# Generating "squeezed" superpositions of coherent states using photon addition and subtraction

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We study how photon addition and subtraction can be used to generate squeezed superpositions of coherent states (SSCSs) in free-traveling fields with high fidelities and large amplitudes. It is shown that an arbitrary $N$-photon subtraction results in the generation of a SSCS with nearly the perfect fidelity ($F>0.999$) regardless of the number of photons subtracted. In this case, the amplitude of the SSCS increases as the number of the subtracted photons gets larger. For example, two-photon subtraction from a squeezed vacuum state of 6.1 dB can generate a SSCS of $\alpha=1.26$, while in the case of the four-photon subtraction a SSCS of a larger amplitude $\alpha=1.65$ is obtained under the same condition. When a photon is subtracted from a squeezed vacuum state and another photon is added subsequently, a SSCS with a lower fidelity ($F\approx0.96$) yet higher amplitude ($\alpha\approx2$) can be generated. We analyze some experimental imperfections including inefficiency of the detector used for the photon subtraction.

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## I. INTRODUCTION

The development of the quantum theory of light has deepened our understanding of nonclassical properties of optical fields. Recently, superpositions of coherent states (SCSs) in free-traveling fields [1,2] have attracted special attention due to their remarkable usefulness. When their amplitudes are large, the SCSs show typical properties of macroscopic quantum superpositions, and because of this, they are often called "Schrödinger cat states" recalling the famous cat paradox [3]. The SCSs enable one to perform many interesting studies for fundamental tests of quantum theory [4–6]. Furthermore, it has been found that SCSs are useful for various applications in quantum-information processing [7–12]. The power of this approach lies in the fact that all four Bell states can be discriminated in a deterministic way only using a beam splitter and photon counting [8,9], which is obviously not the case for the single-photon based approach.

In spite of the manifold usefulness of the SCSs, until recently, the generation of free-traveling SCSs has been known to be difficult. There have been schemes to generate such SCSs using strong nonlinear interactions [1,13] or photon number resolving detectors [14,15], which are not feasible using current technology. Recently, more realistic schemes have been suggested by several authors [16–21]. For example, a scheme using weak Kerr nonlinearities and simple optical elements was suggested [18] based on a previous proposal where strong Kerr nonlinearities are required [13]. As another example, a simple observation was made that SCSs with small amplitudes, such as $\alpha<1.2$, are well-approximated by squeezed single photons [16]. It was also pointed out that squeezed single photons can be obtained by subtracting or adding one photon from pure squeezed vacuums [22]. Meanwhile, single-photon-subtracted squeezed states, which are close to small SCSs ($\alpha\lesssim1$), have been generated by several experimental groups [23–26] and theoretical analysis has been performed [27–29]. Recently, squeezed SCSs (SSCSs) were generated and detected [30], where the size of the states ($\alpha=1.6$) was reasonably large for fundamental tests of quantum theory and quantum-information processing, for which the states are suited despite their squeezing [31]. A scheme using time separated two-photon subtraction was suggested [32] and experimentally demonstrated [33] to generate SCSs of large amplitudes. Despite all the recent progress, however, the fidelities of the generated states are yet to be improved for practical quantum-information processing.

The directions of the development for the generation of SCSs are twofold. First, one needs to generate SCSs with larger amplitudes ($\alpha\gtrsim2$) for macroscopic tests of quantum theory. Second, for quantum-information processing, it is important to generate SCSs with higher fidelity $F>0.99$ while $\alpha\approx1.6$ is an appropriate value [34]. The SSCSs, generated in a recent experiment [30], are simply a squeezed version of the SCSs. Interestingly, the direction of the squeezing in Ref. [30] makes the SSCSs more robust against decoherence than the regular SCSs [35]. The SSCSs can be useful in some protocols as they are [31], and if required, it may be possible to unsqueeze them by means of the squeezing transformation [19,20,36].

In this paper, we are interested in finding methods to generate SSCSs with larger amplitudes and higher fidelities using photon subtraction and addition. Here, we show that the two photon subtraction enables one to produce the SSCS with very high fidelity, and the combination of the subtraction and addition can return the SSCS with a lower fidelity yet higher amplitude.

We also find that consecutive applications of photon subtraction (or subtracting a well-defined number of photons) from a squeezed vacuum state result in the generation of a SSCS with nearly the perfect fidelity regardless of the number of photons subtracted. The amplitude of the SSCS increases as the number of the subtracted photons gets larger. This paper is organized as follows. In Sec. II, we investigate combinations of the ideal single photon addition and subtraction. In Sec. III, some experimental imperfections are analyzed for the realization of the states discussed in Sec. II. In

 

Sec. IV, we numerically show that an arbitrary $N$ photon subtraction, regardless of $N$, can be used to generate a SSCS with an extremely high fidelity as $F>0.999$. We conclude with final remarks in Sec. V.

## II. PHOTON ADDITION TO AND SUBTRACTION FROM SQUEEZED VACUUM

### A. Combinations of photon addition and subtraction

The SCSs are defined as

$$|\mathrm{SCS}_{\varphi}\rangle = \mathcal{N}_{\varphi}(|\alpha\rangle + e^{i\varphi}|-\alpha\rangle), \tag{1}$$

where $\mathcal{N}_{\varphi}$ is a normalization factor, $|\pm\alpha\rangle$ is a coherent state of amplitude $\pm\alpha$, and $\varphi$ is a real local phase factor. The SCSs such as $|\mathrm{SCS}_{\pm}(\alpha)\rangle=\mathcal{N}_{\pm}(|\alpha\rangle\pm|-\alpha\rangle)$ are called even and odd SCSs, respectively, because the even (odd) SCS always contains an even (odd) number of photons. The squeezed vacuum state, $|S(r)\rangle$, with the squeezing parameter $r$ can be obtained by applying the squeezing operator,

$$\hat{S}(r) = e^{r(\hat{a}^2-\hat{a}^{\dagger 2})/2}, \tag{2}$$

where $\hat{a}$ ($\hat{a}^{\dagger}$) is the bosonic annihilation (creation) operator, to the vacuum state. In the number state basis, it can be represented as

$$|S(r)\rangle = \sqrt{\mathrm{sech}\, r}\sum_{k=0}^{\infty}\frac{\sqrt{(2k)!}}{k!}\left[-\frac{1}{2}\tanh r\right]^k|2k\rangle, \tag{3}$$

where $r$ was assumed to be real. As seen in Eq. (3), the squeezed vacuum state contains only an even number of photons. A squeezed single photon, known as a good approximation of a small SCS, can be obtained by adding a photon to a squeezed vacuum as

$$\hat{a}^{\dagger}\hat{S}(r)|0\rangle = \cosh r\hat{S}(r)|1\rangle, \tag{4}$$

where the right-hand side is unnormalized due to the characteristics of the creation operator $a^{\dagger}$. We note that Eq. (4) can easily be shown using the following unitary transformations [37]:

$$\hat{S}^{\dagger}(r)\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{a}^{\dagger}\sinh r,$$

$$\hat{S}^{\dagger}(r)\hat{a}^{\dagger}\hat{S}(r) = \hat{a}^{\dagger}\cosh r - \hat{a}\sinh r. \tag{5}$$

It is also known that the squeezed single photon can also be obtained by subtracting a photon from a squeezed vacuum as

$$\hat{a}\hat{S}(r)|0\rangle = -\sinh r\hat{S}(r)|1\rangle. \tag{6}$$

This may cause us to conjecture that when the photon addition and subtraction are applied successively to the squeezed vacuum, an approximate even SCS may be generated. We can consider four immediate cases, namely, addition and subtraction ($\hat{a}\hat{a}^{\dagger}$), subtraction and addition ($\hat{a}^{\dagger}\hat{a}$), successive additions [$(\hat{a}^{\dagger})^2$], and successive subtractions ($\hat{a}^2$). From Eqs. (4) and (6), it is straigtforward to see that $(\hat{a}^{\dagger})^2$ will result in the same state produced using $\hat{a}^{\dagger}\hat{a}$. It is also easy to see that $\hat{a}\hat{a}^{\dagger}$ will cause the same effect with $\hat{a}^2$. Therefore we shall consider only two cases, $\hat{a}^{\dagger}\hat{a}$ and $\hat{a}^2$, among the four based on the fact that the photon subtraction is relatively easier to perform than the photon addition.

Now suppose an *ideal* situation that a photon is subtracted from the squeezed vacuum state and then another photon is subsequently added. The resulting state, which we shall call photon-subtracted-and-added squeezed state (PSAS), is obtained by applying the annihilation and creation operators, $\hat{a}^{\dagger}\hat{a}$, to the squeezed vacuum state $|\hat{S}(r)\rangle$. After a straightforward calculation using Eqs. (5) and the normalization, the PSAS appears to be

$$|\psi_{\hat{a}^{\dagger}\hat{a}}\rangle = \mathcal{N}_{\hat{a}^{\dagger}\hat{a}}\hat{S}(r)[|0\rangle - \sqrt{2}(\tanh r)^{-1}|2\rangle], \tag{7}$$

where $\mathcal{N}_{\hat{a}^{\dagger}\hat{a}}=\{1+2(\tanh r)^{-2}\}^{-1/2}$. In the same manner, the two-photon subtracted squeezed state (TPSS) can be obtained as

$$|\psi_{\hat{a}^2}\rangle = \mathcal{N}_{\hat{a}^2}\hat{S}(r)(|0\rangle - \sqrt{2}\tanh r|2\rangle), \tag{8}$$

with $\mathcal{N}_{\hat{a}^2}=\{1+2(\tanh r)^2\}^{-1/2}$.

### B. Fidelities against ideal states

The fidelity, $F=|\langle\psi|\psi_t\rangle|^2$, is a measure of how close a state $|\psi\rangle$ is to the target state $|\psi_t\rangle$. It is unity when the two states are identical, while it is zero when the two are orthogonal to each other. The fidelity between the TPSS (or PSAS) and the ideal squeezed (or regular) SCS can be obtained as follows. A even SSCS can be expressed as

$$|\mathrm{SSCS}\rangle = \mathcal{N}_{+}\hat{S}(r')[|\alpha\rangle + |-\alpha\rangle], \tag{9}$$

where $\mathcal{N}_{+}=[2+2e^{-2\alpha^2}]^{-1/2}$. The fidelity $F$ between the TPSS and the even SSCS can be calculated as

$$\begin{aligned} F &= |\langle\psi_{a^2}|\mathrm{SSCS}\rangle|^2 \\ &= 8|\mathcal{N}_{+}\mathcal{N}_{a^2}[\langle 0|\hat{S}^{\dagger}(r-r')|\alpha\rangle \\ &\quad + \nu\langle 2|\hat{S}^{\dagger}(r-r')|\alpha\rangle]|^2 \end{aligned} \tag{10}$$

where $\nu=-\tanh r$. This expression can be evaluated with help of the $x$ representation:

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty}\langle\psi_1|x\rangle\langle x|\psi_2\rangle dx, \tag{11}$$

where the relevant wave functions are

$$\langle x|\hat{S}(r-r')|0\rangle = (\pi g^2)^{-1/4}e^{-x^2/2g^2},$$

$$\langle x|\hat{S}(r-r')|2\rangle = (4\pi g^2)^{-1/4}\left(\frac{2x^2}{g^2}-1\right)e^{-x^2/2g^2},$$

$$\langle x|\alpha\rangle = \pi^{-1/4}e^{-(x-\sqrt{2}\alpha)^2/2}, \tag{12}$$

where $g=\exp[-(r-r')]$. After some calculations, we arrive at the fidelity:

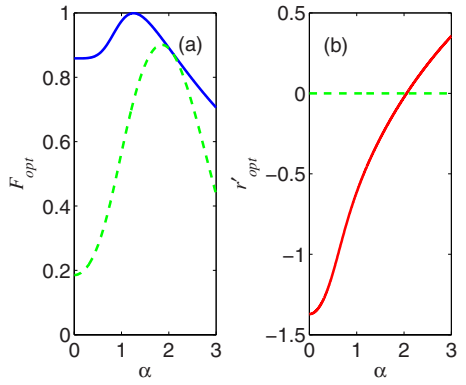


FIG. 1. (Color online) (a) Optimal fidelity $F_{opt}$ between the ideal TPSS ($a^2$) and the squeezed SCS of amplitude $\alpha$ (solid curve) and the optimal fidelity between the ideal TPSS and the corresponding regular SCS (dashed curve). The squeezing parameter $r'_{opt}$ of the initial squeezed state is $r=-0.7$. (b) The squeezing parameter $r'$ of the target SSCS for which the fidelity is optimized.

$$F_{a^2} = \frac{8ge^{-2\alpha^2/(1+g^2)}}{1+g^2}\left|\mathcal{N}_+\mathcal{N}_{a^2}\left\{1+\nu\frac{1+4\alpha^2g^2-g^4}{(1+g^2)^2}\right\}\right|^2. \tag{13}$$

The same approach can be used to derive fidelity of states prepared by combined photon subtraction and addition:

$$F_{a^\dagger a} = \frac{8ge^{-2\alpha^2/(1+g^2)}}{1+g^2}\left|\mathcal{N}_+\mathcal{N}_{a^\dagger a}\left\{1+\mu\frac{1+4\alpha^2g^2-g^4}{(1+g^2)^2}\right\}\right|^2, \tag{14}$$

where

$$\mathcal{N}_{a^\dagger a} = [1+2\mu^2]^{-1/2}, \quad \mu = \frac{1}{\tanh r}. \tag{15}$$

In Figs. 1 and 2, we have used the analytical expressions to show the optimal fidelities for a range of coherent amplitudes. That is, for each $\alpha$, such squeezing $r'$ of the SSCS is found to provide the maximal overlap with the state prepared from the initial squeezed state with $r=-0.7$ ($p$-squeezed state, approximately 6.1 dB of squeezing, which can be realized using current technology). For comparison a fidelity with the corresponding regular SCS is also shown.

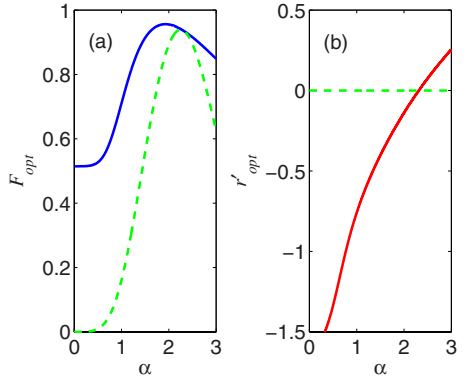


FIG. 2. (Color online) (a) Optimal fidelity $F_{opt}$ between the ideal PSAS ($a^\dagger a$) and the SSCS of amplitude $\alpha$ (solid curve) and the optimal fidelity between the ideal PSAS and the corresponding regular SCS (dashed curve). The squeezing parameter $r'_{opt}$ of the initial squeezed state is $r=-0.7$. (b) The squeezing parameter of the target SSCS for which the fidelity is optimized.

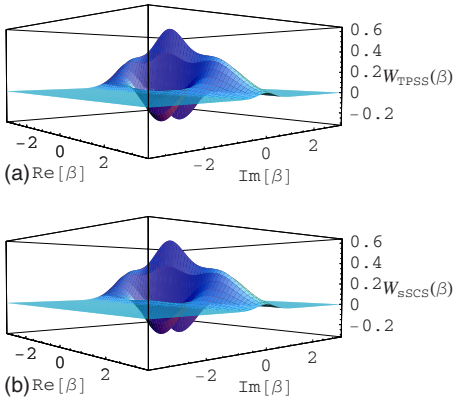


FIG. 3. (Color online) (a) The Wigner function of the TPSS with $r=-0.7$ ($\approx$6.1 dB) and (b) the Wigner function of the ideal SSCS with $\alpha=1.26$ and $r'=-0.425$. The fidelity between the two states is nearly perfect as $F>0.999$.

Figure 1(a) shows the fidelity of the TPSS. It is seen that although the fidelity tops at 0.9 for a regular SCS, when a SSCS is considered, the fidelity of $F=0.999$ can be achieved for $\alpha=1.26$ and $r'=-0.425$. The squeezing parameter $r'$ of the target SSCS that optimizes the fidelity against the corresponding amplitude $\alpha$ is plotted in Fig. 1(b). The fidelity for the PSAS and the optimizing squeezing parameter against $\alpha$ are depicted in Fig. 2. The optimal fidelity is $F=0.956$, which is not as good as the case of the TPSS. However, in this case, the amplitude of the SSCS is larger as $\alpha=1.93$, while the optimal squeezing for the SSCS is only $r'=-0.17$.

The Wigner function of a state with density operator $\rho$ can be obtained from the Fourier transform of its characteristic function $C(\zeta)=\mathrm{Tr}[D(\zeta)\rho]$, where $D(\zeta)=\exp[\zeta\hat{a}^\dagger-\zeta^*\hat{a}]$ is the displacement operator. The Wigner functions of the TPSS and PSAS can be obtained using Eqs. (7) and (8) as

$$W_{\hat{a}^2}(\beta) = \mathcal{N}_{ss}e^{-2Z}(1+2\tanh r\{Z'+\tanh r[1+8Z(Z-1)]\}), \tag{16}$$

$$W_{\hat{a}^\dagger\hat{a}}(\beta) = \mathcal{N}_{as}e^{-2Z}(1+2\coth r\{Z'+\coth r[1+8Z(Z-1)]\}), \tag{17}$$

where $Z=e^{2r}\beta_r^2+e^{-2r}\beta_i^2$ and $Z'=-4e^{2r}\beta_r^2+4e^{-2r}\beta_i^2$. In Figs. 3 and 4, we consider the squeezing parameter $r=-0.7$ (6.1 dB) for the initial squeezed state. Figure 3 shows again that the TPSS is an extremely good approximation of the even SSCS. In Fig. 3, the Wigner functions of the TPSS of $r=-0.7$ and the even SSCS of $r'=-0.425$ look virtually identical and the fidelity between the two states is $F>0.999$. Figure 4 presents the Wigner functions of the PSAS of $r=-0.7$ and the even SSCS of $r'=-0.14$ and $\alpha=2$, where the fidelity between the two states is $F\approx0.955$. This shows that a slight variation of parameters still allows for high fidelity.

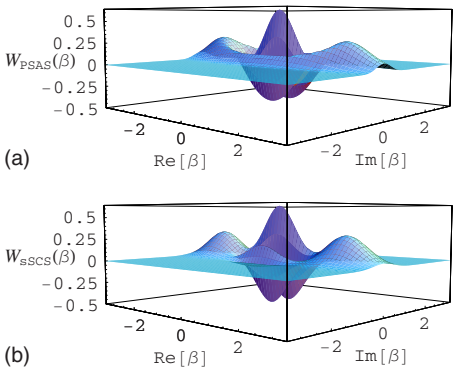


FIG. 4. (Color online) (a) The Wigner function of the PSAS (left) with $r=-0.7$ ($\approx 6.1$ dB) and (b) the Wigner function of the ideal SSCS with $\alpha=2$ and $r'=-0.14$ (right). The fidelity between the two states is $F\approx 0.955$.

To conclude, if one is interested in generating SSCSs of high fidelity, the two-photon subtraction would be a useful scheme, while the photon subtraction and addition would be a better strategy to generate large SSCSs.

## III. EXPERIMENTAL CONSIDERATIONS

### A. Optical operations with ideal avalanche photodetectors

So far, we have considered ideal photon addition and subtraction using annihilation and creation operators. In real experiments, however, they can be implemented only using approximate schemes. The setup for implementation of the annihilation (creation) of a single photon consists of a beam splitter (BS) [noncollinear optical parametric amplification (NOPA)] and an avalanche photodetector. In the Wigner function formalism, the actions of a BS and NOPA, coupling two modes of light, can be characterized with the help of a transformation matrix acting on a vector of variables $(x_1,p_1,x_2,p_2)$ corresponding to quadrature operators $X_j$ and $P_j$ with $[X_j,P_{j'}]=i\delta_{jj'}$:

$$V_{12}=\begin{pmatrix} t & 0 & r & 0 \\ 0 & t & 0 & \chi \\ -r & 0 & t & 0 \\ 0 & -\chi & 0 & t \end{pmatrix}, \tag{18}$$

where the generalized parameters are

$$t=\sqrt{T}, \quad r=\chi=\sqrt{1-T} \tag{19}$$

for $V_{12}$ to describe the action of a beam splitter with transmissivity $T$, and

$$t=\sqrt{G}, \quad r=-\chi=\sqrt{G-1} \tag{20}$$

for $V_{12}$ to describe a NOPA with amplification gain $G$. The three-mode Wigner function for the initial squeezed state and two vacuum ancillas is expressed as

$$W_{\rm tot}(\xi)=\frac{1}{(2\pi)^2\sqrt{|\Sigma_{\rm tot}|}}\exp\left(-\frac{1}{2}\xi\Sigma_{\rm tot}^{-1}\xi^T\right). \tag{21}$$

Here, the vector of variables for the three-mode Wigner function is defined as $\xi=(x_1,p_1,x_A,p_A,x_B,p_B)$, where $x=\beta_r/\sqrt{2}$ and $p=\beta_i/\sqrt{2}$ being compared to the variable, $z$, used in Eqs. (16) and (17), and the subscripts of $x$ and $p$ in order to denote the initial and two ancillary modes, respectively. Furthermore, $|\cdot|$ denotes the determinant of the matrix, the superscript $T$ stands for transposition, and $\Sigma_{\rm tot}$ is the covariance matrix of the state,

$$\Sigma_{\rm tot}={\rm diag}\left(V_x,V_p,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \tag{22}$$

with $V_x$ and $V_p$ being the variances of the initial squeezed state.

The initial state subsequently interacts with the two vacuum modes, transforming the vector of variables into

$$\xi\rightarrow\xi'=V_BV_A\xi, \tag{23}$$

where the transformation matrices $V_A$ and $V_B$ are of the form (18) coupling modes 1 and $A$ and 1 and $B$ with parameters $t_A$, $r_A$, $\chi_A$ and $t_B$, $r_B$, and $\chi_B$, respectively. To complete the transformation, a conditioning measurement is needed. When using two ideal avalanche photodetectors and postselecting the state only when both produce a detection event, we implement a pair of projection operators $\hat{1}-|0\rangle\langle 0|$ and the Wigner function of the output state is transformed to

$$W_{\rm out}(x_1,p_1)=\int W_{\rm tot}(\xi')[1-2\pi W_{\rm vac}(x_A,p_A)] \times[1-2\pi W_{\rm vac}(x_B,p_B)]dx_Adp_Adx_Bdp_B, \tag{24}$$

where $W_{\rm vac}(x,p)=\exp(-x^2-p^2)/\pi$ is the Wigner function of a vacuum state. This integral can be expressed as a sum of four Gaussian integrals,

$$W_{\rm out}(x_1,p_1)=\frac{1}{2\pi\sqrt{|\Sigma_{\rm tot}|}}[I(\Sigma_{nn})-2I(\Sigma_{ny})-2I(\Sigma_{yn}) +4I(\Sigma_{yy})], \tag{25}$$

where $I(\Sigma)$ is shorthand for

$$I(\Sigma)=\frac{1}{(2\pi)^2}\int\exp\left(-\frac{1}{2}\xi\Sigma^{-1}\xi^T\right)dx_Adp_Adx_Bdp_B \tag{26}$$

and $\Sigma_{jj'}$ is a covariance matrix for the particular event when either the two detectors did detect vacuum ($y$) or no measurement has taken place ($n$). The particular covariance matrices can be found as

$$\Sigma_{jj'}=[\Sigma_{\rm tot}^{-1}+\Pi_{jj'}]^{-1}, \tag{27}$$

where the $\Pi_{jj'}$ is the semi-inverted covariance matrix of the vacuum state,

$$\Pi_{nn}={\rm diag}(0,0,0,0,0,0), \quad \Pi_{yy}={\rm diag}(0,0,2,2,2,2),$$
$$\Pi_{yn}={\rm diag}(0,0,2,2,0,0), \quad \Pi_{ny}={\rm diag}(0,0,0,0,2,2). \tag{28}$$

The matrices $\Sigma_{jj'}$ can be decomposed into a block form,

$$\Sigma_{jj'} = \begin{pmatrix} A_{jj'} & C_{jj'} \\ C_{jj'}^T & B_{jj'} \end{pmatrix}, \tag{29}$$

where the submatrix $A_{jj'}$ corresponds to variables $x_1, p_1$, submatrix $B_{jj'}$ to $x_A$, $p_A$, $x_B$, $p_B$, and submatrix $C_{jj'}$ covers relations between these two groups. After the integration, we can arrive at the final Wigner function:

$$W_{\text{out}}(x,p) = \frac{1}{\mathcal{N}\sqrt{|\Sigma_{\text{tot}}|}}[c_{nn}W_{A_{nn}}(x,p) - 2c_{ny}W_{A_{ny}}(x,p) - 2c_{yn}W_{A_{yn}}(x,p) + 4c_{yy}W_{A_{yy}}(x,p)], \tag{30}$$

where

$$W_{\Sigma}(x,p) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(x,p)\Sigma^{-1}(x,p)^T\right] \tag{31}$$

is the Wigner function of a Gaussian state with covariance matrix $\Sigma$, the coefficients are

$$c_{jj'} = \sqrt{|\Sigma_{jj'}|} \tag{32}$$

and

$$\mathcal{N} = (c_{nn} - 2c_{ny} - 2c_{yn} + 4c_{yy})(\sqrt{|\Sigma_{\text{tot}}|})^{-1} \tag{33}$$

is the normalization factor and overall probability of success.

In order to compare the final state (30) with our target state, the even SSCS, we need to employ its Wigner function

$$W_{\text{SCS}}(x,p) = \mathcal{N}_{\text{SCS}}^2 \frac{e^{-gp^2}}{\pi}\left\{\exp\left[-\frac{(x-\sqrt{2g}\alpha)^2}{g}\right] + \exp\left[-\frac{(x+\sqrt{2g}\alpha)^2}{g}\right] + 2e^{-x^2/g}\cos(2\sqrt{2g}\alpha p)\right\}, \tag{34}$$

where $\mathcal{N}_{\text{SCS}} = [2+2e^{-2\alpha^2}]^{-1/2}$, $\alpha$ is the coherent amplitude, and $g = \exp(-2r)$ characterizes squeezing of the state. The fidelity can then be calculated as the overlap between the two Wigner functions:

$$F = 2\pi \int W_{\text{out}}(x,p)W_{\text{SCS}}(x,p)dxdp. \tag{35}$$

Since the Wigner function $W_{\text{out}}(x,p)$ is a sum of Gaussian functions, we can treat the integration in parts and express the final fidelity as

$$F = \frac{1}{\mathcal{N}\sqrt{|\Sigma_{\text{tot}}|}}[c_{nn}f(A_{nn}) - 2c_{ny}f(A_{ny}) - 2c_{yn}f(A_{yn}) + 4c_{yy}f(A_{yy})]. \tag{36}$$

Here $f(\Sigma)$ denotes the fidelity between a Gaussian state with covariance matrix $\Sigma$ and a SSCS. Since all the covariance matrices used are diagonal, we can write these partial fidelities as functions of the diagonal elements as

$$f[\text{diag}(\mu,\nu)] = 2\left[(1+e^{-2\alpha^2})\sqrt{(g+2\mu)\left(\frac{1}{g}+2\nu\right)}\right]^{-1} \times \left[\exp\left(\frac{-2\alpha^2}{1+2\mu/g}\right) + \exp\left(-\frac{2\alpha^2}{1+1/2g\nu}\right)\right]. \tag{37}$$

With Eqs. (36) and (37) we can finally obtain the required fidelity. In analogy with the previous section, the behavior of the state is depicted in Figs. 5 and 6. The comparison reveals that these real states behave in a similar pattern as the ideal ones, the performance is, however, slightly worse. For realistic parameters, $T=0.99$ and $G=1.01$, the optimal fidelities

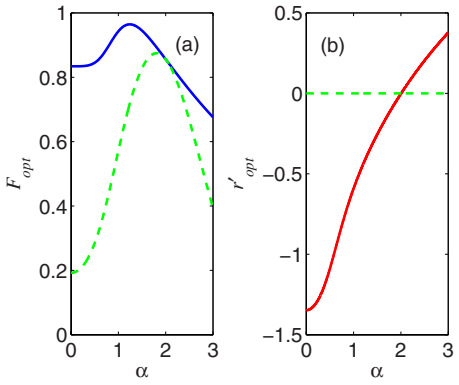


FIG. 5. (Color online) (a) Optimal fidelity $F_{opt}$ between the TPSS ($a^2$) generated using ideal avalanche photon detectors and the SSCS of amplitude $\alpha$ (solid curve) and the optimal fidelity with the corresponding regular SCS (dashed curve). The squeezing parameter of the initial squeezed state is $r=-0.7$. The parameter of real transformation is $T=0.99$. (b) The squeezing parameter of the target SSCS for which the fidelity is optimized.

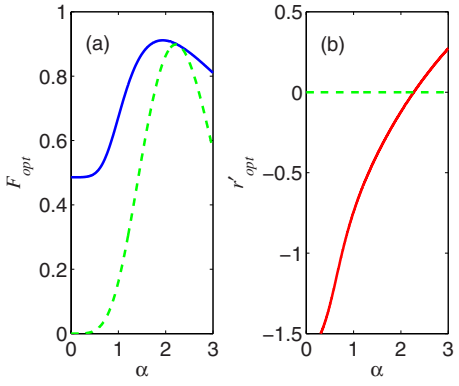


FIG. 6. (Color online) Optimal fidelity $F_{opt}$ between the PSAS ($a^\dagger a$) generated using ideal avalanche photon detectors and the SSCS of amplitude $\alpha$ (solid curve) and the optimal fidelity with the corresponding regular SCS (dashed curve). The squeezing parameter of the initial squeezed state is $r=-0.7$. (b) The squeezing parameter of the target SSCS for which the fidelity is optimized. The parameters of real transformations are $T=0.99$, $G=1.01$.

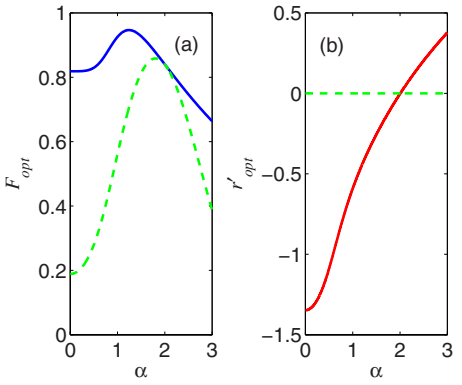


FIG. 7. (Color online) (a) Optimal fidelity $F_{opt}$ between the TPSS ($a^2$) generated using realistic avalanche photon detectors and the SSCS of amplitude $\alpha$ (solid curve) and the optimal fidelity with the corresponding regular SCS (dashed curve). The squeezing parameter of the initial squeezed state is $r=-0.7$. The parameters of realistic transformations are $T=0.99$ and $\eta=0.6$. (b) The squeezing parameter of the target SSCS for which the fidelity is optimized.

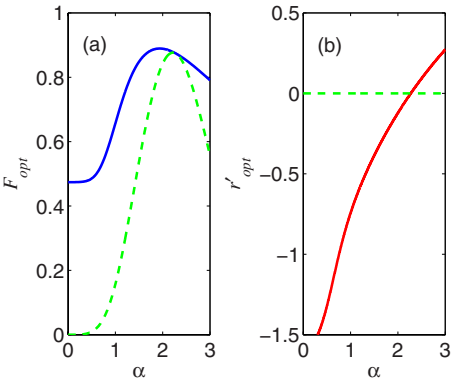


FIG. 8. (Color online) (a) Optimal fidelity $F_{opt}$ between the PSAS ($a^\dagger a$) generated using realistic avalanche photon detectors and the SSCS of amplitude $\alpha$ (solid curve) and the optimal fidelity with the corresponding regular SCS (dashed curve). The squeezing parameter of the initial squeezed state is $r=-0.7$. (b) The squeezing parameter of the target SSCS for which the fidelity is optimized. The parameters of realistic transformations are $T=0.99$, $G=1.01$, and $\eta=0.6$.

are $F\approx 0.96$ for two photon subtraction and $F\approx 0.91$ for subtraction and addition.

### B. Effects of imperfect detectors

We have so far considered ideal avalanche photodetectors with unit quantum efficiency. However, in real experiments, detection efficiency is always limited. A realistic detector with efficiency $\eta$ can be modeled by a beam splitter with transmissivity $\eta$ and vacuum at the idle port inserted in front of an ideal detector. The Wigner function of the output state can then be represented as

$$W_{\text{out}}(x_1,p_1)=\frac{1}{2\pi\sqrt{|\Sigma'_{\text{tot}}|}}[I(\Sigma'_{nn})-2I(\Sigma'_{ny})-2I(\Sigma'_{yn})+4I(\Sigma'_{yy})], \tag{38}$$

where the covariance matrice $\Sigma'_{\text{tot}}$ incorporates the effect of the imperfect detection as

$$\Sigma'_{\text{tot}}=\Xi\Sigma_{\text{tot}}\Xi+(1-\eta)\text{diag}\left(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),$$

$$\Xi=\text{diag}(1,1,\sqrt{\eta},\sqrt{\eta},\sqrt{\eta},\sqrt{\eta}) \tag{39}$$

and the matrices $\Sigma'_{jj'}$ can be obtained from Eqs. (39) and (27). Finally, the fidelities can be arrived at using Eqs. (36) and (37). Figure 7 presents the optimal fidelity of the TPSS generated using avalanche photodetectors with $\eta=0.6$ and a beam splitter of $T=0.99$. Remarkably, under these realistic assumptions, the fidelity $F\approx 0.95$ can still be obtained. The fidelity for PSAS under the same considerations is plotted in Fig. 8 where the optimal fidelity drops down to $F\approx 0.89$. This confirms that two-photon subtraction is a better scheme to generate SSCSs of high fidelity, even though the amplitudes are smaller in this case.

## IV. SUCCESSIVE APPLICATIONS OF PHOTON SUBTRACTION

We have shown that the SSCS is extremely well-approximated by the TPSS. We now show that this can be generalized to arbitrary $N$-photon subtraction with $N\geqslant 3$. Namely, $N$-photon-subtracted squeezed states (NPSSs) are good approximations of the SSCSs, which may be compared with the proposal by Fiurášek *et al.* to generate an arbitrary state by photon subtractions and displacements of a squeezed state [39]. We suppose that a beam splitter of transmmittivity $T$ and an ideal photodetector is used to subtract $N$ photons from a squeezed state. The Wigner function of the NPSS is then obtained as [38]

$$W_N(\beta)=\mathcal{M}_N\exp\left(-\lambda\beta_r^2-\frac{\beta_i^2}{\lambda}\right)\sum_{k=0}^{N}\frac{(-2|\mathcal{R}|)^k}{k![(N-k)!]^2}\times\left|H_{N-k}\left[i\sqrt{\mathcal{R}\lambda}\left(\beta_r+i\frac{\beta_i}{\lambda}\right)\right]\right|^2, \tag{40}$$

where

$$\mathcal{M}_N=\sum_{k=0}^{N/2}\frac{(2|\mathcal{R}|)^{N-2k}}{(N-2k)!(k!)^2}, \tag{41}$$

$$\lambda=(1-\mathcal{R})/(1+\mathcal{R}), \tag{42}$$

$$\mathcal{R}=T|\tanh r|^2, \tag{43}$$

and $H_n[x]$ is the Hermite polynomial. When $N$ is odd (even), the NPSS should be compared with the odd (even) SCS. The fidelity between the NPSS and the SSCS can be obtained using Eq. (35). It is nontrivial to obtain analytical expressions of the fidelity for an arbitrary $N$. As presented in Fig. 9, we numerically assess the fidelity $F$ and plot it from $N=3$ to 8 against the amplitude $\alpha$ of the SSCS. We suppose $\mathcal{R}$ is 0.6, i.e., $r\approx 0.7$ (6.1 dB) when $T=0.99$ and $r=0.75$ (6.5 dB)

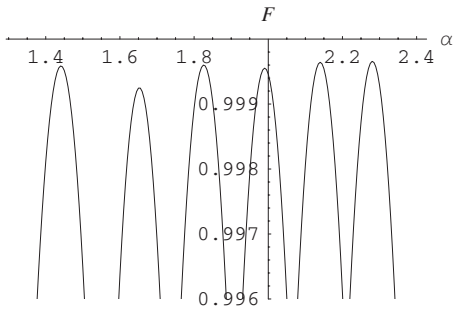


FIG. 9. The optimal fidelity between the NPSSs with $\mathcal{R}(=T\tanh r)=0.6$ and the SSCSs of $r'=0.44$ from $N=3$ (left) to $N=8$ (right). The $x$ axis represents the amplitude $\alpha$ of the target SSCS. The optimal fidelity is $F>0.999$ regardless of the number of $N$.

when $T=0.95$. The degrees of squeezing, 6.1–6.5 dB, are experimentally achievable. Interestingly, the squeezing parameter of the SSCS is $r'=0.44$ regardless of $N$. Our numerical calculation confirms this generalization up to $N=15$. For example, the fidelity is as high as $F>0.999$ for $\alpha=2.52$ ($\alpha=3$) when $N=10$ ($N=15$) with the same squeezing parameter $r'=0.44$ of the SSCS.

## V. REMARKS

We have pointed out the twofold directions of the development for the generation of SCSs. The generation of SCSs with high fidelities ($F>0.99$) and those with large amplitudes ($\alpha\gtrsim 2$) aims at practical applications of quantum-information processing and macroscopic tests of quantum theory. The SSCSs, which were generated in recent experiments [30], may be a good alternative to the SCSs for the aforementioned purposes while their fidelity in those experiments is yet to be improved. In this paper, we have studied how photon addition and subtraction can be used to generate the SSCSs with high fidelities and large amplitudes. We have found that the single photon subtraction and subsequent addition with a squeezed vacuum state can cause the production of the approximate SSCS with $F\approx 0.956$ and $\alpha\approx 2$ when the squeezing degree is about 6.1 dB which is achievable using current technology. Furthermore, we show that $N$ photon subtraction may be used to generate the SSCSs with extremely high fidelities as $F>0.999$. The amplitude of the SSCS increases as the number of the subtracted photons gets larger. For example, $\alpha=1.26$ is obtained for two-photon subtraction while $\alpha=1.68$ for four-photon subtraction when the fidelity is $F>0.999$ for both cases.

We have assessed some experimental imperfections in implementing photon addition and subtraction such as inefficiency of photodetectors and nonunit transmmittivity of beam splitters. It has been shown that the fidelity of $F\approx 0.95$ can still be obtained with detection efficiency $\eta=0.6$ and beam-splitter trasmmittivity $T=0.99$ for two-photon subtraction to generate SSCSs. We believe that our work will immediately motivate experimental efforts to generate high-fidelity SCSs and large-amplitude ones that are useful for various quantum-information applications and fundamental studies.

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