linkage pattern of the relevant ionic states (for $N=4$ ions), dressed with a certain number of vibrational quanta n_p (phonons), coupled by a laser field, which is tuned near to (with a detuning δ) the red-sideband resonance of the center-of-mass vibrational mode. The total number of electronic and vibronic excitations is preserved and it is set to 2. The upper manifold of states, with two ionic excitations (qubits in state $|1\rangle$) and 0 phonons form the database elements used for Grover's search. The Rabi frequencies for the lower transitions are all equal to $g(t)\sqrt{2}$, whereas those for the upper transitions are $g(t)$ (the difference stems from the different phonon numbers).

$$\varphi_d \leqq \varphi \leqq \pi \tag{10}$$

ensures that the number of iterations is n_g (or n_g+1 in exceptional cases). We shall use below this leeway in the value of φ.

III. TRAPPED-ION SYSTEM

The purpose of this paper is to demonstrate how the two Householder reflections $\hat{M}_a(\varphi)$ and $\hat{M}_s(\varphi)$ needed to construct $\hat{G}$ may be performed very simply for databases of nonclassical dimension using a chain of trapped ions.

We consider a linear chain of N identical trapped ions, labeled by an index $k=1,\ldots,N$, each with two relevant internal states $|0_k\rangle$ and $|1_k\rangle$ and corresponding energies satisfying $E_1-E_0=\hbar\omega_0$. The ions interact collectively with a laser pulse (or a pair of pulses depending on the qubit implementation) shining along the trap axis, with angular frequency tuned near to the first motional red sideband of the center-of-mass mode, i.e., $\omega_L=\omega_0-\nu-\delta$, where ν is the trapping frequency, and δ is the detuning from the red-sideband resonance (for simplicity and without loss of generality δ is assumed positive, $\delta>0$), as shown in Fig. 1. We assume that the ions are initially cooled into their ground state of motion [16] and that the trapping frequencies are suitably chosen so that radial motion can be neglected. If the laser field has equal intensity at the position of each ion (for example, if it is directed along the trap axis) then the interaction Hamiltonian in the Lamb-Dicke limit and the rotating-wave approximation is given by [17–19]

$$\hat{H}(t) = \frac{\hbar\eta\Omega(t)}{2\sqrt{N}}\sum_{k=1}^{N}[\sigma_k^+\hat{a}e^{i(\delta t+\phi_k)} + \sigma_k^-\hat{a}^\dagger e^{-i(\delta t+\phi_k)}]. \tag{11}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ are, respectively, the creation and annihilation operators of center-of-mass phonons, while $\sigma_k^+=|1_k\rangle\langle 0_k|$ and $\sigma_k^-=|0_k\rangle\langle 1_k|$ are the internal spin-flip operators. $\eta = \sqrt{\hbar k^2\cos^2\theta_k/2M\nu}$ is the single-ion Lamb-Dicke parameter,

with k being the laser wave number, θ_k the angle between the trap axis and the direction of the laser beam, and M the ion mass. The function $\Omega(t)$ is the (real-valued) time-dependent Rabi frequency of the laser pulse, and the phase factor ϕ_k depends on the equilibrium position of the kth ion. We shall denote the number of internal excitations shared by the ions with n_i and the number of center-of-mass phonons with n_p. The Hamiltonian (11) is valid when the following conditions hold:

$$\delta \ll \nu, \tag{12a}$$

$$\Omega(t) < \nu, \tag{12b}$$

$$\eta\sqrt{n_p + 1} \ll 1. \tag{12c}$$

Conditions (12a) and (12b) ensure that nonresonant transitions to vibrational modes other than the center-of-mass mode have a negligible effect, while condition (12c) is known as the Lamb-Dicke limit. After performing the time-dependent phase transformation

$$\hat{F}(t) = \exp\left(\frac{i}{2}\sum_{k=1}^{N}(\delta t + \phi_k)\sigma_z^{(k)}\right), \tag{13}$$

where $\sigma_z^{(k)} = |1_k\rangle\langle 1_k| - |0_k\rangle\langle 0_k|$ is the Pauli spin matrix for the kth ion, we express the Hamiltonian (11) as

$$\hat{H}_I(t) = \hbar g(t)(\hat{a}^\dagger \hat{J}_- + \hat{a}\hat{J}_+) + \hbar\delta\hat{J}_z, \tag{14}$$

where $\hat{H}_I = \hat{F}^\dagger\hat{H}\hat{F} - i\hbar\hat{F}^\dagger\partial_t\hat{F}$. Here $g(t) = \eta\Omega(t)/2\sqrt{N}$ is the coupling between the internal and motional degrees of freedom. The collective operators

$$\hat{J}_+ = \sum_{k=1}^{N}\sigma_k^+, \quad \hat{J}_- = \sum_{k=1}^{N}\sigma_k^-, \quad \hat{J}_z = \frac{1}{2}\sum_{k=1}^{N}\sigma_k^{(z)}, \tag{15}$$

describe the combined ionic pseudospin of the N ions [20,21]. Since the Tavis-Cummings Hamiltonian (14) commutes with the excitation number operator $\hat{N} = \hat{a}^\dagger\hat{a} + \Sigma_{k=1}^{N}|1_k\rangle\langle 1_k|$, i.e., $[\hat{H}_I, \hat{N}] = 0$, the total number of excitations $n_i + n_p$ is conserved. Below, we will assume that the ion chain is initialized in its motional ground state ($n_p = 0$) and with exactly two ionic excitations ($n_i = 2$). The detuning and Rabi frequency of the laser pulse are chosen to satisfy

$$\nu \gg \delta \gg 1/T, \quad \nu \gg g(t), \tag{16}$$

where T is the pulse duration. Due to the condition $\delta \gg 1/T$, all transitions to states outside of the $(n_i = 2, n_p = 0)$ manifold are suppressed; hence, coherent changes in the motional state of the ions play no role. (Incoherent addition of unwanted motional quanta can be avoided by continuous sympathetic cooling of an auxiliary ion reserved for this purpose [22].)

The operators $\hat{J}_\pm$ and $\hat{J}_z$ satisfy the familiar angular-momentum commutation relations and the dynamics is most conveniently analyzed in a basis consisting of the set of eigenstates of both $\hat{J}_z$ and $\hat{J}^2$, where

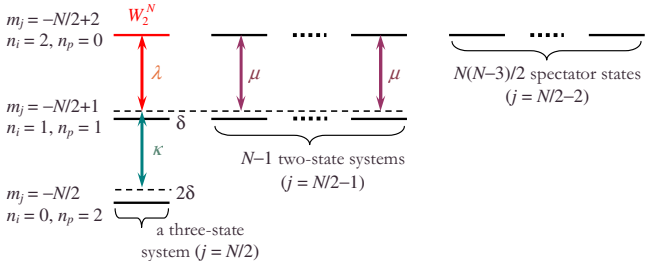

FIG. 2. (Color online) In the basis of the eigenstates of the collective operators $\hat{J}_z$ and $\hat{J}^2$, the coupling scheme represented by the Hamiltonian (14) factorizes into a series of independent chain-wise linkages. Each horizontal manifold corresponds to a given number n_i of the ions being in their excited state. The symmetric Dicke state $|W_2^N\rangle$, in which the system is initialized, is indicated; the identity of the other states is unimportant. κ, λ, and μ indicate the Rabi frequencies for the respective transitions. For simplicity, all energies in the Hamiltonian (14) are shifted by $(N/2-2)\hbar\delta$.

$$\hat{J}^2 = \hat{J}_z^2 + \tfrac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) \tag{17}$$

is the total ionic pseudospin. This change of basis, which is an example of the multilevel Morris-Shore (MS) factorization [23], is illustrated in Fig. 2. To emphasize the angular-momentum structure of the factorized basis states, we choose to label them as $|j, m_j\rangle$, where j and m_j, respectively, label the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$,

$$\hat{J}^2|j, m_j\rangle = j(j+1)|j, m_j\rangle, \tag{18a}$$

$$\hat{J}_z|j, m_j\rangle = m_j|j, m_j\rangle. \tag{18b}$$

In the angular-momentum basis, the original system from Fig. 1 decomposes into a chain of three states, $N-1$ chains of two states, and a set of $N(N-3)/2$ uncoupled single states, as illustrated in Fig. 2. Each chain possesses a definite angular momentum j and each state in a chain has a definite value of m_j, as shown in Fig. 2.

The Rabi frequency in each transition is obtained as a product of the relevant matrix elements of the motional and angular-momentum operators,

$$\langle n_p|\hat{a}^\dagger|n_p - 1\rangle = \sqrt{n_p}, \tag{19a}$$

$$\langle j, m_j - 1|\hat{J}_-|j, m_j\rangle = \sqrt{(j + m_j)(j - m_j + 1)}, \tag{19b}$$

where $|n_p\rangle$ is the Fock state with n_p phonons ($n_p = 0, 1, 2$). As indicated in Fig. 2, there are three distinct couplings,

$$\kappa(t) = g(t)\sqrt{2N}, \tag{20a}$$

$$\lambda(t) = g(t)\sqrt{2(N-1)}, \tag{20b}$$

$$\mu(t) = g(t)\sqrt{N-2}. \tag{20c}$$

In the present context, it is only necessary to identify the states in the longest chain, corresponding to $j = N/2$. We note that the lowest state in this chain is already known to be $|00\cdots0\rangle$, due to our chosen ordering of the states according to

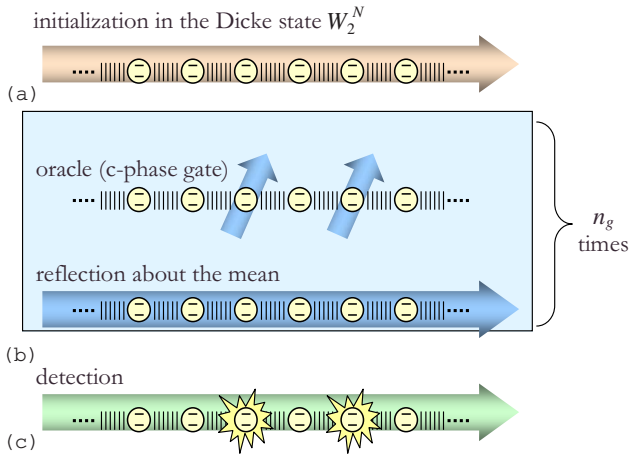

FIG. 3. (Color online) Physical implementation of the quantum search algorithm. (a) The system is initialized in the Dicke state $|W_2^N\rangle$. Stages (b) and (c), respectively, synthesize the reflections $\hat{M}_s$ and $\hat{M}_a$ which together constitute one Grover iteration. Note that the inversion about average operator $\hat{M}_a$ is here implemented in a single step, which is a significant improvement on earlier proposals. (d) The readout stage is performed using a fluorescence measurement simultaneously on the entire chain.

n_i. Because this state is symmetric under ions reordering, and the Hamiltonian (14) is also symmetric under interchange of the ions, it is clear that all states in the longest chain also share this property—the longest chain consists of the symmetric Dicke states $|W_{n_i}^N\rangle$ of N ions sharing n_i internal excitations [20,21,24]. The highest state in this chain is therefore an even superposition of all distinct configurations of N ions sharing two excitations: the Dicke state

$$|W_2^N\rangle = \frac{|110\cdots00\rangle + |101\cdots00\rangle + \cdots + |000\cdots11\rangle}{\sqrt{C_2^N}}, \tag{21}$$

with $C_2^N = N(N-1)/2$. This is exactly the initial state required for Grover's algorithm, $|a\rangle \equiv |W_2^N\rangle$. Conveniently, this state experiences a different coupling strength $\lambda(t)$ than all other factorized basis states. It is this fact which allows for a very simple implementation of the inversion-about-average operator.

Having introduced the necessary formalism we are now ready to show how to generate the two reflections $\hat{M}_a$ and $\hat{M}_s$ that lie at the heart of Grover's algorithm.

IV. IMPLEMENTATION OF QUANTUM SEARCH

A. Description

The quantum register comprises all collective ionic states in which exactly two ion qubits are in state $|1\rangle$, whereas the remaining $N-2$ ions are in state $|0\rangle$. Our proposed implementation of quantum search is sketched in Fig. 3 and consists of four simple operations, all of which are well within current experimental capabilities.

Step 1. The chain of ions is initialized into the symmetric Dicke state $|W_2^N\rangle$. Recently, a range of methods have been proposed for creating such states in a chain of trapped ions [24–28], the simplest of which use adiabatic passage and require only one [24] or two [25] pulses addressing the entire chain.

Step 2. The oracle query is a controlled-phase gate between two of the ions.

Step 3. The inversion-about-average operator is a single off-resonant pulse, with a suitable detuning and Rabi frequency, addressing the entire chain.

Step 4. Readout is achieved by performing a fluorescence measurement on the whole chain, which should identify which two ions are in state $|1\rangle$.

Steps 2 and 3 form the key ingredient of Grover's algorithm and should be repeated n_g (or n_φ) times in order to amplify the marked-state amplitude. They are described in detail in the following two sections.

B. Oracle-query operator

The marked element $|s\rangle$ is a state in which a particular pair of ions are in their state $|1\rangle$, and therefore the oracle operator $\hat{M}_s$ is simply a control-phase gate between these two ions. In fact, our motivation for restricting the database to the states containing just two excitations is that this operation is performed with high accuracy in current experiments [29–31]. For example, Benhelm *et al.* have recently reported a sequentially applied two-ion control-phase gate with $>99\%$ fidelity per gate [29] and it has been suggested that this could be extended to $>99.99\%$ fidelity [32].

When higher numbers of excitations are involved the synthesis of each oracle query becomes more challenging. While any multiqubit operation can efficiently be decomposed using a network or single- and two-qubit gates [1,33], the task of actually realizing such a circuit exceeds the limits of current experimental control even for small numbers of excitations. For example, when the system contains three excitations, the oracle operator can be synthesized using one TOFFOLI gate and two HADAMARD gates. Very recently, it was reported that a TOFFOLI gate between three trapped ions has been realized with a fidelity of 71% [34]. This is a very important development, being the first experimental demonstration of a conditional three-qubit gate using trapped ions. However, the reported fidelity may still be on the low side for application to quantum search if many Grover iterations are required. Therefore, we restrict our proposal to two excitations, although it can be extended to an arbitrary number of excitations.

C. Inversion-about-average operator

Because even for a moderate number of ions N the overall database dimension $\mathcal{N}=N(N-1)/2$ contains a relatively large number of elements, the inversion-about-average operator $\hat{M}_a(\varphi)$ appears to be a complicated operation, since it amounts to a controlled manipulation of all $\mathcal{N}$ amplitudes that describe the state. Fortunately, although this operation looks daunting from a formal perspective, it is possible to

synthesize $\hat{M}_a(\varphi)$ in a single step, by exploiting the symmetry possessed by the Hamiltonian (14). As already noted above, a single laser pulse, which couples equally to all of the ions, has a different effect on the even-superposition state $|a\rangle$ to all other states in the MS basis, i.e., $|a\rangle$ is the only state in the MS basis that experiences a coupling strength of $\lambda(t)$ and is involved in a three-state chain. The other coupled MS states of the $n_i=2$ manifold experience all the same coupling $\mu(t)$ in their respective two-state MS system (Fig. 2).

We assume that the detuning is sufficiently large ($\delta T \gg 1$), so that the transition probabilities in all MS systems vanish and the MS states in the $n_i=2$ manifold acquire only phase shifts: ϕ for $|a\rangle$ and ϕ' for the other coupled states $|\chi_k\rangle$ ($k=1,2,\ldots,N-1$). The propagator within the original $n_i=2$ manifold then reads as [35]

$$\hat{U} = \hat{I} + (e^{i\phi} - 1)|a\rangle\langle a| + (e^{i\phi'} - 1)\sum_{k=1}^{N-1} |\chi_k\rangle\langle\chi_k|$$

$$= \hat{M}_a(\phi)\prod_{k=1}^{N-1} \hat{M}_{\chi_k}(\phi'), \tag{22}$$

where we have used the orthogonality of the MS states. The first factor $\hat{M}_a(\phi)$ in Eq. (22) is exactly the reflection about the mean needed for the Grover's algorithm. In order to make the propagator $\hat{U}$ identical to the desired reflection $\hat{M}_a(\varphi)$ it is necessary that the phase ϕ' is equal to an even integer of π, because $\hat{M}_{\chi_k}(2l\pi)=\hat{I}$, that is

$$\mathrm{mod}(\phi, 2\pi) = \varphi, \tag{23a}$$

$$\mathrm{mod}(\phi', 2\pi) = 0. \tag{23b}$$

We therefore turn our attention to the choice of a laser field that produces such phases.

1. Very large detuning: Adiabatic elimination

In the limit of very large detuning ($\delta \gg g_0, 1/T$) one can eliminate adiabatically the $n_i \neq 2$ manifold and obtain simple expressions for the phases,

$$\phi \approx 2(N-1)g_0^2F^2/\delta + O(g_0^4/\delta^3), \tag{24a}$$

$$\phi' \approx (N-2)g_0^2F^2/\delta + O(g_0^4/\delta^3), \tag{24b}$$

with $F^2=\int_{t_i}^{t_f} f(t')^2 dt'$, where $g(t)=g_0 f(t)$. Conditions (23) require $2(N-1)g_0^2F^2/\delta = 2l_1\pi + \varphi$ and $(N-2)g_0^2F^2/\delta = 2l_2\pi$, with l_1 and l_2 integers. Therefore,

$$\frac{\varphi}{2\pi} = 2l_2 - l_1 + \frac{2l_2}{N-2}. \tag{25}$$

A proper choice of l_2 can produce the desired value of φ with an uncertainty $\Delta\varphi \sim O(1/N)$,

$$\frac{\Delta\varphi}{2\pi} < \frac{2}{N-2}. \tag{26}$$

For $\varphi \approx \pi$ (Grover's phase) we have $l_2=[(N-2)/4]$ and hence $g_0^2/\delta \sim \pi/2$. Hence the uncertainty in the phases will be $O(g_0^4/\delta^3) \sim O(1/\delta)$. Because the Grover algorithm is very sensitive to phase matching [14,15], a very large detuning may be required to make the algorithm work, $\delta T \gtrsim 10^2$, a condition that, after taking various details into account, is further extended towards $\delta T \gtrsim 10^3$. Such a condition may be in conflict with the condition for single-mode interaction, given the variety of modes in the many-ion chain.

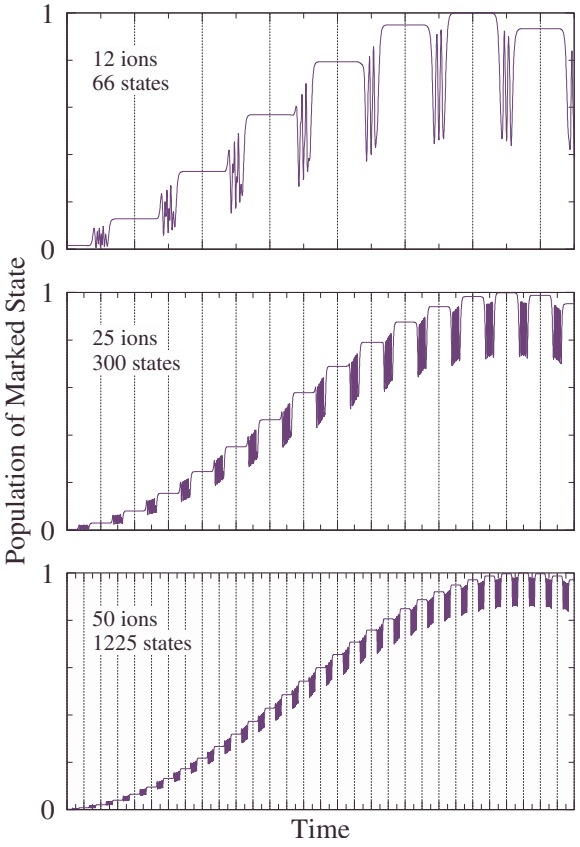

FIG. 4. (Color online) Simulation of the Grover search algorithm with N ions and two excitations [register dimension $\mathcal{N}=N(N-1)/2$], for $N=12$ (upper frame), $N=25$ (middle frame), and $N=50$ (lower frame). The system of ions is assumed to be initialized in the respective symmetric Dicke state $|W_2^N\rangle$. The laser pulses have a Gaussian shape, $g(t)=g_0 e^{-(t-t_n)^2/T^2}$, applied at times $t_n=8nT$. The thin vertical lines display the timing of the oracle call at times $t_o=4(2n+1)T$ and are separated by $8T$; the HR pulses are applied exactly in the middle between two vertical lines. The HR parameters for the reflection about the mean are $\delta T=10$ in all cases and peak Rabi frequencies $g_0 \approx 5.37T^{-1}$ (top), $g_0 \approx 6.79T^{-1}$ (middle), and $g_0 \approx 8.98T^{-1}$ (bottom). The oracle phase is $\varphi \approx -0.94\pi$ (top), $\varphi \approx -0.98\pi$ (middle), and $\varphi \approx \pi$ (bottom); it is chosen to match the phase of the HR. The marked-state population (the fidelity) reaches the value of nearly 99.9% for $n_g=6$, 13, and 26 steps, respectively, in exact agreement with Grover's value. The numerical simulation includes all off-resonant transitions to states with $n_i \neq 2$.

These considerations suggest a simple route to the choice of interaction parameters in situations when a very large detuning can be tolerated. We first select an appropriate integer number l_2 that produces the desired phase φ according to Eq. (25) (e.g., $l_2=[(N-2)/4]$ if $\varphi \approx \pi$). Then from the condition $(N-2)g_0^2F^2/\delta = 2l_2\pi$ we find the value of the ratio g_0^2/δ.

From here, for any fixed large value of δ, we find the peak Rabi frequency g_0.

2. Large detuning: Adiabatic solution

A controlled phase shift can be induced also in the regime $\delta \gg 1/T$ regardless of g_0, i.e., including for $g_0 > \delta$, when an adiabatic elimination cannot be carried out. The physical rationale is the effect of complete population return, which takes place in the adiabatic limit ($\delta T \gg 1$) [36] when the level energies do not cross. The phase shift in each MS two-state system is then

$$\phi' \approx \frac{1}{2}\int_{t_i}^{t_f} [\sqrt{4(N-2)g(t')^2 + \delta^2} - \delta]dt'. \qquad (27)$$

Obviously, in the limit $\delta \gg g_0$ we obtain Eq. (24b); therefore, Eq. (27) is more general. The phase shift ϕ in the three-state MS chain can be calculated as an integral over the respective eigenenergy (not listed here for simplicity). Although certain approximations to these expressions can be derived it is an easy task to evaluate these integrals numerically and choose parameter values that provide the required values of ϕ and ϕ'; we have used this latter approach in the numerical simulations below.

D. Numerical testing

In order to test the above predictions, we have solved the Schrödinger equation numerically, for a Gaussian pulse shape, $g(t)=g_0 e^{-t^2/T^2}$, and a constant detuning δ. Sample results are shown in Fig. 4 for $N=12$, 25, and 50 ions, which imply databases of $\mathcal{N}=N(N-1)/2=66$, 300, and 1225 elements. The fidelity plotted on the vertical axis is the time-dependent population of the marked state ($|1_1 1_2 0_3 \cdots 0_N\rangle$ in the example given). The system of N ions is assumed to be prepared initially in the even Dicke superposition $|W_2^N\rangle$ of the $\mathcal{N}$ collective states, each of which contains exactly two ion qubits in state $|1\rangle$ and the other qubits in state $|0\rangle$. Each Grover iteration consists of a phase shift of the marked state (oracle call), which amounts to a control-phase gate upon the ions in state $|1\rangle$, followed by a single off-resonant pulse of suitable amplitude and detuning, which addresses uniformly, with the same intensity, the entire ion chain (the inversion-about-average operator). The number of steps, for which the algorithm singles out the marked item with almost a certainty (99.9%), is $n_g=6$, 13, and 26, respectively, as predicted by Eq. (6).

V. CONCLUSIONS

The proposed implementation of quantum search with a chain of trapped ions should allow a proof-of-principle demonstration of quantum search with hundreds of database elements with the existing ion trap technology. The proposed implementation does not require complicated multiply conditional gates. The principle that underlies this simplification is to restrict the dynamics to those basis states for which exactly two ions are in their excited states and to take advantage of the symmetry that this subspace exhibits. With this arrangement, the oracle query operator is a control-phase gate between a particular pair of ions, while the inversion-about-average operator is synthesized using a single off-resonant laser pulse that addresses all of the ions simultaneously and uniformly. Our simulations show that the technique should perform very well for a moderately sized quantum register. Motional heating need not have a large effect, since the proposed method can operate alongside continuous sympathetic cooling of an auxiliary ion. Finally, this technique has the potential to be extended to ion trap arrays [37]; work in this direction is under way.

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