Quantum interference effects of a single photon interacting with an atomic chain inside a one-dimensional waveguide

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We investigate the interaction between a single photon and a chain of N equally spaced two-level atoms inside a one-dimensional waveguide. By solving the eigenvectors of the Hamiltonian in the subspace with a single excitation, we address quantum interference effects in two physical processes. First, we analyze the scattering of a single photon when atoms are all initially at their ground states. We discover a set of transmission peaks determined by N. These transmission peaks are narrow and sensitive to the relative position between atoms. Second, we examine the spontaneous emission of a photon from an excited atom in the chain in vacuum. The characteristic atomic decay rates are found to be enhanced and reduced by increasing the number of atoms. In particular, the slowest decay rate scales approximately as N^{-3}.

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I. INTRODUCTION

Since the pioneer work by Purcell [1], modifications of radiative properties of atoms in cavities and waveguides have been a subject of main interest in quantum optics [2]. As photons are ideal carriers of quantum information, an important topic is the transport properties of photons in optical waveguides. Miniaturized waveguides, formed by a line defect in photonic crystals [3] or photonic crystals fibers [4], can have transverse cross sections that are as small as the size of a wavelength square. Therefore photon-atom interactions can be strongly modified as compared with free space systems. We note that the enhancement and suppression of spontaneous emission from an atom inside a waveguide has been studied quite extensively in the literature [5], and the photon scattering problem has been addressed by several authors [6–9]. Specifically, the single-photon single-atom scattering problem in a one-dimensional waveguide was discussed by Shen and Fan who employed a real space model to determine the transport properties of the photon [6], and Domokos *et al.* obtained the Heisenberg operator solution applicable to few-photon pulses [7]. In both studies, the authors have considered the case of a two-level atom, and they reported that a strong modification of photon transmission spectra can be achieved. In particular, a single photon at the resonance atomic transition frequency can be completely reflected.

In this paper we investigate the interaction between a single photon and a finite chain of N atoms inside a one-dimensional waveguide. Our focus is the quantum interference effects from multiple atoms. In the subspace with a single excitation, we are able to determine a complete set of eigenvectors of the Hamiltonian analytically, based on the method in [6]. Such eigenvectors allow us to address the quantum dynamics of a single photon as it propagates through the atoms. In contrast to the case of the single-atom system, we find that a photon can be perfectly transmitted near (but not equal to) the resonance atomic frequency. The positions of such transmission peaks and their widths are sensitive to the spacing between neighboring atoms. Therefore the transmission spectrum of a single photon may be useful in probing the atomic separation in the subwavelength domain.

It is well known that quantum interference between atomic dipoles can affect the spontaneous decay process [10]. With the solution of eigenvectors of the Hamiltonian, we will discuss the dynamics of spontaneous emission when one atom in the chain is initially excited. As we shall see below, the decay of an atom can be described by a set of characteristic (complex) frequencies defined by the poles of eigenvectors. We find that the smallest decay rate scales approximately as N^{-3}, and hence the long time behavior of atomic decay can be significantly suppressed as the number of atoms increases.

II. THE MODEL HAMILTONIAN AND EIGENVECTORS

To begin with, our model depicted in Fig. 1 consists of a chain of N equally spaced two-level atoms interacting with quantized electromagnetic fields in a one-dimensional environment (waveguide). The waveguide is assumed to be infinitely long with negligible lateral loss. The positions of atoms are located at $x=0,L,\ldots,(N-1)L$. Each atom, whose excited and ground states are, respectively, $|e\rangle$ and $|g\rangle$, has the resonant transition frequency ω_A. We assume that ω_A is much larger than the cutoff frequency ω_C of the waveguide, then the dispersion relation of photons at near resonant frequency, $\omega_k \approx \omega_A$, can be taken as linear: $\omega_k = v_g|k|$, where v_g is the group velocity. Under rotating wave approximation, the Hamiltonian of the system is given by ($\hbar=1$) [6]

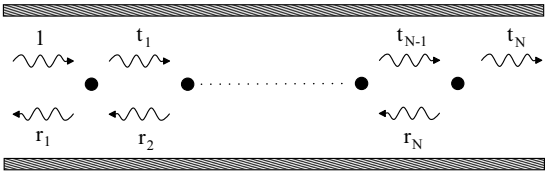

FIG. 1. Schematic drawing of the one-dimensional system in which equally spaced atoms (dots) interact with a photon (wavy lines).

$$\mathcal{H} = \sum_{j=1}^{N} \omega_A |e\rangle_j\langle e| + iv_g \int_{-\infty}^{\infty} dx \left[\psi_L^\dagger \frac{\partial \psi_L}{\partial x} - \psi_R^\dagger \frac{\partial \psi_R}{\partial x} \right] + J \sum_{j=1}^{N} \{ [\psi_R^\dagger(x_j) + \psi_L^\dagger(x_j)] \sigma_{j,-} + \text{H.c.} \}, \tag{1}$$

where J is the dipole interaction strength and $\sigma_{j,\pm}$ is the ladder operator for the jth atom at position x_j. The $\psi_R(x)$ $[\psi_L(x)]$ is the right (left) propagating bosonic field operator for photons. In terms of the plane wave basis, $\psi_R(x)$ $[\psi_L(x)]$ contains modes with wave vectors $k>0$ $(k<0)$ only.

We adopt the position-space approach in [6] to obtain the eigenvectors of the Hamiltonian. Such eigenvectors, defined by $\mathcal{H}|E_k\rangle = E_k|E_k\rangle$, are constructed in the form

$$|E_k\rangle = \int dx [u_{k,R}^*(x)\psi_R^\dagger(x) + u_{k,L}^*(x)\psi_L^\dagger(x)]|0,g\rangle + \sum_{j=1}^{N} e_k^{(j)}|0,e_j\rangle, \tag{2}$$

where k is the quantum number labeling the eigenvector with an eigenenergy $E_k = \hbar\omega_k$. The $|0,g\rangle$ denotes vacuum field and all atoms at ground state; $|0,e_j\rangle$ denotes the vacuum field and jth atom in the excited state with the amplitude $e_k^{(j)}$ and all other atoms in their ground state. To be specific, we consider that the eigenvector in Eq. (2) describes a photon entering the system from the far left, which is being reflected and transmitted (Fig. 1). The $u_{k,R}(x)$ and $u_{k,L}(x)$ are mode functions defined by

$$u_{k,R}^*(x) = \begin{cases} e^{ikx} & x < 0, \\ t_j e^{ik(x-jL)} & (j-1)L < x < jL, \\ t_N e^{ik(x-NL)} & x > (N-1)L \end{cases} \tag{3}$$

and

$$u_{k,L}^*(x) = \begin{cases} r_1 e^{-ikx} & x < 0, \\ r_{j+1} e^{-ik(x-jL)} & (j-1)L < x < jL, \\ 0 & x > (N-1)L, \end{cases} \tag{4}$$

where t_j and r_j are coefficients to be determined. These eigenvectors are orthogonal subjected to the normalization condition

$$\langle E_{k'}|E_k\rangle = 2\pi\delta(k-k'). \tag{5}$$

Note that the completeness relation requires inclusion of the eigenvectors of photons incident from the far right, which can be constructed similarly by symmetry.

By substituting Eq. (2) into $\mathcal{H}|E_k\rangle = E_k|E_k\rangle$, we obtain, for $1 \leq j \leq N$,

$$t_j e^{-ikL} - t_{j-1} + \frac{iJe_k^{(j)}}{v_g} = 0, \tag{6a}$$

$$r_{j+1} e^{ikL} - r_j - \frac{iJe_k^{(j)}}{v_g} = 0, \tag{6b}$$

$$t_{j-1} + r_j - \frac{\Delta_k e_k^{(j)}}{J} = 0. \tag{6c}$$

Here $\Delta_k \equiv \omega_k - \omega_A$ is the detuning, and $t_0 = 1$ and $r_{N+1} \equiv 0$ are defined. It is more convenient to cast Eq. (6) in the form of a transfer matrix so that the coefficients t_j and r_j satisfy

$$\begin{bmatrix} t_j \\ r_{j+1} \end{bmatrix} = \mathbf{M}^j \begin{bmatrix} 1 \\ r_1 \end{bmatrix}, \tag{7}$$

where $\mathbf{M} = \mathbf{GT}$ with

$$\mathbf{G} = \begin{bmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{bmatrix}, \tag{8}$$

$$\mathbf{T} = \begin{bmatrix} 1 - i\dfrac{\Omega}{\Delta_k} & -i\dfrac{\Omega}{\Delta_k} \\ i\dfrac{\Omega}{\Delta_k} & 1 + i\dfrac{\Omega}{\Delta_k} \end{bmatrix}. \tag{9}$$

We see that $\Omega \equiv J^2/v_g$ is the key parameter of the system in the frequency unit. The interaction between atoms and the photon is most effective when the photon frequency is near the atomic frequency with the detuning $|\Delta_k| < \Omega$.

The transfer matrix method is similar to that employed in photonic crystal problems [11]. It can be shown that

$$\mathbf{M}^2 = 2\mathbf{M}\cos\beta - \mathbf{I}, \tag{10}$$

where the angle β is defined by

$$\cos\beta = \cos kL + \frac{\Omega}{\Delta_k}\sin kL. \tag{11}$$

Depending on the right side of Eq. (11), β can be a complex angle and this generally occurs near the resonance with $\Delta_k \approx 0$. For definiteness, the real part of β is chosen in the range $0 \leqslant \mathrm{Re}(\beta) \leqslant \pi$. We also remark that the singular behavior of β as $\Delta_k \to 0$ causes no problem, because the corresponding transfer matrix would give $r_1 \to -1$, i.e., the photon is completely reflected by the first atom.

From Eq. (10) and standard calculations in one-dimensional photonic crystal problems [11], we have

$$\mathbf{M}^j = \frac{1}{\sin\beta}[\mathbf{M}\sin j\beta - \mathbf{I}\sin(j-1)\beta]. \tag{12}$$

Then t_j and r_j are obtained in a closed form:

$$t_j = \frac{1}{s_1}\left\{ \frac{e^{ikL}s_j[e^{ikL}(-i\Omega + \Delta_k)s_{N-1} - \Delta_k s_N]}{e^{ikL}\Delta_k s_{N-1} - (i\Omega + \Delta_k)s_N} - s_{j-1} \right\}, \tag{13}$$

$$r_{j+1} = i\frac{\Omega s_{N-j}}{e^{ikL}\Delta_k s_{N-1} - (i\Omega + \Delta_k)s_N}, \tag{14}$$

$$e_k^{(j)} = \left(\frac{v_g\Omega}{J}\right)\frac{s_{N-j+1} - e^{i\theta}s_{N-j}}{(\Delta_k + i\Omega)s_N - e^{i\theta}\Delta_k s_{N-1}}, \tag{15}$$

where $s_j \equiv \sin j\beta$ is defined. Hence we have solved the eigenvectors explicitly of the Hamiltonian.

For further simplifications, we introduce $\theta \equiv \omega_A L/v_g$. By the fact that $\omega_A \gg \Delta_k$, then $kL = (\omega_A + \Delta_k)L/v_g \approx \theta$. Such an

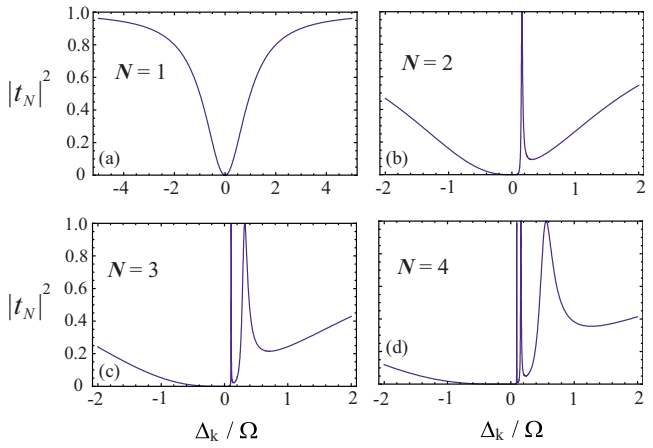

FIG. 2. (Color online) Transmission $|t_N|^2$ for $N=1,2,3,4$; $\theta=0.95\pi$.

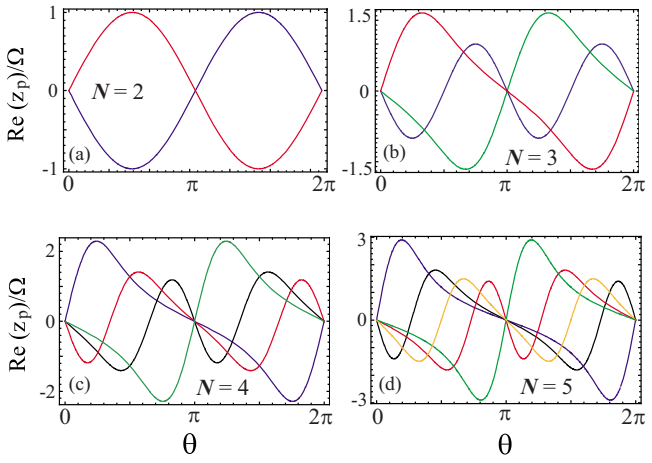

FIG. 3. (Color online) Real part of the poles of $e_k^{(j)}$ as a function of θ for $N=2,3,4,5$.

approximation requires $\Delta_k L/v_g \ll 1$, and this is true for L up to several hundred resonance wavelengths, given that the bandwidth Δ_k of interest here is of the order of the natural linewidth of the atoms.

III. TRANSMISSION SPECTRUM OF A SINGLE PHOTON

Now we consider a single photon incident from the far left with the wave vector k; the scattering of the photon in the long time limit is characterized by the transmission coefficient t_N and reflection coefficient r_1. In the case of the single-atom problem, the transmission is very small near the resonance frequency $\omega_k \approx \omega_A$, and is exactly zero at the resonance [Fig. 2(a)]. However, for $N>1$ atoms, we discover that there are $N-1$ peaks where $|t_N|=1$ near the resonance frequency. This is shown in Fig. 2 for cases with few atoms.

For a given number of atoms N and θ, the peak positions are determined by the condition $r_1=0$, or $\sin N\beta=0$. Together with Eq. (11), the $|t_N|=1$ transmission peaks occur at $\beta=q\pi/N$ with $q=1,2,\ldots,N-1$. This corresponds to the detuned frequencies $\Delta_k=\Delta_q$ given by

$$\Delta_q = \frac{\Omega \sin\theta}{\cos(q\pi/N) - \cos\theta}. \tag{16}$$

Note that the transmission peaks are sensitive to the atomic spacing and hence the transmission spectrum of a single photon can be exploited to probe the atomic separation.

In the case $\theta \approx m\pi$ (m=integer), the atomic spacing is close to an integer multiple of half of the resonance wavelength; the transmission peak at $\Delta_1 \approx 0$ has the narrowest width. By treating $\theta - m\pi$ as a small parameter, we find that the line shape of the narrowest peak, $|t_N|^2$, can be approximated by a Lorenzian function $w_1^2/[(\Delta_k-\Delta_1)^2+w_1^2]$, and the width w_1 is given by

$$w_1 = \frac{\Omega \sin^2(\pi/N)}{N[1+\cos^2(\pi/N)]^3}(\theta - m\pi)^2. \tag{17}$$

This is a result obtained by making a Taylor expansion of t_N around the Δ_1. We see that w_1 can be much smaller than the linewidth of single atom emission Ω in this one-dimensional environment. In particular w_1 decreases as N^{-3} when $N \gg 1$.

We remark that a band gap with zero transmission can be formed in the limit $N \to \infty$ [6]. The band gap is the range where β, defined by Eq. (11), becomes imaginary. This is understood from the fact that β is the Bloch phase [11]. For the system considered here, we obtain the band gap as

$$-\tan(\theta/2) \leqslant \Delta_k/\Omega \leqslant \cot(\theta/2) \tag{18}$$

for $\theta < \pi$, and the inequality signs in Eq. (18) are reversed if $\pi < \theta < 2\pi$.

IV. DYNAMICS OF SPONTANEOUS EMISSION

In this section we turn to the spontaneous emission problem in which the jth atom is initially prepared in the excited state while all others remain at the ground state, and the field is in the vacuum state. By projecting this state onto the eigenvectors solved in Sec. II, the survival probability $P(t)$ that the system remains in its original state is given by

$$P(t) = \left| \frac{1}{2\pi} \int dk |e_k^{(j)}|^2 e^{-i\Delta_k t} \right|^2, \tag{19}$$

where $e_k^{(j)}$ is given by Eq. (15). By inspection, $e_k^{(j)}$ consists of poles z_p on the lower half complex frequency plane. Therefore the integral in $P(t)$ can be evaluated analytically once z_p's are determined. We indicate that the poles of $e_k^{(j)}$ are functions of atomic spacing and particle number. In addition, each $e_k^{(j)}$ has at most N poles and the actual number of poles depends on which particular atom in the chain is concerned.

For $N<10$ atoms, analytic expressions of z_p can be obtained by using MATHEMATICA. In Figs. 3 and 4, we illustrate the real and imaginary parts of z_p as functions of θ for N up to 5. It is interesting that they show an oscillatory pattern with the interatomic separation. Quite generally, z_p are distinct except at certain interatomic distances such as when $\theta = m\pi$. Therefore z_p are simple poles in general, and hence the real part and imaginary part of z_p correspond to a characteristic frequency shift and decay rate, respectively. We can see in Figs. 3 and 4 that the behavior of z_p at $\theta = m\pi$ is particu-

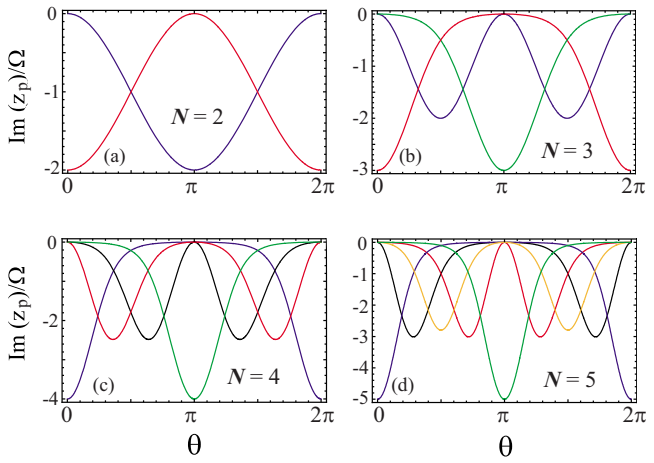

FIG. 4. (Color online) Imaginary part of the poles of $e_k^{(j)}$ as a function of θ for $N=2,3,4,5$.

larly transparent because the only nonzero pole is a simple pole at $z_p=-iN\Omega$. This is a fact that can be obtained analytically in Eq. (15) for a general atom number N, and it suggests that one of the characteristic decay rates is $N\Omega$, which is an enhancement effect due to collective dipoles, when atomic spacing is an integer multiple of half of the resonance wavelength.

For larger N, analytic expressions for the poles are difficult to obtain since higher order polynomial equations have to be solved. Numerical treatment is therefore employed in our calculation for $N \gtrsim 10$. We are interested in the question of how the number of atoms affect the slowest decay rate, which is characterized by the smallest imaginary part of z_p as a function of N. The result is given in Fig. 5 where we find that the smallest characteristic decay rate decreases as $1/N^3$ for a wide range of N. Such an inverse-cubic dependence has been observed for various values of interatomic separation ($\theta=0.2\pi, 0.55\pi, 0.85\pi, 0.95\pi$) with different proportionality constants. The proportionality A is smallest at $\theta=m\pi$ and largest at $\theta=(m+1/2)\pi$.

We remark that in the case $\theta \approx m\pi$, we can derive analytically the $1/N^3$ dependence of the pole. This is achieved by

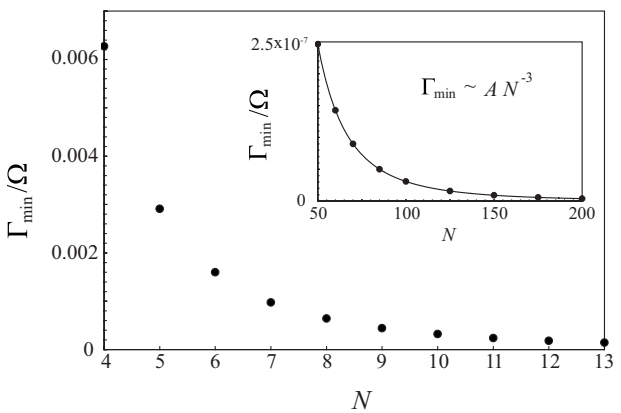

FIG. 5. The smallest imaginary part (in magnitude), $\Gamma_{\min} \equiv \min\{|\mathrm{Im}(z_p)|\}$, among all the poles of $e_k^{(j)}$ as a function of N for the case $\theta=0.95\pi$. The numerical value of $A=0.03\ \Omega$ agrees with analytical results given in text.

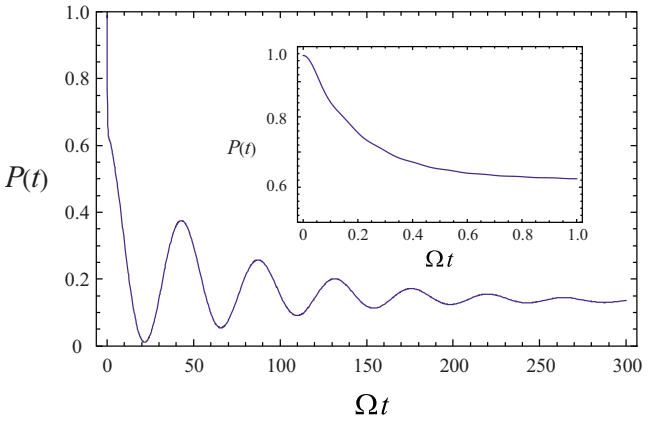

FIG. 6. (Color online) Time dependence of the excited state probability of the middle atom, which is initially excited, in an $N=5$ system with $\theta=0.95\pi$. In this case $z_1/\Omega=-1.213-4.749i$, $z_2/\Omega=0.229-0.0139i$, $z_3/\Omega=0.087-0.000\,29i$. The inset shows a fast decay at short times.

observing that the first transmission peak $\beta=\pi/N$, which is the narrowest one discussed in the previous section, approximately determines the position of the slowest pole. By making a Taylor expansion of the denominator of $e_k^{(j)}$ in power series of Δ_k around the first transmission peak, we can solve the pole, and the result is that the imaginary part is $\pi^2(\theta-m\pi)^2\Omega/8N^3$ as the leading term. This agrees with our numerical calculations in Fig. 5.

As an illustration of the quantum dynamics, we show in Fig. 6 an $N=5$ case, with the middle atom (atom 3) initially excited. In this case, $e_k^{(3)}$ has three poles whose imaginary parts have distinct orders of magnitude as indicated in the caption of the figure. We see that the decay of the atom has several characteristic features. The rapid initial drop at short times corresponds to the decay described by the pole z_1 (inset of Fig. 6). Such a rapid drop is approximately $N=5$ times of the spontaneous decay rate of a single excited atom in the same one-dimensional environment. Between $\Omega t=20$ and 250, the excited state probability of the central atom exhibits oscillations. This is the beating due to interference between the second and third poles, in which the oscillation frequency is the difference of their real parts. At $\Omega t \approx 300$, the oscillations die out and there is about 15% probability of the atom remaining in the excited states. The decay at longer times $\Omega t > 300$ is mainly exponential with a decay time governed by $1/|\mathrm{Im}\, z_3|$, which is associated with a long-lived collective atomic state.

V. CONCLUSION

In conclusion, we have investigated the interaction between a single photon and a chain of two-level atoms in a one-dimensional waveguide. A set of eigenvectors of the system in the single-photon subspace is obtained analytically, which generalizes the work by Shen and Fan [6] to an arbitrary number of atoms. One of the main features in such a multiatom system is the existence of narrow transmission peaks near the atomic resonance frequency, which are absent in the single-atom system. In addition, we have indicated

how the peak positions are sensitive to interatomic separation. Our study also addresses the spontaneous emission of a photon from an excited atom in the atomic chain. By analyzing the poles associated with the eigenvectors, we find that there is a transient period during which the atom could exhibit a rapid initial decay and oscillations, and the long time behavior is a slow exponential decay with the rate proportional to N^{-3} in the large N limit. The long time state corresponds to a singly excited collective atomic state in which the energy can be stored among atoms inside the chain for a considerably long time.

Finally, we comment on two basic assumptions of this work. The first is the linear dispersion relation of photons, which is applicable to situations when the field frequencies are far away from the cutoff frequency of the waveguide. The violation of such a condition would mean that $u_{k,R}(x)$ and $u_{k,L}(x)$ in Eq. (2) obey a higher order differential equation, instead of a first order one. The general solution is still an open problem and it is important because the smaller group velocity v_g near the cutoff frequency would yield a stronger atom-field coupling. The second assumption of our model is that the radiation emitted by atoms is entirely captured by the waveguide modes. It should be expected that quantum noise in nonwaveguide modes would wash out some of the interference effects found in this paper. However, the recent progress of the photonic crystal waveguide could be a step toward the goal of minimizing the radiation loss to the surrounding, as some theoretical studies have indicated that the high spontaneous emission β factor can be achieved for systems with a quantum dot doped inside a photonic crystal waveguide [12]. In the future we hope to address the above group dispersion and decoherence issues.

ACKNOWLEDGMENT

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