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Evolution of entanglement for quantum mixed states

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A simple relation is introduced for concurrence to describe how much the entanglement of a bipartite system is at least left if either (or both) subsystem undergoes an arbitrary physical process. This provides a lower bound for concurrence of mixed states (pure states are included) in contrast to the upper bound given by Konrad *et al.* [Nat. Phys. **4**, 99 (2008)]. Our results are also suitable for general high-dimensional bipartite quantum systems.

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I. INTRODUCTION

Quantum entanglement is an important physical resource in quantum information and computation tasks such as quantum teleportation [1], quantum key distribution [2], quantum computation [3], and so on. In realistic quantum-information processing, entanglement must be prepared or distributed beforehand by two distant parties in which one or more physical systems must be transmitted by a quantum channel. However, unlike classical systems, quantum systems are usually fragile. It is inevitable that environment (channel) will influence the systems of interest more or less and induce decoherence because of the interaction with the systems, so that entanglement is destroyed to some extent before use [4–6]. It is a key task to evaluate the shared entanglement after the influence of environment.

In usual, one must deduce the time evolution of entanglement of the composite system from the time evolution of the quantum state under consideration [7], when the subsystems of the composite quantum system undergo a physical process. That is to say, for the different potential initial states one must repeat the same procedure every time. Quite recently, an important step has been taken by Konrad *et al.* [8] who provided an explicit evolution equation of quantum entanglement quantified by the remarkable concurrence [9] for a bipartite quantum state of qubits. It has been shown that given any one-sided quantum channel, the concurrence of output state corresponding to any initial pure input state of interest can always be equivalently obtained by the product of the concurrence of input state and that of the output state with the maximally entangled state as an input state. However, for a two-sided quantum channel or the initial mixed states, the product of the two concurrences only provides an upper bound for the concurrence of interest. It is obviously important to find a lower bound of the concurrence in order to well grasp the entanglement of output states.

In fact, quantum-mechanics and quantum-information processing (QIP) are not constrained to pure states as well as $(2\otimes 2)$-dimensional quantum systems. Because of imperfect experiments and inevitable disturbance of environments, mixed (entangled) states are ubiquitous, which is inevitable for QIP to cope with. Furthermore, on the one hand, when we face the quantum features, especially entanglement, of mesoscopic even macroscopic quantum systems, such as Bose-Einstein condensates [10], we must deal with high-dimensional density matrices. On the other hand, it has been shown that some QIP tasks based on high-dimensional entangled states are more efficient than those of qubits. For example, cryptographic protocols are more secure based on a quantum channel of qutrits [11–13]. Teleportation can be implemented in faith even though a nonmaximally entangled quantum channel is shared [14]. Therefore, for these large systems, it is necessary to investigate the entanglement of high-dimensional quantum systems (multiparties included).

However, quantification of entanglement, as a precondition of studying entanglement, is generally a hard problem which does not only lie in the poor practicability for high-dimensional quantum systems [15,16] but also the nonlinearity on density matrices for usual entanglement measures [17,18]. Therefore, it is often suggested to derive a lower bound to evaluate the entanglement (entanglement can be better evaluated, if both upper and lower bounds are given). In this paper, we consider the evolution of entanglement with either (or both) subsystem undergoing an arbitrary quantum channel. With concurrence as entanglement measure, we find a lower bound of the concurrence of any output states for a given quantum channel based on the evolution of a probe state as input states. It is shown that our lower bound is not restricted to the bipartite systems of qubits. In particular, for the one-sided quantum channel, our lower bound has a concise form. Furthermore, we also show that it is not necessary to choose the maximally entangled state as the probe states. The paper is organized as follows. In Sec. II, we give a lower bound for the concurrence of bipartite quantum systems; in Sec. III, we show that the lower bound can be obtained in terms of the initial input state and the output state with the probe state as initial states. Finally the conclusion is drawn.

II. LOWER BOUND FOR CONCURRENCE

An $(N_1\otimes N_2)$-dimensional bipartite quantum pure state can be written as

$$|\psi\rangle_{AB}=\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\psi_{ij}|ij\rangle, \tag{1}$$

where $|ij\rangle$ denotes the computational basis and $N_1\times N_2$ matrix ψ (i.e., $|\psi\rangle_{AB}$ without $|\cdots\rangle_{AB}$) represents the matrix nota-

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tion [19,20] of $|\psi\rangle_{AB}$ with matrix element $\psi_{ij}=\langle ij|\psi\rangle_{AB}$ and $\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}|\psi_{ij}|^2=1$. Consider the Schmidt decomposition, $|\psi\rangle_{AB}$ can also be given by

$$|\psi\rangle_{AB}=\sum_{i=0}^{R-1}\lambda_i|ii\rangle, \quad \sum_i \lambda_i^2=1, \tag{2}$$

with λ_i being real in decreasing order and $R=\min\{N_1,N_2\}$. The concurrence of $|\psi\rangle_{AB}$ can be defined [21,22] as

$$C(|\psi\rangle_{AB})=\sqrt{\sum_{\substack{i,j=0(3)\\ i\neq j}}^{N_1-1}\sum_{\substack{p,q=0\\ p\neq q}}^{N_2-1}|\psi_{ip}\psi_{jq}-\psi_{iq}\psi_{jp}|^2} \tag{3}$$

$$=\sqrt{4\sum_{\substack{i,j=0\\ i<j}}^{R-1}\lambda_i^2\lambda_j^2}. \tag{4}$$

The equivalence of Eqs. (3) and (4) lies in that quantum pure states are related to its Schmidt decomposition by local unitary transformations which do not contribute to concurrence. In particular, if $N_1=N_2=2$, Eq. (4) can be reduced to $C(|\psi\rangle_{AB})=2\lambda_1\lambda_2=2|\det(\psi)|$, which will be used later. With the definitions given by Eqs. (3) and (4), we obtain the following theorem.

Theorem 1. For any $(N_1\otimes N_2)$-dimensional bipartite quantum pure state $|\psi\rangle$,

$$C(|\psi\rangle)\geq\sqrt{\frac{2R}{R-1}}\left(\max_{|\phi\rangle\in\mathcal{E}}|\langle\psi|\phi\rangle|^2-\frac{1}{R}\right) \tag{5}$$

with $\mathcal{E}$ denoting the set of $(N_1\otimes N_2)$-dimensional maximally entangled states.

Proof. Since $\max_{|\phi\rangle\in\mathcal{E}}|\langle\psi|\phi\rangle|^2$ is not changed by any local unitary transformation on $|\psi\rangle$, $|\psi\rangle$ can always be understood in the form of Schmidt decomposition, i.e., $|\psi\rangle_{AB}=\sum_{i=0}^{R-1}\lambda_i|ii\rangle$. The maximally entangled state can be written as $|\phi\rangle=\frac{1}{\sqrt{R}}\sum_{i=0}^{R-1}(U_1\otimes U_2)|ii\rangle$, with U_1 and U_2 denoting local unitary transformations. Suppose $a_i=\langle ii|\phi\rangle$, $a_i\in[0,\frac{1}{\sqrt{R}}]$, then $\max_{|\phi\rangle\in\mathcal{E}}|\langle\psi|\phi\rangle|^2$ can be rewritten as

$$\max_{|\phi\rangle\in\mathcal{E}}|\langle\psi|\phi\rangle|^2=\max_{|\phi\rangle\in\mathcal{E}}\left|\sum_{i=0}^{R-1}a_i\lambda_i\right|^2\leq\max_{|\phi\rangle\in\mathcal{E}}\left(\sum_{i=0}^{R-1}|a_i\lambda_i|\right)^2$$
$$\leq\frac{1}{R}\left(\sum_{i=0}^{R-1}\lambda_i\right)^2=\frac{1}{R}\left(1+2\sum_{i,j=0;i<j}^{R-1}\lambda_i\lambda_j\right) \tag{6}$$

$$\leq\frac{1}{R}\left(1+2\sqrt{\frac{R(R-1)}{2}\sum_{i,j=0;i<j}^{R-1}\lambda_i^2\lambda_j^2}\right) \tag{7}$$

$$=\frac{1}{R}\left(1+\sqrt{\frac{R(R-1)}{2}}C(|\psi\rangle)\right). \tag{8}$$

We arrive at the inequality (7) based on the inequality $(\sum_{i=1}^n x_i)^2/n\leq\sum_{i=1}^n x_i^2$. Thus from inequality (8), we can obtain

$$C(|\psi\rangle)\geq\sqrt{\frac{2R}{R-1}}\left(\max_{|\phi\rangle\in\mathcal{E}}|\langle\psi|\phi\rangle|^2-\frac{1}{R}\right). \tag{9}$$

We would like to emphasize that the equal sign in Eq. (9) is always achieved for the pure states of two qubits. That is to say, the right-hand side of Eq. (9) is just the concurrence of two-qubit pure states. However, for a general high-dimensional quantum systems, the equal sign holds only for maximally entangled states [in this case the equal sign in Eq. (7) holds], which shows that the inequality (9) provides a lower bound for concurrence in high dimension. ■

Since the maximum of Eq. (6) is obtained with $|a_i|=\frac{1}{\sqrt{R}}$, $|\phi\rangle$ can be conveniently chosen as $|\tilde{\phi}\rangle=\frac{1}{\sqrt{R}}\sum_{i=0}^{R-1}|ii\rangle$. Therefore, a lower bound of concurrence can be given by

$$C(|\psi\rangle)\geq\sqrt{\frac{2R}{R-1}}\left(\mathrm{Tr}(\rho|\tilde{\phi}\rangle\langle\tilde{\phi}|)-\frac{1}{R}\right) \tag{10}$$

with $\rho=|\psi\rangle\langle\psi|$. It is obvious that the lower bound in Eq. (10) is less than that in Eq. (9). But it can be used conveniently because there does not exist maximization problem.

The inequality (10) can be immediately generalized to mixed states. Concurrence for any $(N_1\otimes N_2)$-dimensional mixed state ρ is defined as $C(\rho)=\min\sum p_iC(|\varphi_i\rangle)$, where the minimum is taken over all possible decompositions such that $\rho=\sum p_i|\varphi_i\rangle\langle\varphi_i|$, $\sum p_i=1$. Based on the optimal decomposition $\rho=\sum q_i|\chi_i\rangle\langle\chi_i|$ such that $C(\rho)=\sum q_iC(|\chi_i\rangle)$, one can get

$$C(\rho)=\sum q_iC(|\chi_i\rangle)$$
$$\geq\sqrt{\frac{2R}{R-1}}\sum q_i\left(\mathrm{Tr}[|\chi_i\rangle\langle\chi_i||\tilde{\phi}\rangle\langle\tilde{\phi}|]-\frac{1}{R}\right)$$
$$=\sqrt{\frac{2R}{R-1}}\left(\mathrm{Tr}[\rho|\tilde{\phi}\rangle\langle\tilde{\phi}|]-\frac{1}{R}\right). \tag{11}$$

Inequality (11) holds for any bipartite quantum states which provides the key result that will be used later.

III. EVOLUTION OF CONCURRENCE

A. One-sided quantum channel

Next, we will show that Eq. (11) can be captured by the evolution of some probe states. Let us first consider an $(N\otimes N)$-dimensional bipartite quantum state ρ with only one subsystem undergoing a quantum channel represented by the superoperator $\$_1$, then the final state can be given by $\rho_f=\frac{(\$_1\otimes\mathbf{1})\rho}{p}$, where $p=\mathrm{Tr}[(\$_1\otimes\mathbf{1})\rho]$ is the probability for channel $\$_1$ which corresponds to non-trace-preserving channel [12]. Any quantum state can be expanded in a representation spanned by maximally entangled states given by

$$|\Phi_j\rangle=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}e^{i(2j_0k\pi/n)}|k\rangle|k\oplus j_1\rangle, \quad j=Nj_0+j_1, \tag{12}$$

where $j_0,j_1=0,1,\ldots,N-1,|k\rangle$ is the computational basis and $\oplus$ denotes the addition modulo N. A maximally entangled state $|\Phi_m\rangle$ can also be written as

$$|\Phi_m\rangle = (\Phi_m P^{-1} \otimes \mathbf{1})|P\rangle, \tag{13}$$

where $|P\rangle=\Sigma_{i,j=0}^{N-1}a_{ij}|ij\rangle$, called probe quantum state in this paper, is a generic entangled pure state with full-rank P [which can be explicitly written as

$$P = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0(N-1)} \\ a_{10} & a_{11} & \cdots & a_{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(N-1)0} & a_{(N-1)1} & \cdots & a_{(N-1)(N-1)} \end{pmatrix},$$

or can be directly obtained by the method provided below Eq. (1)] and P^{-1} denotes the inverse matrix of P. Φ_m are simple unitary transformations determined by Eq. (12). For example, for the state of a pair of qubits, Φ_m correspond to the three Pauli matrices and the identity, respectively. Since any quantum state $|\psi\rangle$ can be given [19], based on maximally entangled state $|\tilde{\phi}\rangle$, by

$$|\psi\rangle = \sqrt{R}(\psi \otimes \mathbf{1})|\tilde{\phi}\rangle = \sqrt{R}(\mathbf{1} \otimes \psi^T)|\tilde{\phi}\rangle, \tag{14}$$

$|\psi\rangle$ can also be written as

$$|\psi\rangle = (\psi P^{-1} \otimes \mathbf{1})|P\rangle = [\mathbf{1} \otimes \psi^T (P^{-1})^T]|P\rangle, \tag{15}$$

where the superscript T denotes transpose. Hence, we have

$$\begin{aligned} \mathrm{Tr}[\rho_f|\tilde{\phi}\rangle\langle\tilde{\phi}|] &= \frac{1}{p}\mathrm{Tr}[(\$_1 \otimes \mathbf{1})\rho|\tilde{\phi}\rangle\langle\tilde{\phi}|] \\ &= \frac{1}{p}\mathrm{Tr}[\rho(\$_1^\dagger \otimes \mathbf{1})(|\tilde{\phi}\rangle\langle\tilde{\phi}|)] \\ &= \frac{1}{p}\mathrm{Tr}[S\rho^* S(\$_1 \otimes \mathbf{1})(|\tilde{\phi}\rangle\langle\tilde{\phi}|)], \end{aligned} \tag{16}$$

where S is the swapping operator defined as $S|j\rangle|k\rangle=|k\rangle|j\rangle$ and we apply Eq. (14) and $S|\tilde{\phi}\rangle=|\tilde{\phi}\rangle$ to the third equal sign. Equation (16) can be understood by the Kraus representation of superoperator $\$_1$ [23]. Based on Eq. (13) and Eq. (15), Eq. (16) can arrive at

$$\begin{aligned} \mathrm{Tr}[\rho_f|\tilde{\phi}\rangle\langle\tilde{\phi}|] &= \frac{1}{p}\mathrm{Tr}\, S\rho^* S(\$_1 \otimes \mathbf{1})\{[\mathbf{1} \otimes \tilde{\phi}^T(P^{-1})^T] \\ &\quad \times |P\rangle\langle P|\mathbf{1} \otimes (P^{-1})^*\tilde{\phi}^*\} \\ &= \frac{1}{p_t R}\mathrm{Tr}\, S\rho^* S[\mathbf{1} \otimes (P^{-1})^T] \\ &\quad \times \left(\frac{(\$_1 \otimes \mathbf{1})|P\rangle\langle P|}{p'}\right)[\mathbf{1} \otimes (P^{-1})^*], \end{aligned} \tag{17}$$

where $p'=\mathrm{Tr}(\$_1 \otimes \mathbf{1})|P\rangle\langle P|$, $p_t=p/p'$ and the asterisk denotes conjugate operation. An alternative derivation can be done by substituting $\rho=\Sigma p_i|\varphi_i\rangle\langle\varphi_i|$ into the first line of Eq. (16). In a representation of maximally entangled states, we can have

$$\begin{aligned} p_t &= \mathrm{Tr}[(\$_1 \otimes \mathbf{1})\rho]/p' \\ &= \mathrm{Tr}\sum_m [(\$_1 \otimes \mathbf{1})\rho]|\Phi_m\rangle\langle\Phi_m|/p' \\ &= \mathrm{Tr}\sum_m \left(\frac{(\$_1 \otimes \mathbf{1})|P\rangle\langle P|}{p'}\right) \\ &\quad \times [\Phi_m \otimes (P^{-1})^*]S\rho^* S[\Phi_m^\dagger \otimes (P^{-1})^T]. \end{aligned} \tag{18}$$

In fact, if the reduced density of the initial state ρ is considered, p_t has an alternative and concise form. Let $\rho_A =\mathrm{Tr}_B\rho_{AB}=\mathrm{Tr}_B\rho$ with Tr_B denoting trace over subsystem B. Consider a decomposition of $\rho=\Sigma p_i|\varphi_i\rangle\langle\varphi_i|$, ρ_A can be rewritten as $\rho_A=\Sigma p_i\varphi_i\varphi_i^\dagger$. Thus p_t can also be given by

$$p_t = \mathrm{Tr}\left(\frac{(\$_1 \otimes \mathbf{1})|P\rangle\langle P|}{p'}\right)\{\mathbf{1} \otimes [P^{-1}\rho_A(P^{-1})^\dagger]^*\}. \tag{19}$$

Equation (19) has a concise form without summation. Substituting Eq. (18) or Eq. (19) into Eq. (17), one can find that $\mathrm{Tr}[\rho_f|\tilde{\phi}\rangle\langle\tilde{\phi}|]$ has been given by a simple algebra on the evolution of the probe state $|P\rangle$ and the original density matrix. That is to say, the lower bound of concurrence in Eq. (11) can be captured by the evolution of the given probe state which can be formally written as

$$C[(\$_1 \otimes \mathbf{1})\rho] \geqslant \sqrt{\frac{2R}{R-1}}\left\{\mathrm{Tr}\left[f\left(\frac{(\$_1 \otimes \mathbf{1})|P\rangle\langle P|}{p'}\right)\rho^*\right] - \frac{1}{R}\right\}, \tag{20}$$

where $f(x)=\frac{1}{p_t R}S[\mathbf{1}\otimes(P^{-1})^T][x][\mathbf{1}\otimes(P^{-1})^*]S$. It is obvious that the lower bound of concurrence is determined by the evoluted probe state.

B. Two-sided quantum channel

Equation (20) can immediately be generalized to the case of a two-sided quantum channel, however the form might not be as simple as Eq. (17). Since the lower bound given in Eq. (17) is valid for mixed initial states, the lower bound for two-sided quantum channel can be easily obtained by replacing ρ in Eq. (17) by $(\mathbf{1}\otimes\$_2)\rho$. Thus $\mathrm{Tr}[|\tilde{\phi}\rangle\langle\tilde{\phi}|(\$_1\otimes\$_2)\rho]$ can be written as

$$\begin{aligned} \mathrm{Tr}[|\tilde{\phi}\rangle\langle\tilde{\phi}|(\$_1 \otimes \$_2)\rho] &= \mathrm{Tr}[|\tilde{\phi}\rangle\langle\tilde{\phi}|(\$_1 \otimes \mathbf{1})(\mathbf{1} \otimes \$_2)\rho] \\ &= \frac{1}{R}\mathrm{Tr}\sum_k \{[\mathbf{1} \otimes (P^{-1})^T][(\$_1 \otimes \mathbf{1})|P\rangle\langle P|][\mathbf{1} \otimes (P^{-1})^*]S[(\mathbf{1} \otimes \$_2)|\Psi_{\rho k}\rangle\langle\Psi_{\rho k}|]^* S\}, \end{aligned} \tag{21}$$

where we have replaced ρ by a potential decomposition of $\rho=\Sigma_k|\Psi_{\rho k}\rangle\langle\Psi_{\rho k}|$ which is especially referred to as the eigenvalue decomposition for simplicity. Applying Eq. (15) to $|\Psi_{\rho k}\rangle$, Eq. (21) leads to

$$\mathrm{Tr}[|\tilde{\phi}\rangle\langle\tilde{\phi}|(\$_1\otimes\$_2)\rho]=\frac{1}{R}\mathrm{Tr}\sum_k\{[\mathbf{1}\otimes(P^{-1})^T][(\$_1\otimes\mathbf{1})|P\rangle\langle P|][\mathbf{1}\otimes(P^{-1})^*]S(\Psi_{\rho k}P^{-1}\otimes\mathbf{1})^*[(\mathbf{1}\otimes\$_2)|P\rangle\langle P|]^*[(P^{-1})^\dagger\Psi_{\rho k}^\dagger\otimes\mathbf{1}]^*S\}$$

$$=\frac{1}{R}\mathrm{Tr}\sum_k\{[(\$_1\otimes\mathbf{1})|P\rangle\langle P|]^*(\mathbf{1}\otimes P^{-1}\Psi_{\rho k}P^{-1})S[(\mathbf{1}\otimes\$_2)|P\rangle\langle P|]S[\mathbf{1}\otimes(P^{-1}\Psi_{\rho k}P^{-1})^\dagger]\}. \quad (22)$$

Substituting Eq. (21) into Eq. (17), one can find that the lower bound of concurrence $C[(\$_1\otimes\$_2)\rho]$ can be captured by the evolutions of the probe state under the two quantum channels. In order to avoid the decomposition of the initial state ρ, one can expand $[(\mathbf{1}\otimes\$_2)|P\rangle\langle P|]$ in the representation of maximally entangled states. Thus, Eq. (22) can be rewritten as

$$\mathrm{Tr}[|\tilde{\phi}\rangle\langle\tilde{\phi}|(\$_1\otimes\$_2)\rho]=\frac{1}{R}\mathrm{Tr}\sum_{mnk}\langle\Phi_m|[(\mathbf{1}\otimes\$_2)|P\rangle\langle P|]|\Phi_n\rangle[(\$_1\otimes\mathbf{1})|P\rangle\langle P|]^*[(\mathbf{1}\otimes P^{-1}\Psi_{\rho k}P^{-1})]S|\Phi_m\rangle\langle\Phi_n|S[\mathbf{1}\otimes(P^{-1}\Psi_{\rho k}P^{-1})^\dagger]$$

$$=\frac{1}{R}\mathrm{Tr}\sum_{mn}\langle\Phi_m|[(\mathbf{1}\otimes\$_2)|P\rangle\langle P|]|\Phi_n\rangle[(\$_1\otimes\mathbf{1})|P\rangle\langle P|]^*\{[\Phi_m^T(P^{-1})^T\otimes P^{-1}]\}S\rho S[(P^{-1})^*\Phi_n^*\otimes(P^{-1})^\dagger]. \quad (23)$$

Finally, it is worth noting that the maximally entangled state is a special choice of our probe states. Furthermore, the value of the lower bound of concurrence does not depend on the choice of probe state. It is obvious that Eq. (17) and Eq. (18) correspond to the trace-preserving quantum channels. The general results for non-trace-preserving channels are omitted here, which can be directly given by adding some normalization constants like the case of one-sided quantum channel.

For integrity, we show that the upper bound given in Ref. [8] can be captured by the given probe state $|P\rangle$, but the value of the bound is not changed. Suppose that $|\psi\rangle$ is a bipartite quantum state of qubits, then the concurrence can be given by

$$C(|\psi\rangle)=2|\det(\psi)|. \quad (24)$$

If one of the subsystems undergoes a quantum channel $\$_1$, the final state can be given by $\rho_f=(\$_1\otimes\mathbf{1})|\psi\rangle\langle\psi|/p_1$, with $p_1=\mathrm{Tr}(\$_1\otimes\mathbf{1})|\psi\rangle\langle\psi|$. Thus we have

$$\rho_f(\sigma_y\otimes\sigma_y)\rho_f^*(\sigma_y\otimes\sigma_y)=\left(\frac{\det(\psi)}{p_1p_2\det(P)}\right)^2\times\rho_P(\sigma_y\otimes\sigma_y)\rho_P^*(\sigma_y\otimes\sigma_y), \quad (25)$$

with $\rho_P=(\$_1\otimes\mathbf{1})|P\rangle\langle P|/p_2$ and $p_2=\mathrm{Tr}(\$_1\otimes\mathbf{1})|P\rangle\langle P|$. Based on Eq. (25), we can obtain

$$C(\rho_f)=\left|\frac{\det(\psi)}{\det(P)}\right|C(\rho_P)=\frac{C(|\psi\rangle)C(\rho_P)}{2|\det(P)|}. \quad (26)$$

It is obvious for a mixed initial state ϱ that Eq. (25) is extended to

$$C(\varrho_f)\leqslant\frac{C(\varrho)C(\rho_P)}{2|\det(P)|}. \quad (27)$$

For two-sided quantum channel, one can easily obtain

$$C(\varrho_f)\leqslant\frac{C(\varrho)C(\rho_{P1})}{2|\det(P)|}\frac{C(\rho_{P2})}{2|\det(P)|}, \quad (28)$$

with $\rho_{P1}=(\$_1\otimes\mathbf{1})|P\rangle\langle P|/p_2$, $\rho_{P2}=(\mathbf{1}\otimes\$_2)|P\rangle\langle P|/p_2'$ and $p_2'=\mathrm{Tr}(\mathbf{1}\otimes\$_2)|P\rangle\langle P|$.

Thus concurrence (especially bipartite concurrence of qubits) can be better evaluated by the lower bound and the upper bound given in Eq. (28) than by only one bound. From Eq. (20) as well as Eq. (22), one might think that it is not so convenient compared with the upper bound in Ref. [8], because the upper bound is a simple linear relationship between the concurrence of evolved probe state and that of the initial states. However, generally speaking, concurrence is not a direct observable [24–26], therefore, to evaluate concurrence of a state in practice, one must evaluate the quantum state by quantum state tomography [27] and then turn to a mathematical procedure. In other words, it is inevitable for Ref. [8] to evaluate quantum states in a practical scenario. In this sense, we think that their practicability is almost the same. What is more, our lower bound has obvious advantages: (1) The lower bound has the consistent spirit with entanglement measures of mixed states for which the lower bounds (infimum) are usually needed; (2) bounds, especially lower bounds, with elegant forms like Ref. [8] might be difficult to provided. In particular, so far there have not been analytic results of entanglement measure (in particular, concurrence included) for general high-dimensional quantum systems, therefore, if the bounds of entanglement for high-dimensional mixed states still include the calculation of high-dimensional entanglement measures, it only formally provides an elegant relationship, but it has usually poor practicability. (3) The derivation based on our extended lower bound provides a universal method for all analytic bounds of entanglement measure, with which one can only focus all the attention on the tightness of the bounds, but there might be a great deal of difference on the complexity of the final results between different lower bounds.

IV. SIMPLE APPLICATION

As an application, we only consider a bipartite system of qubits, because there exist analytic concurrence and acceptable upper bounds for bipartite mixed state of qubits. Thus one can directly find the tightness of our lower bound by comparing it with the concurrence and the upper bound. The bipartite quantum state we considered is given by

$$\rho = x\rho_r + \frac{(1-x)}{4}1, \quad x \in [0,1] \tag{29}$$

with

$$\rho_r = \begin{pmatrix} 0.4322 & 0.2113 & 0.1073 & 0.3369 \\ 0.2113 & 0.1845 & 0.0406 & 0.1798 \\ 0.1073 & 0.0406 & 0.0504 & 0.1144 \\ 0.3369 & 0.1798 & 0.1144 & 0.3330 \end{pmatrix} \tag{30}$$

randomly generated by MATLAB 6.5. We suppose that each subsystem of ρ undergoes an amplitude-damping quantum channel given in Kraus representation [28] as

$$\$_1 : M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{0.8} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \sqrt{0.2} \\ 0 & 0 \end{pmatrix}; \tag{31}$$

$$\$_2 : \widetilde{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{0.7} \end{pmatrix}, \quad \widetilde{M}_2 = \begin{pmatrix} 0 & \sqrt{0.3} \\ 0 & 0 \end{pmatrix}. \tag{32}$$

The final state can be written as

$$\rho_f = (\$_1 \otimes \$_2)\rho. \tag{33}$$

The upper bound, the concurrence itself and the lower bound are given in Fig. 1, respectively, from which we can find that our lower bound is a good evaluation of concurrence. The probe state can be chosen freely.

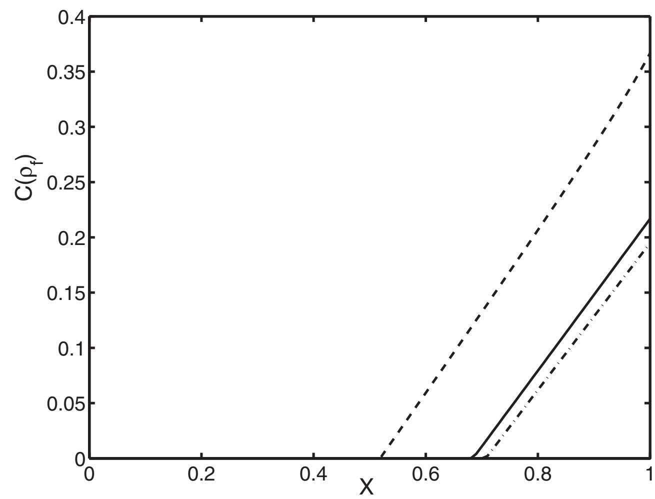

FIG. 1. (Dimensionless) The upper bound of concurrence (dashed line) given in Ref. [12], the concurrence (solid line) and the lower bound (dashed-dotted line) of quantum state ρ_f vs x. ρ_f is the final output state of an initial state ρ with the subsystems undergoing a quantum channel $\$_1$ and $\$_2$, respectively.

V. CONCLUSION AND DISCUSSION

We have presented a lower bound of concurrence. In particular, when the quantum state under consideration evolves under a quantum channel, the lower bound of concurrence can be completely captured by the evolution of probe states. Thus, the evolution of concurrence for any initial state can be well evaluated by the lower bound and the upper bound. Furthermore, the lower bound is also suitable for high-dimensional quantum state. We would like to emphasize that, even though the form of the lower bound seems not to be as elegant as that of the upper bound, but their practicality is almost the same.

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