Recoil corrections to decay rates of hydrogenic ions

G. S. Adkins
Department of Physics and Astronomy, Franklin & Marshall College, Lancaster, Pennsylvania 17604, USA

J. Sapirstein
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556, USA
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Calculations of recoil corrections to first order in $1/M$, where M is the mass of the nucleus, require an exact treatment of the electron propagator along with the evaluation of up to two transverse photon exchange diagrams. The real part of this effect was calculated in a previous paper, and in this paper we evaluate the imaginary part. Comparison with the known low-Z recoil results is made, and the behavior of the effect at high Z is studied.

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I. INTRODUCTION

In a recent paper [1] the formulas that give the first-order recoil correction to energy levels of hydrogenic ions valid to all orders in $Z\alpha$, where Z is the charge of the nucleus were rederived using diagrammatic techniques. This problem was first treated by Braun [2], and has since been extensively studied by Shabaev [3] and collaborators [4]. Equivalent formulas were also found by Pachucki and Grotch [5]. While the deriviation was quite complex, the end result was relatively simple, involving only three expressions, which can be arranged by the number of transverse photons exchanged. The complexity of the derivation is associated with the fact that infinite sets of Coulomb photon exchange diagrams all contribute at the same order. However, these sets can be accounted for by replacing the free electron propagator with a bound electron propagator, the so-called Dirac-Coulomb propagator, which satisfies the equation

$$\left[\left(E+\frac{Z\alpha}{|\vec{x}|}\right)\gamma_0+i\vec{\gamma}\cdot\vec{\nabla}_x-m\right]S_F(E;\vec{x},\vec{y})=\delta^3(\vec{x}-\vec{y}). \quad (1)$$

In terms of this, the three recoil contributions to the energy shift of an electron in a valence state v, evaluated in Coulomb gauge, are

$$\Delta E_C=-\frac{(4\pi Z\alpha)^2 i}{M}\int\frac{d^3k d^3q d^3r d^3l}{(2\pi)^{12}}\int\frac{dq_0}{2\pi}\frac{1}{(q_0+i\epsilon)^2}\times\frac{(\vec{l}-\vec{r})\cdot(\vec{q}-\vec{k})}{|\vec{l}-\vec{r}|^2|\vec{q}-\vec{k}|^2}\bar{\psi}_v(\vec{l})\gamma_0 S_F(\epsilon_v+q_0;\vec{r},\vec{q})\gamma_0\psi_v(\vec{k}), \quad (2)$$

$$\Delta E_{1T}=-\frac{2i(4\pi Z\alpha)^2}{M}\int\frac{d^3k d^3q d^3r d^3l}{(2\pi)^{12}}\int\frac{dq_0}{2\pi}\frac{1}{q_0+i\epsilon}\times\frac{(l-r)_j D_{ij}(q_0,\vec{q}-\vec{k})}{|\vec{l}-\vec{r}|^2}\bar{\psi}_v(\vec{l})\gamma_0 S_F(\epsilon_v+q_0;\vec{r},\vec{q})\gamma_i\psi_v(\vec{k}), \quad (3)$$

and

$$\Delta E_{TT}=\frac{(4\pi Z\alpha)^2 i}{M}\int\frac{d^3k d^3q d^3r d^3l}{(2\pi)^{12}}\int\frac{dq_0}{2\pi}D_{ik}(q_0,\vec{l}-\vec{r})\times D_{jk}(q_0,\vec{q}-\vec{k})\bar{\psi}_v(\vec{l})\gamma_i S_F(\epsilon_v+q_0,\vec{r},\vec{q})\gamma_j\psi_v(\vec{k}). \quad (4)$$

In the above D_{ij} is the transverse photon propagator, given by

$$D_{ij}(k_0,\vec{k})=\frac{\delta_{ij}-\frac{k_i k_j}{|\vec{k}|^2}}{k_0^2-|\vec{k}|^2+i\delta}. \quad (5)$$

Its Fourier transformed form is

$$D_{ij}(k_0,\vec{k})=-\frac{1}{4\pi}\int d^3r e^{-i\vec{k}\cdot\vec{r}}\frac{1}{r}\left\{\frac{r^2\delta_{ij}-3x_i x_j}{k_0^2 r^4}+e^{i|k_0|r}\left[\frac{\delta_{ij}}{k_0^2 r^2}(-1+i|k_0|r+k_0^2r^2)+\frac{x_i x_j}{k_0^2 r^4}(3-3i|k_0|r-k_0^2r^2)\right]\right\}. \quad (6)$$

(The $|k_0|$ notation is shorthand for $\sqrt{k_0^2+i\delta}$, but for the real values of k_0 used in this calculation there is no difference). If we define two basic functions

$$A(x)=\frac{1}{x^2}[1+e^{ix}(-1+ix+x^2)],$$

$$B(x)=\frac{1}{x^2}[-3+e^{ix}(3-3ix-x^2)] \quad (7)$$

we can express the transverse photon propagator more compactly as

$$D_{ij}(k_0,\vec{k})=-\frac{1}{4\pi}\int d^3r e^{-i\vec{k}\cdot\vec{r}}\frac{1}{r}\left[A(|k_0|r)\delta_{ij}+B(|k_0|r)\frac{x_i x_j}{r^2}\right], \quad (8)$$

and in the following we will use this abbreviated form. In Ref. [1] we evaluated the real part of three terms with finite basis set techniques [6] for all n=1, 2, and 3, states, finding good agreement with previous calculations [4].

Use of the spectral representation of the Dirac-Coulomb propagator allows the all-Coulomb term to be manipulated into the form

$$\Delta E_C = -\frac{1}{M}\sum_{m}^{E_m<0} \langle v|\vec{p}|m\rangle \cdot \langle m|\vec{p}|v\rangle, \tag{9}$$

which we note is manifestly real. As we are interested in the imaginary part of the energy, it will not be considered further here.

In the previous work we were solely concerned with energy shifts. However, as is also the case for the similar calculation of the self-energy diagram contributing to the Lamb shift, the remaining two of the above equations have both real and imaginary parts. The imaginary part corresponds to a recoil correction to the decay rate, and is the subject of this paper.

Several papers have addressed the issue of recoil corrections to decay rates. The nonrelativistic result

$$\Gamma_{\rm NR} = \Gamma_0\left(\frac{m_r}{m}\right)^3 (1+Zm/M)^2 \tag{10}$$

was presented by Karshenboim [7], who referenced earlier work by Fried and Martin [8], Bacher [9], and Drake [10]. While his paper presented formulas valid for any value of Z, he concentrated on deriving the $(Z\alpha)^2$ corrections instead of a full numerical evaluation, which is carried out in this paper. We note that from an experimental point of view these recoil corrections are of limited interest, given that it is rare to be able to measure a decay rate with greater than one percent accuracy, but, as emphasized in Ref. [7], for muonic atoms the effect is more pronounced.

The plan of this paper is the following. In the next section we treat ΔE_{1T}, which has the leading $Z\alpha$ behavior $m^2/M(Z\alpha)^5$ for both the real and imaginary part. At low Z the imaginary part must reproduce the recoil correction to the well-known nonrelativistic decay rate of hydrogenic ions. We check that this is so, and in addition study what happens as Z increases. In the next section ΔE_{TT} is treated. This begins in order $m^2/M(Z\alpha)^7$. In the conclusion we present a discussion of our results.

II. ONE TRANSVERSE PHOTON EXCHANGE

Use of a spectral decomposition for the electron propagator allows us to write Eq. (3) as

$$\Delta E_{1T} = -\frac{2i(4\pi Z\alpha)^2}{M}\sum_m \int \frac{d^3k d^3q d^3r d^3l}{(2\pi)^{12}} \times \int \frac{dq_0}{2\pi}\frac{1}{q_0+i\epsilon}\frac{1}{\epsilon_v+q_0-\epsilon_m(1-i\delta)} \times \frac{(l-r)_j D_{ij}(q_0,\vec{q}-\vec{k})}{|\vec{r}-\vec{l}|^2}\psi_v^\dagger(\vec{l})\psi_m(\vec{r})\psi_m^\dagger(\vec{q})\alpha_i\psi_v(\vec{k}). \tag{11}$$

Transforming to coordinate space gives

$$\Delta E_{1T} = -\frac{2(Z\alpha)^2}{M}\sum_m \int d^3x d^3y \times \int \frac{dq_0}{2\pi}\frac{1}{q_0+i\epsilon}\frac{1}{\epsilon_v+q_0-\epsilon_m(1-i\delta)}\frac{y_j}{y^3 x} \times \psi_v^\dagger(\vec{y})\psi_m(\vec{y})\psi_m^\dagger(\vec{x})\alpha_i\psi_v(\vec{x}) \times\left[A(|q_0|x)\delta_{ij}+B(|q_0|x)\frac{x_i x_j}{x^2}\right]. \tag{12}$$

The result of carrying out an angular momentum reduction is

$$\Delta E_{1T} = \frac{2i(Z\alpha)^2}{M}\sum_{[m]}\frac{C_1^2(vm)}{2j_v+1} \times \int\frac{dq_0}{2\pi}\frac{1}{q_0+i\epsilon}\frac{1}{\epsilon_v+q_0-\epsilon_m(1-i\delta)} \times \int_0^\infty \frac{dy}{y^2}R_{vm}(y)\int_0^\infty\frac{dx}{x} \times[\sqrt{3}Q_{mv}^1(x)A(|q_0|x)-S_{mv}(x)B(|q_0|x)], \tag{13}$$

where

$$R_{mv}(x) = g_m(x)g_v(x)+f_m(x)f_v(x),$$

$$S_{mv}(x) = g_m(x)f_v(x)-g_v(x)f_m(x),$$

$$Q_{mv}^1(x) = \frac{1}{\sqrt{3}}\{-g_m(x)f_v(x)+g_v(x)f_m(x)+(\kappa_m-\kappa_v) \times[g_m(x)f_v(x)+g_v(x)f_m(x)]\}. \tag{14}$$

Here $C_1(vm)$ is a shorthand for $C_1(\kappa_v\kappa_m)$, where

$$C_l(\kappa_m\kappa_n) \equiv (-1)^{j_m+1/2}\sqrt{(2j_m+1)(2j_n+1)} \times\begin{pmatrix} l & j_m & j_n \\ 0 & -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\Pi(l_m,l_n,l), \tag{15}$$

with the parity factor Π vanishing if the sum of the angular momenta is odd. The bracket around m means that the magnetic quantum number is not included, as it has already been summed over to obtain this coefficient.

In Ref. [1] we evaluated the real part of this in two ways. In the first we used a slightly different form for the correction that involved an integration over photon energy. For excited states the denominator of the integrand passed through zeros, and we evaluated the integral by taking principal parts. This is equivalent to taking the real part of the integral, and thus cannot be used to evaluate recoil corrections to the decay rate. Here we use the second approach, in which a Wick rotation of the q_0 integration in the above expression was carried out. For excited states this procedure involved encircling poles associated with states of lower energy, and these pole terms had both real and imaginary parts. The explicit expression for the pole terms can be written as

TABLE I. Recoil correction from ΔE_{1T} to the decay rate of $2p_{1/2}$, $2p_{3/2}$, $3s$, and $3p_{1/2}$ states. Units $2Zm/M\Gamma_0$, with Γ_0 given in Eq. (17).

	$2p_{1/2}\to 1s$	$2p_{3/2}\to 1s$	$3s\to 2p_{1/2}$	$3s\to 2p_{3/2}$	$3p_{1/2}\to 1s$	$3p_{1/2}\to 2s$
$Z=1$	1.000 043	1.000 009	0.333 335	0.666 746	1.000 036	1.000 053
$Z=5$	1.001 073	1.000 234	0.333 385	0.668 664	1.000 902	1.001 323
$Z=10$	1.004 307	1.000 935	0.333 538	0.674 684	1.003 616	1.005 310
$Z=20$	1.017 449	1.003 741	0.334 102	0.699 180	1.014 623	1.021 556
$Z=30$	1.040 125	1.008 423	0.334 869	0.741 525	1.033 518	1.049 740
$Z=40$	1.073 612	1.014 982	0.355 523	0.804 126	1.061 199	1.091 705
$Z=50$	1.119 961	1.023 399	0.335 494	0.890 615	1.099 081	1.150 464
$Z=60$	1.182 366	1.033 612	0.333 756	1.006 130	1.149 317	1.230 795
$Z=70$	1.265 837	1.045 457	0.328 421	1.157 659	1.215 178	1.340 381
$Z=80$	1.378 515	1.058 575	0.315 838	1.354 371	1.301 773	1.492 066
$Z=90$	1.534 442	1.072 193	0.288 450	1.607 598	1.417 491	1.708 783
$Z=100$	1.760 191	1.084 647	0.228 925	1.929 281	1.577 269	2.035 688

$$\Delta E_{1T} = \frac{2(Z\alpha)^2}{M}\sum_{[m]\leq v}\frac{C_1^2(vm)}{2j_v+1}\frac{1}{\omega_{mv}}\int_0^\infty \frac{dy}{y^2}R_{vm}(y) \times \int_0^\infty \frac{dx}{x}[\sqrt{3}Q^1_{mv}(x)A(-\omega_{mv}x) - S_{mv}(x)B(-\omega_{mv}x)], \tag{16}$$

where $\omega_{mv}=\epsilon_m-\epsilon_v$ is the value of q_0 at each enclosed pole. We note that this is a negative quantity, so that an explicit minus sign has been included in the arguments of the A and B terms in order to enforce the absolute value sign in Eq. (6). We also note the presence of two "half poles" associated with the $q_0+i\epsilon$ denominator and cases where ω_{mv} vanishes due to Dirac degeneracy. For these terms the pole is avoided with a half circle, which gives a relative factor of 1/2 compared to the completely encircled cases. However, these terms are purely real and are not treated here.

The numerical evaluation of this expression is straightforward: one simply replaces the real parts of the A and B functions with the corresponding imaginary parts. The remaining part of the Wick rotation involves an integral along the imaginary q_0 axis. As it is real, it makes no contribution to the decay rate, which is tabulated in Tables I and II for the states with principal quantum number up to $n=3$. The ground state is left out because it is stable, and the metastable $2s$ state is left out because it has no recoil corrections in order m/M, a fact discussed in the conclusions. The recoil corrections to the decay rate from one transverse photon diagrams are presented in terms of the nonrelativistic limit, which we now discuss.

Nonrelativistic limit. Tables I and II present the contribution to the decay rate, obtained by multiplying the imaginary part of the energy by $-2/\hbar$, as the coefficient of $2Zm/M\Gamma_0$, where Γ_0 is the well known [11] nonrelativistic decay rate, being specifically

$$\Gamma_0(2p-1s) = \left(\frac{2}{3}\right)^8\frac{mc^2Z^4\alpha^5}{\hbar},$$

$$\Gamma_0(3s-2p) = \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^8\frac{mc^2Z^4\alpha^5}{\hbar},$$

TABLE II. Recoil correction from ΔE_{1T} to the decay rate of $3p_{3/2}$, $3d_{3/2}$, and $3d_{5/2}$ states. Units $2Zm/M\Gamma_0$, with Γ_0 given in Eq. (17).

	$3p_{3/2}\to 1s$	$3p_{3/2}\to 2s$	$3d_{3/2}\to 2p_{1/2}$	$3d_{3/2}\to 2p_{3/2}$	$3d_{5/2}\to 2p_{3/2}$
$Z=1$	1.000 021	1.000 001	0.833 359	0.166 668	1.000 005
$Z=5$	1.000 523	1.000 032	0.833 980	0.166 700	1.000 115
$Z=10$	1.002 094	1.000 118	0.835 925	0.166 801	1.000 462
$Z=20$	1.008 419	1.000 328	0.843 763	0.167 204	1.001 850
$Z=30$	1.019 109	1.000 171	0.857 046	0.167 882	1.004 173
$Z=40$	1.034 386	0.998 796	0.876 107	0.168 840	1.007 445
$Z=50$	1.054 567	0.994 808	0.901 429	0.170 089	1.011 683
$Z=60$	1.080 059	0.986 003	0.933 652	0.171 643	1.016 915
$Z=70$	1.111 341	0.968 932	0.973 570	0.173 519	1.023 169
$Z=80$	1.148 913	0.938 163	1.022 095	0.175 738	1.030 483
$Z=90$	1.193 142	0.885 030	1.080 117	0.178 328	1.038 897
$Z=100$	1.243 844	0.795 522	1.148 058	0.181 322	1.048 460

$$\Gamma_0(3p-1s)=\frac{1}{96}\frac{mc^2Z^4\alpha^5}{\hbar},$$

$$\Gamma_0(3p-2s)=\frac{16}{3}\left(\frac{2}{5}\right)^9\frac{mc^2Z^4\alpha^5}{\hbar},$$

$$\Gamma_0(3d-2p)=96\left(\frac{2}{5}\right)^{11}\frac{mc^2Z^4\alpha^5}{\hbar}. \quad (17)$$

The origin of this factor, the Zm/M part of Eq. (10), is as follows. The above decay rates are allowed dipole transitions arising from the minimal substitution

$$\frac{\vec{p}^2}{2m}\rightarrow\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m}, \quad (18)$$

specifically corresponding to the cross term, which depends on the electron mass, photon polarization, and electron momentum as

$$-\frac{e}{mc}\vec{p}\cdot\vec{\epsilon}. \quad (19)$$

However, the photon can also be emitted from the nucleus. In this case there are two factors of minus one, one from the opposite charge of the nucleus and one because in the center of mass system its momentum is the negative of the electron momentum. In addition there is a factor Z associated with the nuclear charge, and the electron mass is replaced by the nuclear mass. The net effect is to add to the electron matrix element in Eq. (19) a term

$$-\frac{eZ}{Mc}\vec{p}\cdot\vec{\epsilon}, \quad (20)$$

which can be accounted for by multiplying Eq. (19) by $1+Zm/M$. As the decay rate involves the square of the matrix element and we are neglecting $(m/M)^2$ terms, the recoil correction to the nonrelativistic decay rate should be as given above. As Z increases deviations from the nonrelativistic limit are to be expected, and this is seen in Tables I and II. Further discussion will be given in the concluding section.

III. TWO TRANSVERSE PHOTON EXCHANGE

The coordinate space version of Eq. (4) is

$$\Delta E_{2T}=\frac{i(Z\alpha)^2}{M}\int\frac{dq_0}{2\pi}\sum_m\frac{1}{\epsilon_v+q_0-\epsilon_m(1-i\delta)}\times\int d^3x d^3y\frac{1}{xy}\left[A(|q_0|x)\delta_{ik}+B(|q_0|x)\frac{x_ix_k}{x^2}\right]\left[A(|q_0|y)\delta_{jk}+B(|q_0|y)\frac{y_jy_k}{y^2}\right]\psi_v^\dagger(\vec{x})\alpha_i\psi_m(\vec{x})\psi_m^\dagger(\vec{y})\alpha_j\psi_v(\vec{y}). \quad (21)$$

In Ref. [1] the q_0 integral was evaluated with the same Wick rotation as described above. As with the one-transverse photon term, this resulted in a purely real part involving an integration along the complex axis, and pole terms with both real and imaginary parts. There are four parts to the pole terms, which, after an angular momentum reduction, can be written as

$$\Delta E_{TT}(a)=-\frac{3(Z\alpha)^2}{M}\sum_m{}'\frac{C_1^2(vm)}{2j_v+1}\int_0^\infty dxQ_{vm}^1(x)\times A(-\omega_{mv}x)\int_0^\infty dyQ_{mv}^1(y)A(-\omega_{mv}y), \quad (22)$$

$$\Delta E_{TT}(b)=\frac{\sqrt{3}(Z\alpha)^2}{M}\sum_m{}'\frac{C_1^2(vm)}{2j_v+1}\int_0^\infty dxQ_{vm}^1(x)\times A(-\omega_{mv}x)\int_0^\infty dyS_{mv}(y)B(-\omega_{mv}y), \quad (23)$$

$$\Delta E_{TT}(c)=\frac{\sqrt{3}(Z\alpha)^2}{M}\sum_m{}'\frac{C_1^2(vm)}{2j_v+1}\int_0^\infty dxS_{vm}(x)\times B(-\omega_{mv}x)\int_0^\infty dyQ_{mv}^1(y)A(-\omega_{mv}y), \quad (24)$$

and

$$\Delta E_{TT}(d)=-\frac{(Z\alpha)^2}{M}\sum_m{}'\frac{C_1^2(vm)}{2j_v+1}\int_0^\infty dxS_{vm}(x)\times B(-\omega_{mv}x)\int_0^\infty dyS_{mv}(y)B(-\omega_{mv}y). \quad (25)$$

We note that an extra factor of $1/(2\pi)^3$ is present in Eqs. (80)–(83) in Ref. [1] compared to these expressions: the present result is the correct one. As A and B are both complex, both their real and imaginary parts were needed in the calculation of the energy shifts, and only a slight rearrangement of the calculation is needed to determine the imaginary part, and thus the decay rate. This rate begins in order $m^2/M(Z\alpha)^7$, down by a factor of $(Z\alpha)^2$ from the leading contribution. We present in Tables III and IV its value in terms of Γ_0.

IV. CONCLUSION

We first discuss the fact that our results are incomplete, as can be seen from Eq. (10), which has in the nonrelativistic limit both the $2Zm/M$ term treated here and a scaling factor $(m_r/m_e)^3$ which would contribute $-3m/M$ had we included it. However, this contribution is well known to arise from accounting for recoil in the self-energy diagram. The imaginary part of this diagram is, of course, the decay rate when multiplied by $-2i$, and the fact that recoil can be accounted for by including the above mentioned scaling factor is discussed in many references.

TABLE III. Recoil correction from ΔE_{TT} to the decay rate of $2p_{1/2}$, $2p_{3/2}$, $3s$, and $3p_{1/2}$ states. Units $2Zm/M\Gamma_0$, with Γ_0 given in Eq. (17).

	$2p_{1/2}\rightarrow 1s$	$2p_{3/2}\rightarrow 1s$	$3s\rightarrow 2p_{1/2}$	$3s\rightarrow 2p_{3/2}$	$3p_{1/2}\rightarrow 1s$	$3p_{1/2}\rightarrow 2s$
$Z=1$	−0.000 047	−0.000 017	0.000 008	0.000 013	−0.000 053	−0.000 020
$Z=5$	−0.001 166	−0.000 416	0.000 204	0.000 315	−0.001 332	−0.000 500
$Z=10$	−0.004 679	−0.001 663	0.000 821	0.001 262	−0.005 343	−0.002 011
$Z=20$	−0.018 960	−0.006 637	0.003 371	0.005 109	−0.021 593	−0.008 217
$Z=30$	−0.043 606	−0.014 878	0.007 934	0.011 700	−0.049 437	−0.019 166
$Z=40$	−0.080 016	−0.026 310	0.015 047	0.021 251	−0.090 121	−0.035 888
$Z=50$	−0.130 456	−0.040 824	0.025 630	0.033 919	−0.145 626	−0.060 106
$Z=60$	−0.198 471	−0.058 271	0.041 229	0.049 603	−0.218 968	−0.094 639
$Z=70$	−0.289 655	−0.078 448	0.064 510	0.067 559	−0.314 735	−0.144 195
$Z=80$	−0.413 116	−0.101 078	0.100 297	0.085 685	−0.440 041	−0.216 987
$Z=90$	−0.584 556	−0.125 754	0.158 000	0.099 179	−0.606 397	−0.328 391
$Z=100$	−0.833 519	−0.151 827	0.258 058	0.097 970	−0.833 734	−0.510 319

We next discuss the behavior of the recoil corrections to decay rates along the isoelectronic sequence. At low Z the two-transverse photon diagram is suppressed by two orders of α, and the recoil correction is close to the factor $2Zm/M$ discussed above. We note that this result is considerably simpler than the recoil correction for the real part of the energy, the so-called Salpeter correction [12]. That correction, while part of it is proportional to Zm/M times the Lamb shift, also has contributions that are not proportional. For the $3s\rightarrow 2p$ decay we note the 2/3 vs 1/3 weighting and for the $3d\rightarrow 2p$ the 5/6 vs 1/6 weighting, both of which follow from the Clebsh-Gordon factor $C_1(vm)$.

We next turn to the high Z behavior of recoil corrections to decay rates, though of course the higher nuclear mass will make experimental detection difficult. While the deviation from the nonrelativistic result for $2p_{3/2}\rightarrow 1s$ is never greater than 10%, the $2p_{1/2}\rightarrow 1s$ result is more sensitive to relativistic effects, changing by almost a factor of 2 by $Z=100$. A similar effect is seen when comparing decays of the $3p_{3/2}$ states to $1s$ and $2s$ compared to the decays of $3p_{1/2}$ states. The enhanced sensitivity of $p_{1/2}$ states to relativistic effects is commonly encountered, and is connected to the fact that the lower component of these states is a penetrating s state. Finally, d states have nonpenetrating lower components, and their decay rates deviate relatively little from the nonrelativistic expression.

With regard to experiment, we note an interesting recent work by Volotka *et al.* [13], which was concerned with a discrepancy in an M1 decay in boronlike argon, explictly investigating whether recoil corrections could account for it, with a negative result. We again note the strong enhancement of recoil effects for muonic atoms.

An interesting feature of our calculation has to do with the $2s$ decay rate. It vanishes identically in our formalism, which is valid to first order in m/M. However, in the nonrecoil limit it is well known that, while highly suppressed at low Z, this state has a one-photon $M1$ decay mode, which dominates the two-photon mode at high Z. However, our result for one-

TABLE IV. Recoil correction from ΔE_{TT} to the decay rate of $3p_{3/2}$, $3d_{3/2}$, and $3d_{5/2}$ states. Units $2Zm/M\Gamma_0$, with Γ_0 given in Eq. (17).

	$3p_{3/2}\rightarrow 1s$	$3p_{3/2}\rightarrow 2s$	$3d_{3/2}\rightarrow 2p_{1/2}$	$3d_{3/2}\rightarrow 2p_{3/2}$	$3d_{5/2}\rightarrow 2p_{3/2}$
$Z=1$	−0.000 018	−0.000 014	−0.000 007	−0.000 002	−0.000 005
$Z=5$	−0.000 444	−0.000 361	−0.000 185	−0.000 060	−0.000 129
$Z=10$	−0.001 775	−0.001 444	−0.000 742	−0.000 241	−0.000 518
$Z=20$	−0.007 094	−0.005 802	−0.003 005	−0.000 964	−0.002 070
$Z=30$	−0.015 945	−0.013 142	−0.006 895	−0.002 177	−0.004 654
$Z=40$	−0.028 309	−0.023 569	−0.012 603	−0.003 888	−0.008 267
$Z=50$	−0.044 168	−0.037 188	−0.020 417	−0.006 114	−0.012 904
$Z=60$	−0.063 508	−0.054 054	−0.030 749	−0.008 873	−0.018 557
$Z=70$	−0.086 327	−0.074 050	−0.044 176	−0.012 191	−0.025 219
$Z=80$	−0.112 634	−0.096 654	−0.061 491	−0.016 099	−0.032 880
$Z=90$	−0.142 435	−0.120 441	−0.083 784	−0.020 638	−0.041 531
$Z=100$	−0.175 659	−0.142 010	−0.112 533	−0.025 854	−0.051 158

transverse photon energy shift is purely real for this state, indicating that recoil corrections enter in order $(m/M)^2$ or higher. This follows because of the factor $C_1(vm)$, which vanishes by parity for $2s \rightarrow 1s$.

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