Scattering of scalar light fields from collections of particles

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Using the angular spectrum representation of fields and the first Born approximation we develop a theory of scattering of scalar waves with any spectral composition and any correlation properties from collections of particles which have either deterministic or random distributions of the index of refraction and locations. An example illustrating the far-field intensity and the far-field spectral degree of coherence produced on scattering of a model field from collections of several particles with Gaussian potentials is considered.

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I. INTRODUCTION

Scattering of electromagnetic waves from collections of particles is of considerable interest in many areas such as medical diagnostics and imaging, remote sensing in the atmosphere and ocean, to name a few. Most techniques for analyzing interaction of waves with such media are based on the theory of radiative transfer which does not take into account the interference effects and therefore are not suitable for calculations in all scattering regimes. However, in some situations, for instance, when the dimensions of particles are on the same order as the typical transverse spatial correlation length of the incident field, interference effects should be taken into account.

On a rigorous basis scattering of stochastic fields from random media was first treated in Refs. [1–3] (see also Ref. [4]). Inverse problem, i.e., the problem of finding the statistical properties of scatterers from the statistical properties of the scattered field was addressed in Ref. [5] (see also Ref. [6]). Analysis of scattering from collections of particles was carried out in Refs. [7,8] but was limited to incident polychromatic plane waves and to spectral properties of scattered light. Later new techniques such as variable coherence tomography [9] and variable coherence microscopy [10] were proposed and tested experimentally. Both methods use correlation properties of an incident field to sense the statistical properties of a particulate medium.

In a recent publication [11] a theory was developed that makes it possible to study scattering of scalar fields of arbitrary spectral composition and coherence properties from deterministic and random continuous scatterers. The combination of scattering matrix theory and the angular spectrum decomposition of fields employed in Ref. [11] made the treatment of complex phenomenon of scattering complete and simple. In this paper we will extend the theory in Ref. [11] to scattering of arbitrary scalar fields from collections of discrete particles of both deterministic and random nature.

II. REVIEW OF THE SCATTERING THEORY

Following Ref. [11] (see also Refs. [4,12]), we will first review the general theory of scattering of scalar fields with arbitrary spectral and coherence properties from static deterministic or random media. Let us first consider a monochromatic scalar field at a point with position vector $\mathbf{r}$ and frequency ω, $U^{(i)}(\mathbf{r};\omega)e^{-i\omega t}$, propagating into the half-space $z>0$. Its space-dependent part can be represented in the form of the angular spectrum of plane waves

$$U^{(i)}(\mathbf{r};\omega)=\int\int a^{(i)}(\mathbf{u};\omega)e^{ik(\mathbf{u}_\perp\cdot\mathbf{r}+u_z z)}d^2\mathbf{u}_\perp, \tag{1}$$

where integration extends over the (u_x,u_y) plane. Here $k=\omega/c$ is the wave number, c being the speed of light in vacuum; $\mathbf{u}=(u_x,u_y,u_z)$ is a unit vector, $\mathbf{u}_\perp=(u_x,u_y,0)$, and

$$u_z=\sqrt{1-|\mathbf{u}_\perp|^2},\quad \text{when } |\mathbf{u}_\perp|\leq 1 \quad \text{(homogeneous waves)}, \tag{2a}$$

$$=i\sqrt{|\mathbf{u}_\perp|^2-1},\quad \text{when } |\mathbf{u}_\perp|>1 \quad \text{(evanescent waves)}. \tag{2b}$$

It was shown in Ref. [11] that the total field produced on scattering, being the sum of the incident and the scattered fields can be calculated by the formula

$$U^{(t)}(\mathbf{r};\omega)=\int\int a^{(t)}(\mathbf{u};\omega)e^{ik(\mathbf{u}_\perp\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}d^2\mathbf{u}'_\perp, \tag{3}$$

where the scattering amplitude of the total scattered field is given by the expression

$$a^{(t)}(\mathbf{u},\omega)=\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)a^{(i)}(\mathbf{u}',\omega), \tag{4}$$

$\mathbb{S}$ being the spectral scattering matrix. Positive or negative sign in Eq. (3) must be chosen for forward-scattering and back-scattering portions of the scattered field, respectively. Integration in Eq. (3) is performed only over the homogeneous part of the angular spectrum.

If the field incident on the scatterer is stochastic, wide-sense statistically stationary, then its second-order, two-point spatial correlation properties (in frequency domain) can be characterized by the cross-spectral density function (see Ref. [4] Sec. 4.1)

$$W^{(i)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle U^{(i)*}(\mathbf{r}_1;\omega)U^{(i)}(\mathbf{r}_2;\omega)\rangle \tag{5}$$

or by its angular correlation function

$$\mathcal{A}^{(i)}(\mathbf{u}_1,\mathbf{u}_2;\omega)=\langle a^{(i)*}(\mathbf{u}_1;\omega)a^{(i)}(\mathbf{u}_2;\omega)\rangle, \tag{6}$$

which can be shown to be the four-dimensional Fourier transform of $W^{(i)}$ [12]. The cross-spectral density matrix of the total scattered field then becomes

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle U^{(t)*}(\mathbf{r}_1;\omega)U^{(t)}(\mathbf{r}_2;\omega)\rangle$$
$$=\int\int\int\int \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)$$
$$\times e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_1 d^2\mathbf{u}_2 d^2\mathbf{u}_{1\perp}' d^2\mathbf{u}_{2\perp}', \quad (7)$$

where

$$\mathbb{M}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega)=\mathbb{S}^*(\mathbf{u}_1,\mathbf{u}_1';\omega)\mathbb{S}(\mathbf{u}_2,\mathbf{u}_2';\omega) \quad (8)$$

is the pair scattering matrix [11], integrations in Eq. (7) are being taken only over homogeneous waves.

Under the first Born approximation the scattering matrix may be expressed in terms of the scattering potential in a simple manner. Let $n(\mathbf{r},\omega)$ be the refractive index distribution throughout the scatterer. The scattering potential $F(\mathbf{r},\omega)$ is then given by the formula (see Ref. [12], Sec. 13.1)

$$F(\mathbf{r},\omega)=\frac{k^2}{4\pi}[n^2(\mathbf{r},\omega)-1], \quad \mathbf{r}\epsilon D,$$
$$=0, \quad \text{otherwise}, \quad (9)$$

where D denotes the domain occupied by the scatterer. Then, the pair-scattering matrix of the total field takes the form

$$\mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)=\tilde{F}^*[k(\mathbf{u}_1-\mathbf{u}_1'),\omega]\tilde{F}[k(\mathbf{u}_2-\mathbf{u}_2'),\omega], \quad (10)$$

where tilde denotes the two-dimensional Fourier transform.

In the case when the scatterer is random the expression (8) generalizes to

$$\mathbb{M}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega)=\langle\mathbb{S}^*(\mathbf{u}_1,\mathbf{u}_1';\omega)\mathbb{S}(\mathbf{u}_2,\mathbf{u}_2';\omega)\rangle_{rm}, \quad (11)$$

where $\langle\cdots\rangle_{rm}$ denotes the average taken over the ensemble of realizations of the scattering medium.

In particular, in the far field of the scatterer the total field $U^{(t)}$ and the cross-spectral density function $W^{(t)}$ along directions specified by unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ reduce to the forms [11]

$$U^{(t)}(r\mathbf{u},\omega)\approx \pm\frac{2\pi i u_z}{k}\frac{e^{ikr}}{r}\int \mathbb{S}(\mathbf{u},\mathbf{u}',\omega)a^{(i)}(\mathbf{u}',\omega)d^2\mathbf{u}_\perp', \quad (12)$$

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)\approx \pm\frac{4\pi^2}{k^2r^2}u_{z1}u_{z2}\int\int \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)$$
$$\times\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)d^2\mathbf{u}_1'd^2\mathbf{u}_2'. \quad (13)$$

With the help of the cross-spectral density function $W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)$ [see Eq. (7)] we may at once determine the spectrum $S^{(t)}(\mathbf{r};\omega)$ and the spectral degree of coherence $\mu^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)$ of the total field using the formulas (see Ref. [13], Sec. 4.3.2):

$$S^{(t)}(\mathbf{r};\omega)=W^{(t)}(\mathbf{r},\mathbf{r};\omega), \quad (14)$$

$$\mu^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\frac{W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S^{(t)}(\mathbf{r}_1;\omega)}\sqrt{S^{(t)}(\mathbf{r}_2;\omega)}}. \quad (15)$$

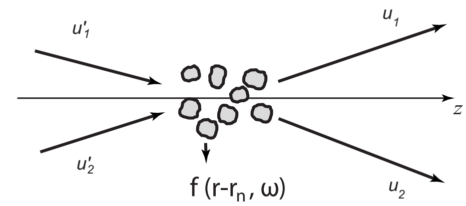

FIG. 1. Notation relating to the scattering of two correlated plane waves from a collection of particles.

III. SCATTERING FROM COLLECTIONS OF PARTICLES

Suppose that light is being scattered from a static collection of particles of L different types which occupy domain D (see Fig. 1). To characterize the response of such a collection to the incident light we will use the discrete-particle model, in which the scattering potential $F(\mathbf{r},\omega)$ of the collection can be represented by a finite sum of potentials of individual scatterers, i.e.,

$$F(\mathbf{r},\omega)=\sum_{l=1}^{L}\sum_{m=1}^{M_l} f_l(\mathbf{r}-\mathbf{r}_m,\omega), \quad (16)$$

where $\mathbf{r}_m$ is the location of a scattering center, f_l is the scattering potential of the scatterer of type l, M_l is the number of particles of type l.

In the case when the collection is static but random one can characterize its response with the help of the correlation function of the scattering potential which reduces for the particulate medium to the form (Ref. [4], Sec. 6.3.1)

$$C_F(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle F^*(\mathbf{r}_1,\omega)F(\mathbf{r}_2,\omega)\rangle_{rm}$$
$$=\sum_{l=1}^{L}\sum_{j=1}^{L}\sum_{m=1}^{M_l}\sum_{n=1}^{N_j}\langle f_l^*(\mathbf{r}-\mathbf{r}_m,\omega)f_j(\mathbf{r}-\mathbf{r}_n,\omega)\rangle_{rm}. \quad (17)$$

If all particles in the collection are identical the summations over indexes l and j in the expressions (16) and (17) may be omitted and they become

$$F(\mathbf{r},\omega)=\sum_{n=1}^{N} f(\mathbf{r}-\mathbf{r}_n,\omega), \quad (18)$$

$$C_F(\mathbf{r}_1,\mathbf{r}_2,\omega)=\sum_{m=1}^{M}\sum_{n=1}^{N}\langle f^*(\mathbf{r}_1-\mathbf{r}_m,\omega)f(\mathbf{r}_2-\mathbf{r}_n,\omega)\rangle_{rm}. \quad (19)$$

In scattering from collections of particles by scalar wavefields several cases should be differentiated. The incident field can be either deterministic or stochastic and, in addition, particles can form deterministic collection (Sec. III A) or random collection (Sec. III B).

A. Deterministic collection of scatterers

We begin by considering the simplest situation when the incident field, say $U^{(i)}(\mathbf{r},\omega)$, is deterministic and it is scattered from a deterministic collection of particles, i.e., the

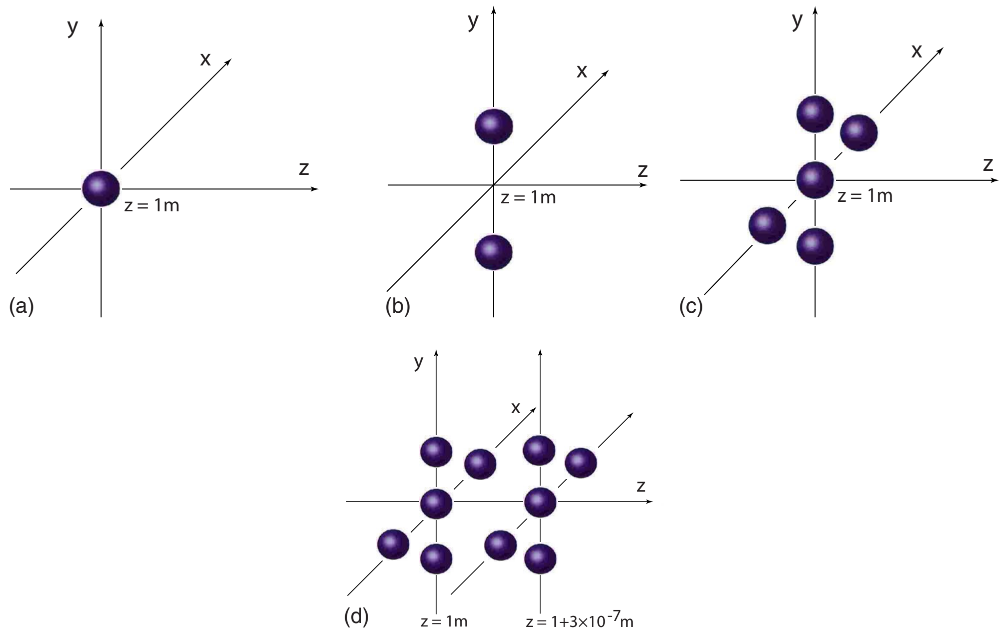

FIG. 2. (Color online) The coordinates of the particles (a) $(0,0,1m)$; (b) $(0,3\times10^{-7}m,1m)$, $(0,-3\times10^{-7}m,1m)$; (c) $(0,0,1m)$, $(0,3\times10^{-7}m,1m)$, $(0,-3\times10^{-7}m,1m)$, $(3\times10^{-7}m,0,1m)$, $(-3\times10^{-7}m,0,1m)$; (d) $(0,0,1m)$, $(0,3\times10^{-7}m,1m)$, $(0,-3\times10^{-7}m,1m)$, $(3\times10^{-7}m,0,1m)$, $(-3\times10^{-7}m,0,1m)$, $(0,0,1+3\times10^{-7}m)$, $(0,3\times10^{-7}m,1+3\times10^{-7}m)$, $(0,-3\times10^{-7}m,1+3\times10^{-7}m)$, $(3\times10^{-7}m,0,1+3\times10^{-7}m)$, $(-3\times10^{-7}m,0,1+3\times10^{-7}m)$.

distribution of the refractive index within particles and their locations are deterministic. In this case the scattering potential of the system of n (identical) particles is given by Eq. (18).

In this case, within the validity of the first Born approximation the scattering matrix takes the form [Ref. [11], Eq. (35)]

$$\mathrm{S}(\mathbf{u},\mathbf{u}',\omega)=\mathcal{F}[F(\mathbf{r},\omega)]=\sum_{n=1}^{N}\mathcal{F}[f(\mathbf{r}-\mathbf{r}_n,\omega)], \quad (20)$$

where $\mathcal{F}$ denotes three-dimensional Fourier transform.

On performing Fourier transforms of potentials of individual particles with the help of variables $\mathbf{R}_n=\mathbf{r}-\mathbf{r}_n$ $(n=1,\ldots,N)$ we find that

$$\mathrm{S}(\mathbf{u},\mathbf{u}',\omega)=\sum_{n=1}^{N}e^{-i\mathbf{r}_n\cdot\mathbf{K}}\tilde{f}[\mathbf{K},\omega], \quad (21)$$

where $\mathbf{K}=k(\mathbf{u}-\mathbf{u}')$ is the so-called momentum-transfer vector.

The total scattered field produced on scattering can be found with the help of Eqs. (3), (4), and (21) to be

$$U^{(t)}(\mathbf{r},\omega)=\sum_{n=1}^{N}\int\int e^{-i\mathbf{r}_n\cdot\mathbf{K}}\tilde{f}[\mathbf{K},\omega]\times a^{(i)}(\mathbf{u}',\omega)e^{-ik(\mathbf{u}_\perp\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}d^2\mathbf{u}'_\perp. \quad (22)$$

In the far-zone of the scatterer the total field reduces to the expression involving single integral, i.e.,

$$U^{(t)}(r\mathbf{u},\omega)=\pm\frac{2\pi i u_z}{k}\frac{e^{ikr}}{r}\sum_{n=1}^{N}e^{-ik\mathbf{r}_n\cdot\mathbf{u}}\int e^{ik\mathbf{r}_n\cdot\mathbf{u}'}\tilde{f}[\mathbf{K},\omega]\times a^{(i)}(\mathbf{u}',\omega)d^2\mathbf{u}'_\perp. \quad (23)$$

We will now consider the case when the field incident on the system of particles is stochastic and is characterized by the cross-spectral density function $W^{(i)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$. The cross-spectral density function of the total scattered field is then given by expression (7). Noting that [see Eq. (21)]

$$\mathrm{S}^*(\mathbf{u}_1,\mathbf{u}'_1,\omega)\mathrm{S}(\mathbf{u}_2,\mathbf{u}'_2,\omega)=\sum_{m=1}^{M}\sum_{n=1}^{N}e^{-i[\mathbf{K}_2\cdot\mathbf{r}_n-\mathbf{K}_1\cdot\mathbf{r}_m]}\tilde{f}^*(-\mathbf{K}_1,\omega)\tilde{f}(\mathbf{K}_2,\omega), \quad (24)$$

where $\mathbf{K}_\alpha=k(\mathbf{u}_\alpha-\mathbf{u}'_\alpha)$ $(\alpha=1,2)$, we find, on substituting from Eq. (24) into Eq. (7) that

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\sum_{m=1}^{M}\sum_{n=1}^{N}\int\int\int\int e^{-i[\mathbf{K}_2\cdot\mathbf{r}_n-\mathbf{K}_1\cdot\mathbf{r}_m]}\times\tilde{f}^*(-\mathbf{K}_1,\omega)\tilde{f}(\mathbf{K}_2,\omega)\mathcal{A}^{(i)}(\mathbf{u}'_1,\mathbf{u}'_2,\omega)\times e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_1d^2\mathbf{u}_2d^2\mathbf{u}'_{1\perp}d^2\mathbf{u}'_{2\perp}. \quad (25)$$

In the far zone of the scattering volume the last equation reduces to the formula

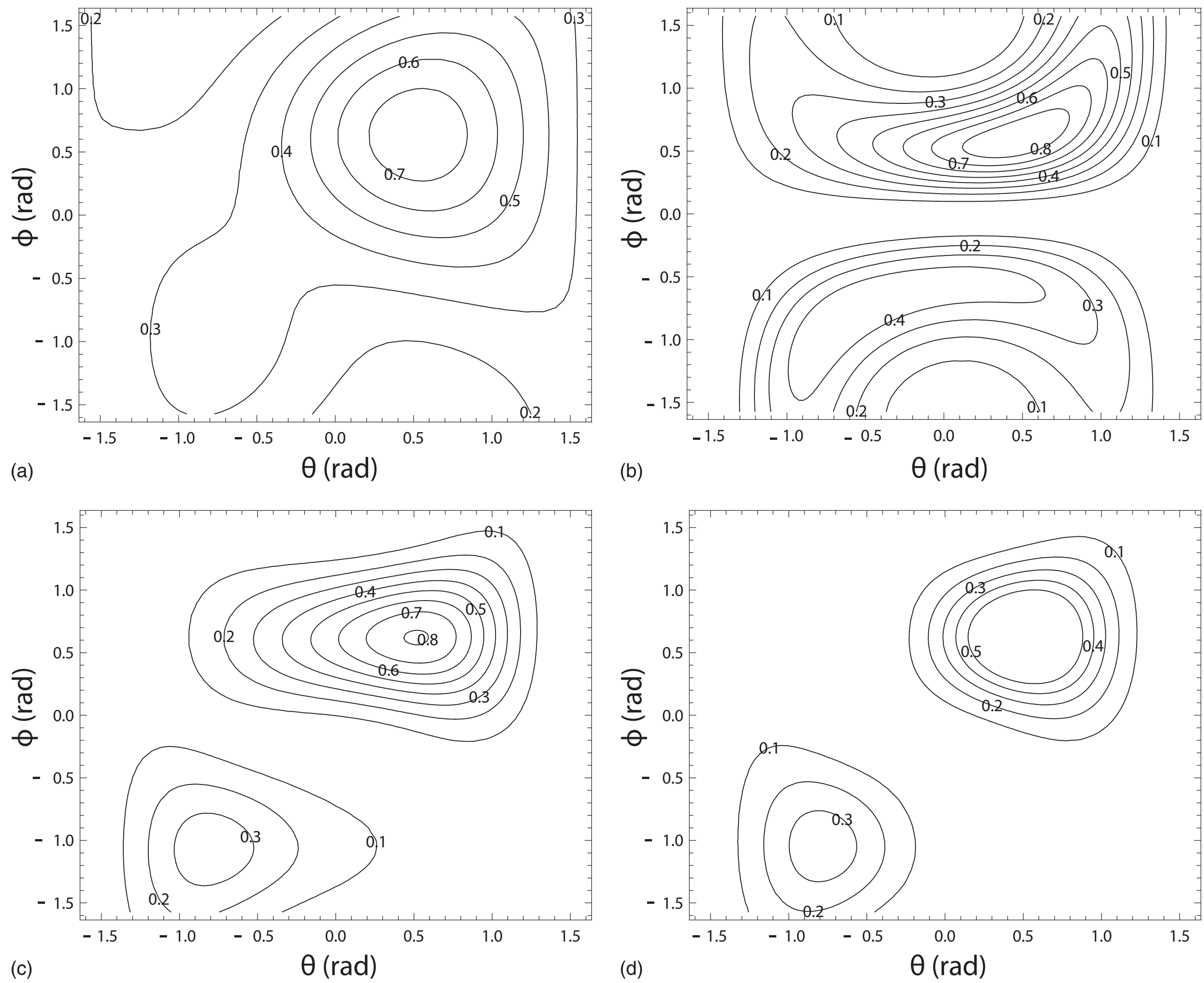

FIG. 3. Contours of the spectral density of the far field produced by scattering of two correlated plane waves on particles with Gaussian potential. The parameters σ and Δ are kept fixed: $k\sigma=1$, $k\Delta=1$, (a) one particle; (b) two particles; (c) five particles; (d) ten particles.

$$W^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2, \omega) = \pm \frac{4\pi^2}{k^2 r^2} u_{1z} u_{2z} \sum_{m=1}^{M} \sum_{n=1}^{N} \int \int e^{-i[\mathbf{K}_2 \cdot \mathbf{r}_n - \mathbf{K}_1 \cdot \mathbf{r}_m]} \times \tilde{f}^*(-\mathbf{K}_1, \omega) \tilde{f}(\mathbf{K}_2, \omega) \times \mathcal{A}^{(i)}(\mathbf{u}_1', \mathbf{u}_2', \omega) d^2\mathbf{u}_{1\perp}' d^2\mathbf{u}_{2\perp}'. \tag{26}$$

B. Random collections of scatterers

Let us now assume that the incident field $U^{(i)}(\mathbf{r},\omega)$ is deterministic but it is scattered from random collection of identical particles. In general, second-order statistical properties of such a collection at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ may be characterized by a correlation function (19). In the case when the scattering medium is random it can be shown [see Ref. [11], Eq. (43)] that within the validity of the first Born approximation the pair-scattering matrix is related to the correlation function [see Eq. (17)] of the scattering potential by the formula

$$\mathbb{M}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1', \mathbf{u}_2') = \tilde{C}_F(\mathbf{K}_1, \mathbf{K}_2, \omega), \tag{27}$$

where, as before, $\mathbf{K}_\alpha = k(\mathbf{u}_\alpha - \mathbf{u}_\alpha')$, $(\alpha=1,2)$ are the momentum-transfer vectors and tilde stands for six-dimensional Fourier transform.

By random collection of particles we mean the collection whose elements have deterministic potential but their positions are distributed randomly in a given volume. Under this assumption, on substituting from Eq. (19) into Eq. (27) and evaluating the Fourier transforms of individual particles in the collection with the help of the variables $\mathbf{R}_{2n} = \mathbf{r}_2 - \mathbf{r}_n$ and $\mathbf{R}_{1m} = \mathbf{r}_1 - \mathbf{r}_m$, we find that

$$\mathbb{M}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1', \mathbf{u}_2') = \tilde{C}_f(-\mathbf{K}_1, \mathbf{K}_2, \omega) \mathbb{Q}(\mathbf{K}_1, \mathbf{K}_2). \tag{28}$$

Here $\tilde{C}_f$ are given by the expressions

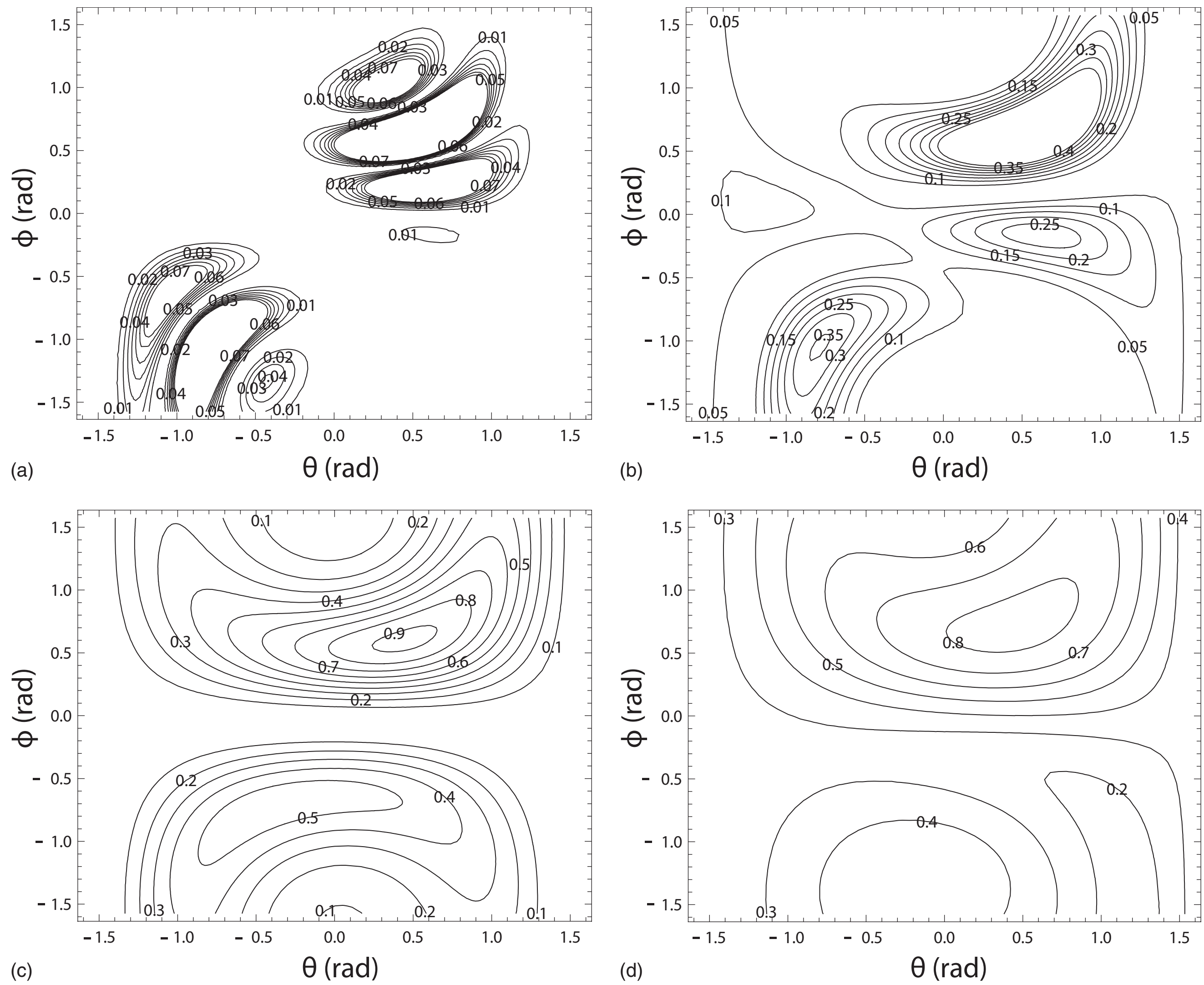

FIG. 4. Contours of the spectral density of the far field produced by scattering of two correlated plane waves on two particles with Gaussian potential. The parameter Δ is kept fixed: $k\Delta=1$, (a) $k\sigma=3$, (b) $k\sigma=1.5$, (c) $k\sigma=0.9$, (d) $k\sigma=0.5$.

$$\widetilde{C}_f(\mathbf{K}_1,\mathbf{K}_2,\omega)=\int\int\langle f^{*}(\mathbf{R}_{1m},\omega)f(\mathbf{R}_{2n},\omega)\rangle \times e^{-i[\mathbf{K}_2\cdot\mathbf{R}_{2n}-\mathbf{K}_1\cdot\mathbf{R}_{1m}]}d^3\mathbf{R}_{1m}d^3\mathbf{R}_{2n} \qquad (29)$$

and

$$\mathbb{Q}(\mathbf{K}_1,\mathbf{K}_2)=\left\langle\sum_m\sum_n e^{-i[\mathbf{K}_2\cdot\mathbf{r}_n-\mathbf{K}_1\cdot\mathbf{r}_m]}\right\rangle_{rm} \qquad (30)$$

where the later is call the "pair-structure factor." The pair structure factor may be considered as a two-point generalization of the structure factor [see Eq. (5) of Ref. [7]] frequently used for characterization of collection of particles. Unlike structure factor that describes the way in which the intensity of the incident beam is scattered in space, pair-structure factor provides information about spatial correlations of the scattered field.

The cross-spectral density function of the field scattered from the random collection of particles can then be found by the formula

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2,\omega) =\int\int\int\int\widetilde{C}_f(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbb{Q}(\mathbf{K}_1,\mathbf{K}_2)a^{(i)*}(\mathbf{u}_1',\omega) \times a^{(i)}(\mathbf{u}_2',\omega)e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_1d^2\mathbf{u}_2d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'. \qquad (31)$$

In the far zone of the scatterer the previous expression reduces to the form

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2,\omega) =\pm\frac{4\pi^2}{k^2r^2}u_{1z}u_{2z}\int\int\widetilde{C}_f(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbb{Q}(\mathbf{K}_1,\mathbf{K}_2) \times a^{(i)*}(\mathbf{u}_1',\omega)a^{(i)}(\mathbf{u}_2',\omega)d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'. \qquad (32)$$

In case when the incident field is random the last two formulas have the same form, except for the product $a^{(i)*}(\mathbf{u}_1',\omega)a^{(i)}(\mathbf{u}_2',\omega)$ of amplitudes of the incident light must

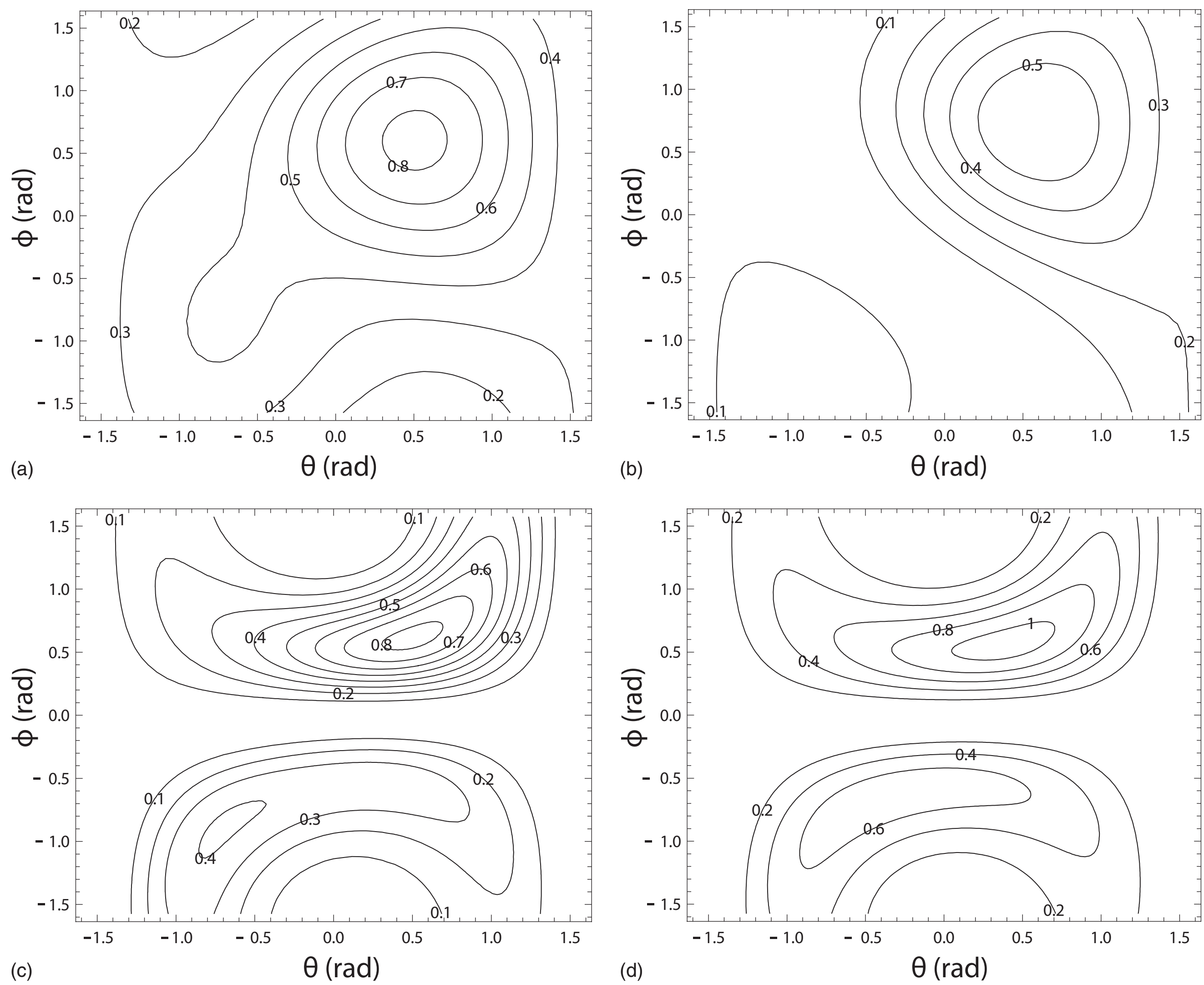

FIG. 5. Contours of the spectral density of the far field, produced by scattering of two correlated plane waves on two particles with Gaussian potential. The parameter σ is kept fixed: $k\sigma=1$, (a) one particle, $k\Delta=10$; (b) one particle, $k\Delta=0.1$; (c) two particles, $k\Delta=10$; (d) two particles, $k\Delta=0.1$.

be substituted by their correlation function $\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2',\omega)$. We then find that the cross-spectral density function becomes

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \int\int\int\int \widetilde{C}_f(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbb{Q}(\mathbf{K}_1,\mathbf{K}_2) \times \mathcal{A}^{(i)*}(\mathbf{u}_1',\mathbf{u}_2',\omega)e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_1 d^2\mathbf{u}_2 d^2\mathbf{u}_{1\perp}' d^2\mathbf{u}_{2\perp}'. \tag{33}$$

In the far zone of the scatterer the last formula reduces to the expression

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2,\omega) = \pm\frac{4\pi^2}{k^2r^2}u_{1z}u_{2z}\int\int \widetilde{C}_f(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbb{Q}(-\mathbf{K}_1,\mathbf{K}_2) \times \mathcal{A}^{(i)*}(\mathbf{u}_1',\mathbf{u}_2',\omega)d^2\mathbf{u}_{1\perp}' d^2\mathbf{u}_{2\perp}'. \tag{34}$$

IV. EXAMPLE OF TWO PARTIALLY CORRELATED POLYCHROMATIC PLANE WAVES SCATTERED BY A DETERMINISTIC MEDIUM

As an application to the theory discussed in Sec. III we consider the incident field $U^{(i)}$ which consists of two mutually correlated polychromatic plane waves propagating along directions $\mathbf{u}_1'$ and $\mathbf{u}_2'$, and scattered from a collection of spheres. The spectral amplitude $a^{(i)}(\mathbf{u}',\omega)$ of the incident field has the form

$$a^{(i)}(\mathbf{u}',\omega) = a^{(i)}(\mathbf{u}_1',\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1') + a^{(i)}(\mathbf{u}_2',\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2'), \tag{35}$$

$\delta^{(2)}(\mathbf{u})$ being the spherical Dirac delta function [14] (see also Ref. [11], where the same model incident field was initially introduced).

On substituting from Eq. (35) into Eq. (6), we find that the angular correlation function of the incident field takes the form

$$\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)=\mathfrak{a}(\mathbf{u}_1',\mathbf{u}_1';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1') + \mathfrak{a}(\mathbf{u}_2',\mathbf{u}_2';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2') + \mathfrak{a}(\mathbf{u}_1',\mathbf{u}_2';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2') + \mathfrak{a}(\mathbf{u}_2',\mathbf{u}_1';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1'), \tag{36}$$

where the angular correlation function $\mathfrak{a}(\mathbf{u}_1',\mathbf{u}_2';\omega)$ is assumed to be Gaussian, i.e.,

$$\mathfrak{a}(\mathbf{u}_p',\mathbf{u}_q';\omega)=\mathfrak{a}_{pq}e^{(-k^2\Delta^2/2)(\mathbf{u}_q'-\mathbf{u}_p')^2}\ (p,q=1,2), \tag{37}$$

where $\mathfrak{a}_{pq}$ and Δ may depend, in general, on frequency ω.

Suppose that the scatterers are spherical centered at points $\mathbf{r}_c=(x_n,y_n,z_n)$, having a three-dimensional (soft) Gaussian potential

$$f(\mathbf{r}_n;\omega)=B\exp\left[-\frac{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2}{2\sigma^2}\right]. \tag{38}$$

The variance σ^2 is taken to be independent of position but, in general, will depend on the frequency. On calculating three-dimensional Fourier transform of the expression (38) we find that

$$\tilde{F}(\mathbf{K};\omega)=B(2\pi)^{(3/2)}\sigma^3e^{-K^2\sigma^2/2}\sum_{n=1}^{N}e^{ix_nK_x}e^{iy_nK_y}e^{iz_nK_z}. \tag{39}$$

On substituting from Eq. (39) into Eq. (10) and setting $\mathbf{K}=k(\mathbf{u}-\mathbf{u}')$ we find that, within the accuracy of the first Born approximation, the pair scattering matrix has the form

$$\mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)=B^2(2\pi)^3\sigma^6\left(e^{(-k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2}\sum_{n=1}^{N}e^{-ikx_n(u_{1x}-u_{1x}')}e^{-iky_n(u_{1y}-u_{1y}')}e^{-ikz_n(u_{1z}-u_{1z}')}\right)\times\left(e^{(-k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_2')^2}\sum_{n=1}^{N}e^{ikx_n(u_{2x}-u_{2x}')}e^{iky_n(u_{2y}-u_{2y}')}e^{ikz_n(u_{2z}-u_{2z}')}\right). \tag{40}$$

When we substitute from Eqs. (36) and (40) [with the help of Eq. (37)] first into Eq. (13) and then into Eq. (14) we obtain the formula for the spectral density of the far field

$$S^{(t)}(r\mathbf{u};\omega)=\frac{B^2(2\pi)^5\sigma^6u_z^2}{k^2r^2}\Bigg\{e^{-k^2\sigma^2(\mathbf{u}-\mathbf{u}_1')^2}\left(\sum_{n=1}^{N}e^{-ikx_n(u_x-u_{1x}')}e^{-iky_n(u_y-u_{1y}')}e^{-ikz_n(u_z-u_{1z}')}\sum_{n=1}^{N}e^{ikx_n(u_x-u_{1x}')}e^{iky_n(u_y-u_{1y}')}e^{ikz_n(u_z-u_{1z}')}\right)\mathfrak{a}_{11}$$
$$+e^{-k^2\sigma^2(\mathbf{u}-\mathbf{u}_2')^2}\left(\sum_{n=1}^{N}e^{-ikx_n(u_x-u_{2x}')}e^{-iky_n(u_y-u_{2y}')}e^{-ikz_n(u_z-u_{2z}')}\sum_{n=1}^{N}e^{ikx_n(u_x-u_{2x}')}e^{iky_n(u_y-u_{2y}')}e^{ikz_n(u_z-u_{2z}')}\right)\mathfrak{a}_{22}$$
$$+2e^{(-k^2\sigma^2/2)(\mathbf{u}-\mathbf{u}_1')^2}e^{(-k^2\sigma^2/2)(\mathbf{u}-\mathbf{u}_2')^2}e^{(-k^2\Delta^2/2)(\mathbf{u}_2'-\mathbf{u}_1')^2}$$
$$\times\mathrm{Re}\left[\mathfrak{a}_{12}\left(\sum_{n=1}^{N}e^{-ikx_n(u_x-u_{1x}')}e^{-iky_n(u_y-u_{1y}')}e^{-ikz_n(u_z-u_{1z}')}\sum_{n=1}^{N}e^{ikx_n(u_x-u_{2x}')}e^{iky_n(u_y-u_{2y}')}e^{ikz_n(u_z-u_{2z}')}\right)\right]\Bigg\}. \tag{41}$$

We note that on integrating spherical δ functions in Eq. (13) [or, alternatively, Eq. (26) valid specifically for deterministic collections] we could get rid of the double integral in Eq. (41) and write the right-hand side as a linear combination.

In Fig. 2 we show distributions of one, two, five, and ten spheres that we used for all our numerical examples (Figs. 3–6). The parameters used for all of the numerical calculations are $\lambda=0.633\times10^{-6}$ m, $B=1$, $a_1=0.6e^{i\pi/7}$, $a_2=0.9e^{i\pi/6}$.

In Figs. 3–5 we illustrate the behavior of the spectral density of the far field calculated from Eq. (41) and normalized by the factor $B^2(2\pi)^5\sigma^6u_z^2/k^2r^2$. By these sets of contour-plots we show the dependence of spectral density distribution on various parameters of the incident field and of the particle system. Angles θ and ϕ are the polar and the azimuthal angles of the unit vector $\mathbf{u}$ in spherical coordinates, i.e., $u_x=\cos\theta\cos\phi$, $u_y=\cos\theta\sin\phi$, $u_z=\sin\theta$. Angles $\theta_{1,2}'$, $\phi_{1,2}'$ are the polar and azimuthal angles of vectors $\mathbf{u}_{1,2}'$ in spherical coordinates. For Figs. 3–5 we have chosen the directions of the incident field to be $\theta_1'=-\pi/4$, $\phi_1'=-\pi/3$, $\theta_2'=\pi/6$, $\phi_2'=\pi/5$.

In Fig. 3 we show the behavior of spectral density of far fields as a number of particles in the system grows from 1 to 10 (see Fig. 2), provided the size of the individual particles as well as correlation and directions of the incident plane waves are kept fixed. One can see that with the increase of the number of particles from 1 to 5 [see Figs. 3(a)–3(c)] the distribution becomes more localized around two centers corresponding to directions of the incident plane waves. However, for larger number of particles, e.g., 10 [Fig. 3(d)], the localization becomes less pronounced again.

In Fig. 4 the spectral density of the far field is shown for four different values of the scaled size of the particles $k\sigma$. As

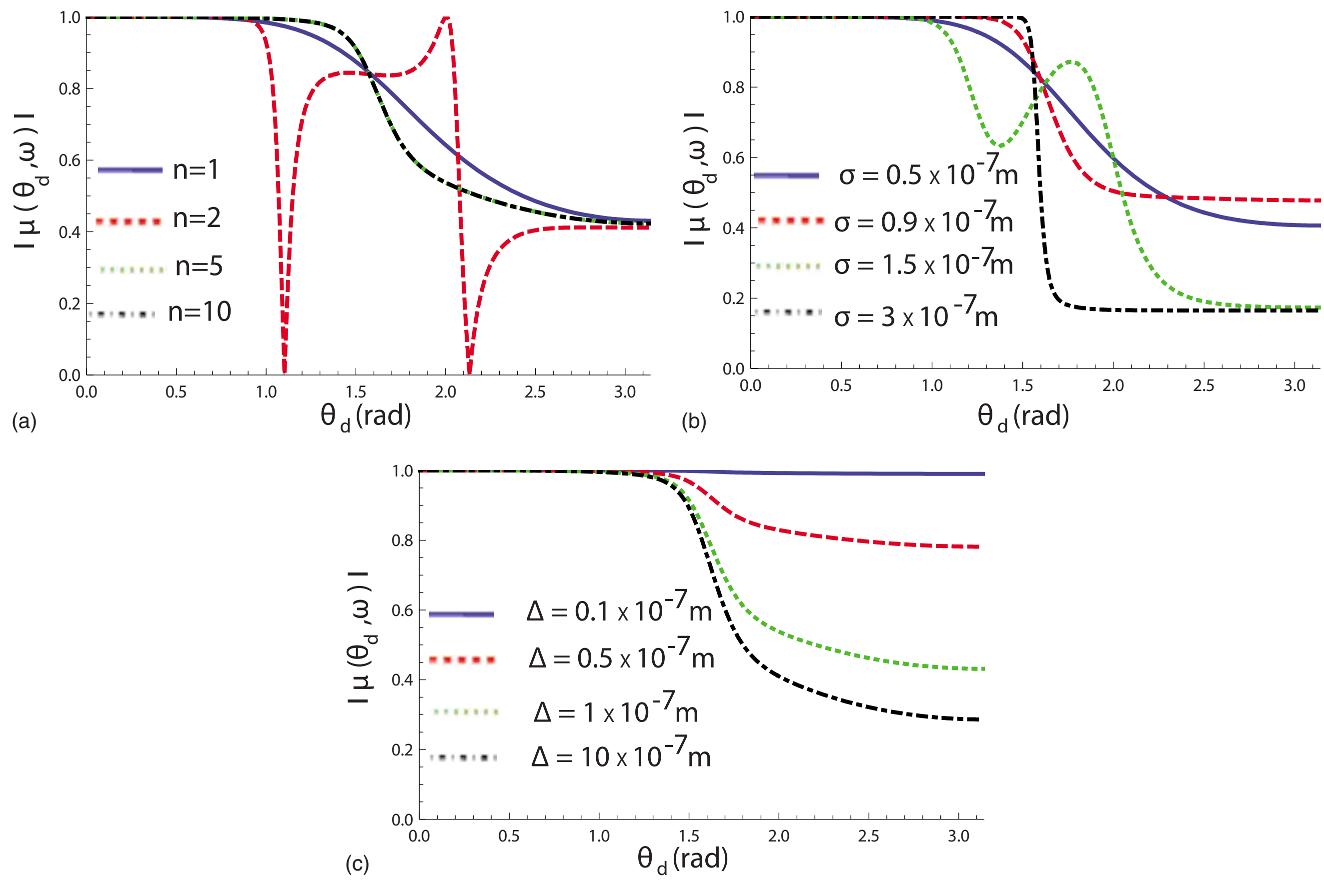

FIG. 6. (Color online) Modulus of the spectral degree of coherence of the far field as a function of $\theta_d = \theta_2 - \theta_1$, where $\theta_1 = 0$, produced by scattering of two correlated plane waves, (a) on different number of particles ($k\sigma = 1$, $k\Delta = 1$); (b) on five particles, for different σ values ($k\Delta = 1$); (c) on five particles, for different Δ values ($k\sigma = 1$).

the size decreases the interference effects disappear [Fig. 4(a)] and the distribution becomes more localized [compare peak values in Figs. 4(a) and 4(b)]. However, with further decrease of $k\sigma$ the localization becomes less noticeable [Figs. 4(c) and 4(d)].

In Fig. 5 we compare the changes in far-field spectral density with the scaled degree of correlation $k\Delta$ of the incident plane waves. Figures 5(a) and 5(b) refer to one-particle system. Figures 5(c) and 5(d) refer to two-particle system. One can readily see that for two-particle system the influence of $k\Delta$ is less evident [compare Figs. 5(c) and 5(d)]. On substituting from Eqs. (37) and (40) [with the help of Eq. (36)] first into Eq. (13) and then into Eq. (15) we now obtain the expression for the spectral degree of coherence of the far field produced on scattering of two correlated plane waves on the collection of spheres with Gaussian potentials

$$
\begin{aligned}
\mu^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2; \omega) = \frac{1}{\sqrt{S^{(t)}(r\mathbf{u}_1;\omega)}\sqrt{S^{(t)}(r\mathbf{u}_2;\omega)}} \Bigg\{ & e^{(-k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2}\left(\sum_{n=1}^{N} e^{-ikx_n(u_{1x}-u_{1x}')}e^{-iky_n(u_{1y}-u_{1y}')}e^{-ikz_n(u_{1z}-u_{1z}')}\right) \\
& \times e^{(-k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_1')^2}\left(\sum_{n=1}^{N} e^{ikx_n(u_{2x}-u_{1x}')}e^{iky_n(u_{2y}-u_{1y}')}e^{ikz_n(u_{2z}-u_{1z}')}\right)\mathfrak{a}_{11} \\
& + e^{(-k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_2')^2}\left(\sum_{n=1}^{N} e^{-ikx_n(u_{1x}-u_{2x}')}e^{-iky_n(u_{1y}-u_{2y}')}e^{-ikz_n(u_{1z}-u_{2z}')}\right) \\
& \times e^{(-k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_2')^2}\left(\sum_{n=1}^{N} e^{ikx_n(u_{2x}-u_{2x}')}e^{iky_n(u_{2y}-u_{2y}')}e^{ikz_n(u_{2z}-u_{2z}')}\right)\mathfrak{a}_{22} \\
& + \Bigg[e^{(-k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2}\left(\sum_{n=1}^{N} e^{-ikx_n(u_{1x}-u_{1x}')}e^{-iky_n(u_{1y}-u_{1y}')}e^{-ikz_n(u_{1z}-u_{1z}')}\right)
\end{aligned}
$$

$$\times e^{(-k^2\sigma^2/2)(\mathbf{u}_2 - \mathbf{u}_2')^2}\left(\sum_{n=1}^{N} e^{ikx_n(u_{2x}-u_{2x}')}e^{iky_n(u_{2y}-u_{2y}')}e^{ikz_n(u_{2z}-u_{2z}')}\right)\mathfrak{a}_{12}$$
$$+ e^{(-k^2\sigma^2/2)(\mathbf{u}_1 - \mathbf{u}_2')^2}\left(\sum_{n=1}^{N} e^{-ikx_n(u_{1x}-u_{2x}')}e^{-iky_n(u_{1y}-u_{2y}')}e^{-ikz_n(u_{1z}-u_{2z}')}\right)$$
$$\times e^{(-k^2\sigma^2/2)(\mathbf{u}_2 - \mathbf{u}_1')^2}\left(\sum_{n=1}^{N} e^{ikx_n(u_{2x}-u_{1x}')}e^{iky_n(u_{2y}-u_{1y}')}e^{ikz_n(u_{2z}-u_{1z}')}\right)\mathfrak{a}_{21}\Bigg] e^{(-k^2\Delta^2/2)(\mathbf{u}_1' - \mathbf{u}_2')^2}\Bigg\}. \quad (42)$$

We note here again that on integrating spherical δ functions in Eq. (13) (or, alternatively, Eq. (26) valid specifically for deterministic collections) we could get rid in Eq. (42) of the double integral and write the right-hand side as a linear combination.

Figure 6 shows the behavior of the modulus of the spectral degree of coherence $|\mu^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega)|$ of the far-field calculated from Eq. (42). We assume that the plane waves are incident on the collection of spheres along directions specified by polar angles $\phi_1'=\pi/2$, $\phi_2'=-\pi/2$ and azimuthal angles $\theta_1'=\theta_2'=0.3$ rad. The modulus of the degree of coherence of the scattered field was calculated as a function of the angle $\theta_d=\theta_2-\theta_1$, while the other angles were kept fixed: $\theta_1=0$, $\phi_1=\pi/2$, $\phi_2=\pi/2$.

In Fig. 6(a) the behavior of $|\mu^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega)|$ for four collections of particles (see Fig. 2) is plotted. The appearance of interference effects is obviously seen starting from the case $n=2$. Figure 6(b) shows the influence of different values of σ on scattering from five particles when $k\Delta=1$. In Fig. 6(c) we illustrate the effect of the scaled correlation parameter $k\Delta$ of the incident plane waves on the scattered spectral density scattered from five particles, while the scaled size of the spheres $k\sigma$ is kept fixed.

V. CONCLUDING REMARKS

We have developed the theory for far-field scattering of electromagnetic fields of deterministic and random nature from collections of discrete particles which have either deterministic or random locations. The theory is based on the scattering matrices approach and uses the first Born approximation but is not limited to it. It enables scattering of fields with practically arbitrary spectral and coherence properties from particulate media in the most rigorous possible manner, while previously the majority of calculations were done for a single monochromatic or, at best, a polychromatic plane wave. From the example that we have considered, that involves scattering of two mutually correlated plane waves on several particles with Gaussian potentials, it is seen how the spectral density and the state of coherence of the far field depend on directions of the plane waves, their degree of correlation and, of course, on all the properties of the collection of scatterers.

Although our analysis is limited only to collections for which the boundaries of the individual particles are soft and multiple scattering effects are neglected. In many practical cases our calculations are very relevant, e.g., in scattering of a random light beam from a tenuous collection of cells suspended in a solution. If the size of a cell is on the order of the wavelength then interference effects dominate multiple scattering effects and the first Born approximation is sufficient for obtaining an adequate solution.

This approach may be later extended to beamlike fields of interest, for example, Gaussian beam, Bessel beam, etc., or any general, three-dimensional (nonparaxial) fields. Also, we have only introduced analytical formulas describing how scattering from random collections of particles may be carried out in a similar fashion as from deterministic collections. Since, however, this subject involves completely different aspects of what was considered in this paper we will address it in detail in a later publication.

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