Salecker-Wigner-Peres clock and double-barrier tunneling

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We reexamine the Salecker-Wigner-Peres clock formalism and show that it can be directly applied to the phenomenon of tunneling. Then we apply this formalism to the determination of the tunneling time of a nonrelativistic wave packet, sharply concentrated around a tunneling energy, incident on a symmetric double-barrier potential. In order to deepen the discussion about the generalized Hartman effect, we consider the case in which the clock runs only when the particle can be found inside the region *between* the barriers and show that, whenever the probability to find the particle in this region is non-negligible, the corresponding time (which in this case turns out to be a dwell time) increases with barrier spacing.

I. INTRODUCTION

An unambiguous definition of tunneling time is an important problem in quantum mechanics, due to both its applications and its relevance to the foundations of the theory. It is, however, a problem which has eluded physicists since the beginnings of the quantum theory. Many attempts have been made to define such a time scale (see [1–4] for reviews). However, most of these definitions (phase time, dwell time, Larmor time, etc.), while being valid to describe some specific characteristic of the tunneling process, also present difficulties if one tries to interpret them as traversal times in general.

Perhaps the most striking of the above-mentioned difficulties concerns the issue of superluminality, a direct consequence of the Hartman effect [5], which asserts that the phase time saturates for opaque barriers. More recently, the Hartman effect has been considered for the double-barrier potential, and it was verified that in the opaque limit the phase time does not depend on the spacing between the barriers either, a phenomenon referred to as the *generalized Hartman effect* [6] (also see [7–9]). Although there are no real paradoxes associated with these phenomena, since it is well known that the group velocity cannot be associated with the signal velocity in such a situation [10], the subject has originated intense debate in the literature (see, for example, [11] and references cited therein). Independently of these controversial interpretations, it is in fact a counterintuitive result that in the opaque limit the phase time does not depend on the spacing between the barriers, because one could expect the group velocity to have the usual physical meaning in that *free* region. Subsequent investigations, both in the nonrelativistic [12] and in the relativistic [13] cases, indicate that this lack of dependence may be an artifact of the opaque limit, but the subject still deserves further investigation.

A fruitful avenue of investigation of tunneling times considers the use of quantum clocks. A quantum clock is a secondary dynamical system weakly coupled to the system of interest and having a degree of freedom evolving uniformly in time. One of the most prominent works along this line leads to Larmor time [14,15], but other clocks are possible (see, for example, [2] and references cited therein). Here we are particularly interested in the clock formalism introduced by Salecker and Wigner [16] and later reexamined by Peres [17], who used it to investigate, among other problems, the time of flight for a nonrelativistic particle (see also [18]). The extension for a relativistic particle was later done by Davies [19].

In [17] Peres also introduced a "time operator" (*not* canonically conjugate to the clock's Hamiltonian), whose expectation values do not lead to sensible results for the tunneling time in the presence of a localized potential, as was later shown by Leavens and McKinnon [20,21]. To overcome such a difficulty, these authors proposed a modification of the original Salecker-Wigner-Peres (SWP) formalism by the introduction of a calibration procedure. However, in his treatment of the time of flight problem Peres [17] did not use directly such an operator, but defined the time given by the clock (t_c) as the derivative of the phase shift of the wave function with respect to the perturbation potential. In this work we demonstrate that Peres' original approach, contrary to what is usually stated, can be directly applied to the tunneling time problem, without the need for calibration. In Sec. II we present a brief review of the SWP formalism and clarify some important issues related to it. In particular, we present a simple proof of the general result that t_c (averaged over the scattering channels) is exactly equal to the well-known dwell time (a result obtained by Leavens and McKinnon [21] through the expectation values of the Peres' "time operator" only *after* calibration). In Sec. III we apply the SWP formalism to the tunneling through a symmetric double barrier and analyze the dependence of t_c with the spacing between the barriers. It must be noticed that although in this case the (transmitted or reflected) time resulting from the SWP clock is exactly the dwell time, this formalism proves to be operationally better suited to address the question of independence or not of the tunneling time with respect to the barrier spacing in the limit of opaque barriers, providing a simpler procedure for the direct calculation of the time spent

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by the wave packet only *between* the two barriers. Therefore, such an approach allows us to deepen discussions about the generalized Hartman effect. In Sec. IV we discuss the results and their interpretation. In the Appendix we list the explicit expressions for some terms appearing in the expressions for the times obtained in Sec. III.

II. SWP CLOCK AND THE TUNNELING TIME PROBLEM

The *free* SWP clock consists of a quantum rotor, which for a Hilbert space of dimension N has Hamiltonian given by [17]

$$H_c = \omega J, \qquad (1)$$

with $J = -i\hbar \frac{\partial}{\partial \theta}$, $\omega = \frac{2\pi}{N\tau}$, and τ is the clock's resolution. The energy eigenstates are $u_m = \frac{e^{im\theta}}{\sqrt{2\pi}}$, $\theta \in [0, 2\pi]$, with eigenvalues $\mathcal{V}_m = m\hbar\omega$ ($m = -j, \ldots, j$). Another convenient orthonormal basis for the clock's Hilbert space is [17]

$$v_k(\theta) = \frac{1}{\sqrt{2\pi N}} \frac{\sin\left[\frac{N}{2}(\theta - \omega k\tau)\right]}{\sin\left[\frac{1}{2}(\theta - \omega k\tau)\right]}, \quad \theta \in [0, 2\pi), \qquad (2)$$

with $k = 0, \ldots, N-1$. These are the eigenstates of the Hermitian operator

$$T = \sum_{k=0}^{N-1} k\tau \mathcal{P}_k, \quad \text{with } \mathcal{P}_k v_l = \delta_{kl} v_l, \qquad (3)$$

with eigenvalues $k\tau$. The above operator plays the role of a "time" operator, despite not being canonically conjugate to the Hamiltonian. The motivation for this identification is that, for large N, the wave functions v_k are sharply peaked around $\theta = \omega k\tau$, with a width $\frac{2\pi}{N}$, and their time evolution is given by

$$\exp\left(-\frac{i}{\hbar} H_c t\right) v_k(\theta) = v_k(\theta - \omega t). \qquad (4)$$

Thus, $v_k(\theta)$ evolves *rigidly* and *uniformly* within the interval $[0, 2\pi]$. In particular, for large N the peaks translate from $\omega k\tau$ to $\omega(k\tau + t)$ (mod 2π).

It must be noticed that it is only for times $t = n\tau$, with n an integer, that

$$v_k(\theta - \omega n\tau) = v_{(k+n)(\text{mod}N)}(\theta),$$

such that the whole set of eigenfunctions $v_k(\theta)$ can be obtained from any of them (say, v_0) through a sequence of (discrete) time translations. This fact is at the origin of the discrepancies found by Leavens [20] between the *intrinsic* time t in Eq. (4) and the expectation value of Eq. (3) whenever t is not an integer multiple of τ. To overcome such a difficulty, Leavens introduced a *calibration procedure*, later revised by Leavens and McKinnon [21], designed in such a way that after calibration the clock times (given as an average over an ensemble of freely running clocks) coincided with the intrinsic ones.

On the other hand, the above properties of the wave functions v_k under time evolution allowed Peres to consider them as the proper clock's hand, with the clock's "reading" given by the angle ωt_c (the translation of the wave function's peak). In applying this approach to the one-dimensional scattering of a particle of mass μ by a localized potential $V(z)$ confined within the region $0 < z < L$, the clock-system coupling can be designed to measure the time the wave packet spends within an *arbitrary* region $z_1 < z < z_2$. In this case the Hamiltonian for the coupled system is given by [17]

$$H = H_s + P(z)H_c, \qquad (5)$$

where $H_s = \frac{p^2}{2\mu} + V(z)$, H_c is the clock Hamiltonian (1), and $P(z)$ is a projection operator into the interval (z_1, z_2). Let us consider a particle in a stationary state of energy E incident from the left (the results remain valid for a wave packet strongly concentrated around E). For the initial (free) clock state we choose, following Peres, $v_0(\theta)$. Then, assuming that the highest eigenvalue of the clock, $\mathcal{V}_m$, is negligible when compared to all the relevant energy scales in the problem, the *final* (asymptotic) state of the whole system is given by [17,20]

$$\Psi(z, \theta) = \begin{cases} Te^{ikz} v_0(\theta - \omega t_c^T), & z \geq \max\{z_2, L\}, \\ \\ Re^{-ikz} v_0(\theta - \omega t_c^R), & z \leq \min\{z_1, 0\}, \end{cases}$$

where T and R are the transmission and reflection coefficients (which depend on the energy E) for the system in the absence of the clock. From the above expressions and Eq. (4), one identifies the *Peres' transmission* and *reflection times*, respectively, by

$$t_c^T(E) = -\hbar \left.\frac{\partial \phi_T^{(m)}}{\partial \mathcal{V}_m}\right|_{\mathcal{V}_m=0} \quad \text{and} \quad t_c^R(E) = -\hbar \left.\frac{\partial \phi_R^{(m)}}{\partial \mathcal{V}_m}\right|_{\mathcal{V}_m=0}, \qquad (6)$$

where $\phi_T^{(m)}$ ($\phi_R^{(m)}$) is the phase delay of transmission (reflection) in the presence of the clock and the superscript (m) indicates the mth clock's eigenstate. In deriving the above result, it is assumed that in the vanishingly weak-coupling limit $T^{(m)} \approx |T| e^{i\phi_T^{(m)}}$ and $R^{(m)} \approx |R| e^{i\phi_R^{(m)}}$ [17].

A relation between the dwell time and Eq. (6) can be obtained by following steps similar to those Winful used to derive a relation between the dwell time and the phase times [22] (see also [23]). In order to do this, one must realize that for a stationary incident particle and when the clock is in its mth stationary state the problem is reduced to the solution of the time-independent Schrödinger equation with a localized potential $V^{(m)}(z) = V(z) + \mathcal{V}_m \mathcal{P}(z)$ [17,20], whose solution outside the potential region is given by

$$\psi^{(m)}(z) = \begin{cases} T^{(m)} e^{ikz}, & z \geq \max\{z_2, L\}, \\ \\ e^{ikz} + R^{(m)} e^{-ikz}, & z \leq \min\{z_1, 0\}, \end{cases} \qquad (7)$$

where $k = \frac{1}{\hbar}\sqrt{2\mu E}$. Considering the Schrödinger equation with the potential $V^{(m)}(z)$ and its complex conjugate, we obtain, after taking the vanishingly weak-coupling limit $\mathcal{V}_m \to 0$,

$$\mathcal{P}(z)\psi^*\psi = -\frac{\hbar^2}{2\mu}\frac{\partial}{\partial z}\left[\left(\frac{\partial\psi^*}{\partial z}\right)\left(\frac{\partial\psi^{(m)}}{\partial\mathcal{V}_m}\right)_{\mathcal{V}_m=0}\right.$$
$$\left. - \psi^*\left(\frac{\partial^2\psi^{(m)}}{\partial\mathcal{V}_m\partial z}\right)_{\mathcal{V}_m=0}\right], \quad (8)$$

where ψ denotes the wave function after the limit $\mathcal{V}_m \to 0$. Integrating the above expression over the region (z_1, z_2), we obtain $\frac{-2\mu}{\hbar^2}\int_{z_1}^{z_2}dz|\psi|^2 = [\cdots]_{z_2} - [\cdots]_{z_1}$, where $[\cdots]$ corresponds to the term in brackets in the above expression, which is constant for all $z \ge z_2$ and for all $z \le z_1$, because $\mathcal{P}(z) = 0$ in those regions. Taking advantage of this fact, we can use any value of $z > z_2$ to compute the set of brackets $[\cdots]_{z_2}$. For convenience we choose z in the region corresponding to the transmitted wave. In the same way, we can choose z in the incident-reflection region to compute the set of brackets $[\cdots]_{z_1}$. This procedure, together with Eq. (7), yields

$$\frac{\mu}{i\hbar^2 k}\int_{z_1}^{z_2}dz|\psi|^2 = T^*\left(\frac{\partial T^{(m)}}{\partial\mathcal{V}_m}\right)_{\mathcal{V}_m=0} + R^*\left(\frac{\partial R^{(m)}}{\partial\mathcal{V}_m}\right)_{\mathcal{V}_m=0},$$

which can be rewritten as

$$\frac{\mu}{\hbar k}\int_{z_1}^{z_2}dz|\psi|^2 = |T|^2\left[-\hbar\frac{\partial\phi_T^{(m)}}{\partial\mathcal{V}_m}\right]_{\mathcal{V}_m=0} + |R|^2\left[-\hbar\frac{\partial\phi_R^{(m)}}{\partial\mathcal{V}_m}\right]_{\mathcal{V}_m=0}.$$

Identifying the incident flux $j_{in} = \frac{\hbar k}{\mu}$, the left-hand side of the above expression is the well-known expression for the dwell time t_D [1,2]. Finally, from Eq. (6) we obtain

$$t_D = |T|^2 t_c^T + |R|^2 t_c^R. \quad (9)$$

Although the above relation was also obtained by Sokolovski *et al.* [24] through the use of Feynman's path integrals, our proof is worth mentioning due to its simplicity. Analogous results were also obtained in the framework of weak measurements or through the Larmor clock formalism, but involving, in general, complex times [25–27]. Relation (9), however, involves only real times. Leavens and McKinnon [21] showed that using the "time operator," Eq. (3), such a relation can only be obtained after applying their calibration procedure. As another important point concerning the above relation we emphasize that, *as long as* the transmission and reflection times are defined by Eq. (6), no interference term enter it (the analogous relation involving phase times necessarily requires such a term [22]). The validity of such a relation has been much debated in the literature (see, for instance, [1,2,28] for different points of view), and we hope that the above derivation helps to clarify the fact that it all depends on how the transmission and reflection times are defined (see also [3,25] for further discussions).

Finally, we note that when the whole potential $V(z) + \mathcal{V}_m\mathcal{P}(z)$ is *symmetric* the reflected and the transmitted phases differ only by a constant [15], which leads to $t_D = t_c^T = t_c^R$ [4,13]. In such a case it must also be noted that any of the expressions in Eq. (6) constitute an operationally simpler way to calculate the exact dwell time. We shall take advantage of this fact in the next section.

III. DOUBLE-BARRIER TUNNELING

Let us now consider a particle having a given energy E (or a wave packet sharply concentrated around this energy) incident from the left on a symmetric double-barrier potential, given by

$$V(z) = V_0\{\Theta(z)\Theta(a-z) + \Theta(z-d-a)\Theta(d+2a-z)\}. \quad (10)$$

We will consider only the case $E < V_0$, characterizing a tunneling process. Even though we are chiefly concerned with the case in which the clock runs only if the particle is inside the region separating the two barriers ($a < z < a+d$), it is instructive to first consider the clock running when the particle is anywhere within the potential region $(0, 2a+d)$. The solution of the time-independent Schrödinger equation outside the potential region in this case is of the form (7), with the transmission amplitude given by

$$T^{(m)} = 8ikp_m q_m^2 e^{-i(2a+d)k}(2\sin(p_m d)(k^2+q_m^2)(p_m^2+q_m^2)$$
$$- \{(k-iq_m)^2[2p_m q_m\cos(p_m d)$$
$$+ (p_m^2-q_m^2)\sin(p_m d)]e^{-2q_m a}$$
$$+ (q_m-ik)^2[2p_m q_m\cos(p_m d)$$
$$- (p_m^2-q_m^2)\sin(p_m d)]e^{2q_m a}\})^{-1}, \quad (11)$$

where $q_m = \sqrt{2\mu(V_0+\mathcal{V}_m-E)}/\hbar$ and $p_m = \sqrt{2\mu(E-\mathcal{V}_m)}/\hbar$. The corresponding phase is

$$\phi_T^{(m)} = -(2a+d)k - \tan^{-1}\left(\frac{\beta_m}{\alpha_m}\right), \quad (12)$$

with α_m and β_m defined as

$$\alpha_m \equiv 2kq_m[2p_m q_m\cos(p_m d)\cosh(2q_m a)$$
$$+ (q_m^2-p_m^2)\sin(p_m d)\sinh(2q_m a)],$$

$$\beta_m \equiv -(k^2+q_m^2)(p_m^2+q_m^2)\sin(p_m d)$$
$$+ 2p_m q_m(q_m^2-k^2)\cos(p_m d)\sinh(2q_m a)$$
$$+ (q_m^2-p_m^2)(q_m^2-k^2)\sin(p_m d)\cosh(2q_m a). \quad (13)$$

From the symmetry of the total potential (including the term due to the clock), it follows that $t_D = t_c^R = t_c^T$. So using Eq. (6) we obtain

$$t_D = t_c^T = -\frac{\mu}{\hbar(\alpha_0^2+\beta_0^2)}\left(\frac{h_1}{k} - \frac{h_2}{q}\right), \quad (14)$$

where α_0 and β_0 are obtained from Eq. (13) by taking the limit $\mathcal{V}_m \to 0$ [which corresponds to $p_m \to k$ and $q_m \to q = \frac{1}{\hbar}\sqrt{2\mu(V_0-E)}$]. The explicit expressions for α_0 and β_0 as well as for h_1 and h_2 are given in the Appendix.

Now, let us consider the case in which the SWP clock runs only when the particle can be found inside the region *between* the two barriers: namely, in the interval $(a, a+d)$. The (transmitted) Peres' time t_c^{bet} in this case can be obtained from the above results simply by taking the limit $q_m \to q$ in Eq. (12) before taking the derivative in Eq. (6), which gives

$$t_c^{\text{bet}} = -\frac{\mu}{\hbar k}\frac{h_1}{\alpha_0^2 + \beta_0^2}. \qquad (15)$$

Using the above expression, we can rewrite Eq. (14) as

$$t_D = t_c^{\text{bet}} + t_c^{\text{bar}}, \qquad (16)$$

where

$$t_c^{\text{bar}} = \frac{\mu}{\hbar q}\frac{h_2}{\alpha_0^2 + \beta_0^2} \qquad (17)$$

is just the (transmitted) Peres' time obtained by allowing the clock to run only when the particle passes *within any* of the two barriers [which corresponds to take $p_m \to k$ in all the above perturbed expressions, before taking the derivative in Eq. (6)]. From the proof presented in the previous section, together with the symmetry of the total potential, it follows that Eqs. (15) and (17) are the *dwell times* spent in the regions *between* and *within* the two barriers, respectively.

In order to discuss the generalized Hartman effect, we specialize our results to the opaque limit $qa \to \infty$, in which the Peres' time (14) for the whole potential region *saturates* to the value

$$t_D = t_c^T \underset{qa\to\infty}{\longrightarrow} \frac{\mu}{\hbar q^2}\frac{2kq}{(k^2 + q^2)}. \qquad (18)$$

Then, in the opaque limit the behavior of the Peres' time (dwell time) for the *whole* potential region is analogous to that of the phase time, in which it is independent of both the barrier width a and the barrier spacing a, and we obtain a version of the generalized Hartman effect. The above saturated result could also be obtained from the nonrelativistic limit for the dwell time obtained in [13] (in that reference the dwell time was obtained from a relation among the phase and the dwell times involving interference terms [23]). However, methods based on the phase time, which is an asymptotically extrapolated quantity, are not suitable to study the behavior of the time the particle spends *only* inside the region *between* the barriers.

In the opaque limit $qa \to \infty$, expression (15) immediately yields $t_c^{\text{bet}} \to 0$. A more careful analysis taking into account the leading terms in the asymptotic situation in which qa is large, but finite, shows that

$$t_c^{\text{bet}} \underset{qa\gg1}{\longrightarrow} \frac{4\mu q^2}{\hbar}\frac{e^{-2qa}}{(k^2 + q^2)}$$
$$\times \frac{\{2kd(k^2 + q^2) + 4kq\sin^2(kd) + (k^2 - q^2)\sin(2kd)\}}{\{(k^2 - q^2)\sin(kd) - 2kq\cos(kd)\}^2}. \qquad (19)$$

Apart from terms coming from the multiple reflections and/or interference at the barriers, this expression clearly displays an increasing (almost linear) dependence of t_c^{bet} with respect to the barrier spacing d, *for any finite qa*. Figure 1 shows the behaviors of the three times entering expression (16), with increasing d. Figure 1(a) concerns a large but finite barrier width a. We can observe the almost linear increasing of t^{bet} with increasing d. In Fig. 1(b) the barrier width is increased 3 times and we can already observe the
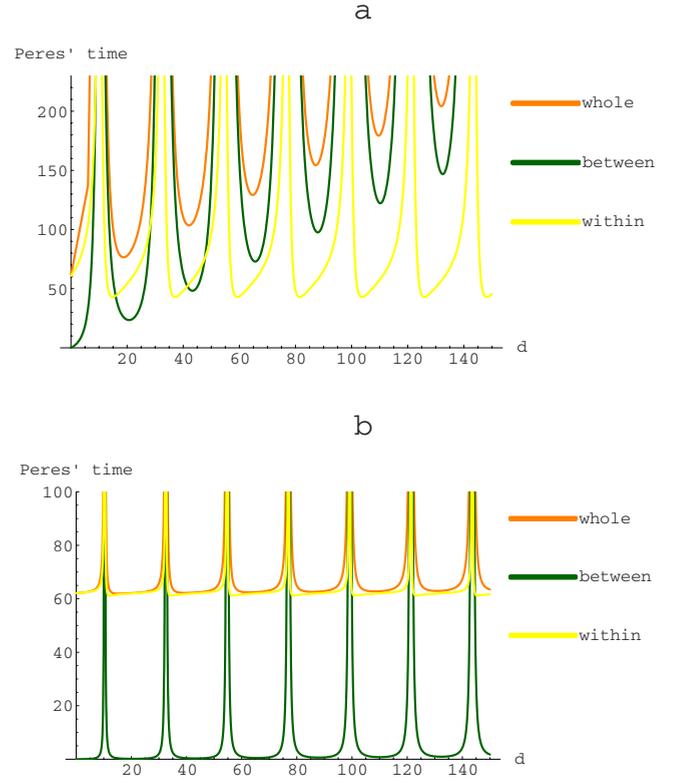

FIG. 1. (Color online) Peres' time dependence with respect to the distance d between the two barriers. In both figures the three clock times appearing in relation (16) are shown, corresponding to the *whole* potential region $(0, 2a+d)$; to the region *between* the barriers, $(a, a+d)$; and to the region *within* the two barriers, $(0, a) \cup (a+d, 2a+d)$. *Natural units* are used ($\hbar = c = 1$), so that mass and energies are expressed in units of the particle mass μ and distances and times are expressed in units of μ^{-1}. In both figures, $E = 0.01$ and $V_0 = 0.018$. (a) $a = 10$ and (b) $a = 30$.

tendency to saturation (except by the peaks) of the three times: t_D and t_c^{bar} tends to the saturated value (18), while t_c^{bet} tends to saturate to zero. Increasing even more the barrier width makes the peaks vanish. It can also be shown that if the clock runs only within the *first* barrier, the corresponding saturated dwell time in the opaque limit is exactly the same as Eq. (18). Summarizing, in the opaque limit $qa \to \infty$ the transmission amplitude beyond the *first* barrier goes to zero and its associated phase becomes meaningless. This fact corroborates the view that the generalized Hartman effect, which asserts the independence of the tunneling time on d, is indeed an artifact of the opaque limit (see also [12,13]).

IV. CONCLUDING REMARKS

In this work we reexamined the SWP clock formalism, which (despite being based on a thought experiment) is useful to understand the fundamental concepts involved in the definition of a time scale for tunneling processes. We showed that in Peres' version it can be directly applied to the problem of tunneling times. We demonstrated that departures from the Peres' approach are not necessary, even in the case of localized potentials, if one focuses, as Peres did, on the

time evolution of the eigenfunctions of his "time operator," instead of focusing on its expectation values, as did Leavens and McKinnon [21].

Using Peres' approach and through a simple extension of a proof originally designed by Winful in the context of phase times [22,23], we have shown that the Peres' time (6), when weighted by the transmission and reflection probabilities, averages exactly to the dwell time [incidentally, for symmetric potentials any of the two expressions (6) provides an operationally simpler way to calculate the dwell time]. On the other hand, we did not address questions regarding the SWP clock's resolution in the presence of a localized potential (see [17,20]) because this issue can be addressed in the framework of the weak measurement theory [29], as suggested in [30,31]. Besides that, in this paper each of the time readings associated with the clock is equivalent to a dwell time, which is a well-established time scale [1–4].

We then applied the SWP formalism to the symmetric double-barrier potential, aiming to analyze the so-called generalized man effect. We calculated explicitly the dwell time and verified that in the opaque limit it does not depend on the barrier separation, confirming the emergence of the generalized man effect also in this case (see also [12]). However, we added an insight into this debated question by allowing the clock to run only inside the region between the barriers, and taking into consideration the leading terms when the barrier width is large, but finite, we unambiguously showed that the dwell time increases "almost linearly" with the barrier spacing d (apart from terms arising from the multiple reflections and interference inside this region). The fact that such a behavior is modulated by an exponential decay $\exp(-2qa)$ can be understood by noticing that the dwell time is an average over the probability of finding the particle in the interest region (and since this probability decays exponentially with qa, so will t_D). Therefore, whenever the probability to find the particle in the region between the barriers is nonnegligible, the corresponding dwell time depends on the barrier spacing.

All the above considerations reinforce the conclusions arrived in [12], in the context of a Fabry-Perot cavity, and in [13], for the relativistic tunneling through double barriers, that the generalized Hartman effect is just a mathematical artifact of the opaque limit: although in that limit the transmission phase is well defined and finite, it is meaningless since T itself goes to zero. Therefore, any time scale defined in terms of such phase (such as phase times and, as seen in Sec. II, dwell times) also becomes meaningless in this limit: it corresponds to the trivial fact that the particle does not penetrate past the first barrier, and therefore it makes no sense to associate any time duration to its passage (or dwelling) in the region between the two barriers.

APPENDIX

In this appendix we present the explicit expressions for α_0, β_0, h_1, and h_2, which appear in the expressions for the Peres' times (14), (15), and (17):

$$\alpha_0 \equiv 2kq[2kq\cos(kd)\cosh(2qa) \\ + (q^2 - k^2)\sin(kd)\sinh(2qa)],$$

$$\beta_0 \equiv -(k^2 + q^2)^2\sin(kd) + 2kq(q^2 - k^2)\cos(kd)\sinh(2qa) \\ + (q^2 - k^2)^2\sin(kd)\cosh(2qa),$$

while $h_1 \equiv \alpha_0\gamma_1 - \beta_0\gamma_2$ and $h_2 \equiv \alpha_0\gamma_3 - \beta_0\gamma_4$, with

$$\gamma_1 \equiv -2k(q^2 + k^2)\sin(kd) - d(q^2 + k^2)^2\cos(kd) \\ + 2q(q^2 - k^2)\cos(kd)\sinh(2qa) \\ - 2qkd(q^2 - k^2)\sin(kd)\sinh(2qa) \\ - 2k(q^2 - k^2)\sin(kd)\cosh(2qa) \\ + d(q^2 - k^2)^2\cos(kd)\cosh(2qa), \tag{A1}$$

$$\gamma_2 \equiv 2kq\{2q\cos(kd)\cosh(2qa) - 2qkd\sin(kd)\cosh(2qa) \\ - 2k\sin(kd)\sinh(2qa) + d(q^2 - k^2)\cos(kd)\sinh(2qa)\}, \tag{A2}$$

$$\gamma_3 \equiv -4q(q^2 + k^2)\sin(kd) + 2k(3q^2 - k^2)\cos(kd)\sinh(2qa) \\ + 4kqa(q^2 - k^2)\cos(kd)\cosh(2qa) \\ + 4q(q^2 - k^2)\sin(kd)\cosh(2qa) \\ + 2a(q^2 - k^2)^2\sin(kd)\sinh(2qa), \tag{A3}$$

$$\gamma_4 \equiv 2k\{4kq\cos(kd)\cosh(2qa) + (3q^2 - k^2)\sin(kd)\sinh(2qa) \\ + 4kq^2a\cos(kd)\sinh(2qa) \\ + 2qa(q^2 - k^2)\sin(kd)\cosh(2qa)\}. \tag{A4}$$

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