State transfer in highly connected networks and a quantum Babinet principle

D. I. Tsomokos,[1] M. B. Plenio,[2,3] I. de Vega,[4] and S. F. Huelga[1]
[1]*Quantum Physics Group, STRI, School of Physics, Astronomy & Mathematics, University of Hertfordshire, Hatfield AL10 9AB, United Kingdom*
[2]*Institute for Mathematical Sciences, Imperial College London, London SW7 2PG, United Kingdom*
[3]*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, United Kingdom*
[4]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, Garching D-85748, Germany*
(Received 16 August 2008; published 5 December 2008)

The transfer of a quantum state between distant nodes in two-dimensional networks is considered. The fidelity of state transfer is calculated as a function of the number of interactions in networks that are described by regular graphs. It is shown that perfect state transfer is achieved in a network of size N, whose structure is that of an $(N/2)$-cross polytope graph, if N is a multiple of 4. The result is reminiscent of the Babinet principle of classical optics. A quantum Babinet principle is derived, which allows for the identification of complementary graphs leading to the same fidelity of state transfer, in analogy with complementary screens providing identical diffraction patterns.

DOI: 10.1103/PhysRevA.78.062310 PACS number(s): 03.67.−a, 75.10.Pq

I. INTRODUCTION

The dynamics of quantum many-body systems offers a rich variety of features. This quantum dynamics is often investigated in one-dimensional chains, which are amenable to exact analytical treatment in some cases [1] and, in other cases, efficient numerical methods have been developed for their simulation [2]. For more general lattice structures, henceforth referred to as *graphs*, few analytical treatments are known. An important problem that arises in this context is the interplay between the dynamics of quantum many-body systems and the properties of the underlying graph, which determines the interaction structure of the many-body system. A variety of interesting phenomena, two examples of which are perfect state transfer [3,4] and the possibility of deciding the graph isomorphism problem [5], have recently been explored in such complex quantum systems.

In the case of graphs with uniform nearest neighbor coupling, *perfect* state transfer (PST) has so far been proven possible only with rings of $N=4$ spins, chains of $N=2$ or 3 spins, and Cartesian products of such graphs, the so-called one-link and two-link hypercubes [4,6,7]. For larger networks, it appears that an increase in the number of spins and the degree of the underlying graph tends to compromise the transmission of quantum information [8–10]. In the static case it has been shown that higher connectivity and associated monogamy constraints frustrate the system and affect its quantum correlations [11]. On the other hand, if natural interactions are abandoned in favor of particular coupling schemes, in which only nearest neighbors interact and the interaction strength depends on their position relative to a fixed point, then perfect state transfer is possible in spin chains with large N [4,12–14].

In this paper we investigate whether it is possible to transfer perfectly a quantum state between two distant nodes of a two-dimensional spin network, in which the interactions between spins are both permanent and homogeneous. We show that PST can be achieved in such a two-dimensional highly connected network of arbitrary size N. This is possible with a unique regular configuration, namely, a two-dimensional graph of the $(N/2)$-cross polytope [15], which is dual to the hypercube in $N/2$ dimensions and isomorphic to a type of circulant graph [16]. It turns out that these findings lead to a natural quantum generalization of a well-known principle in classical optics. Therefore the plan of the paper is the following. In Sec. II we introduce a general spin model, whose defining characteristic is that it preserves the total number of excitations in the network; then in Sec. III we present numerical calculations, which reveal the special properties of $(N/2)$-cross polytope graphs; and in Sec. IV we provide analytical results that support our numerical findings and prove the main result of the paper. In Sec. V, based on the quantum state transfer properties of complementary graphs, we derive a quantum version of the Babinet principle from classical optics. Basic results of Monte Carlo simulations on the influence of static disorder on the system are presented in Sec. VI. Concluding in Sec. VII, we discuss our results.

II. EXCITATION-PRESERVING QUANTUM NETWORK

We begin by considering N spins-$\frac{1}{2}$ situated along a circle, as shown in Fig. 1. It is understood that, if two spins are interacting, a line is drawn between them. The result is a graph $\mathcal{G}=(V,E)$; the vertices $V(\mathcal{G})$ represent the spin sites and the edges $E(\mathcal{G})$ represent pairwise interactions. The nec-

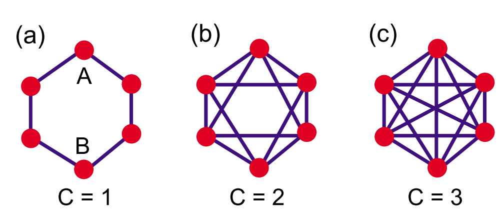

FIG. 1. (Color online) A circulant spin network can be used to transfer quantum states from A to B. In a network of $N=6$ spins there are three possible configurations with connectivity $C=$(a) 1, (b) 2, and (c) 3, as shown. Network (b) is a three-cross polytope graph.

essary information about the graph $\mathcal{G}$ is contained in its adjacency matrix, $A(\mathcal{G})$, whose elements are given by $A_{ij}=1$ if $\{i,j\}\in E(\mathcal{G})$ and are zero otherwise. We consider Hamiltonians of the form ($\hbar=1$)

$$\mathcal{H}=\sum_{k=1}^{N}\omega_k\sigma_k^+\sigma_k^- + \sum_{k\neq l} J_{k,l}(\sigma_k^-\sigma_l^+ + \sigma_k^+\sigma_l^-), \tag{1}$$

where σ_k^+ (σ_k^-) are the raising and lowering operators for site k, ω_k is the local site excitation energy, and $J_{k,l}$ denotes the hopping rate of an excitation between the sites k and l. The dynamics in this system preserves the total excitation number, defined by $\mathcal{N}=\Sigma_{k=1}^{N}\sigma_k^+\sigma_k^-$. During dynamical evolution the state of the network, $|\Psi(t)\rangle=\exp(-i\mathcal{H}t)|\Psi_0\rangle$, where $|\Psi_0\rangle$ is the initial state, always remains in the same excitation sector because $[\mathcal{H},\mathcal{N}]=0$. In what follows we restrict our attention to the single-excitation sector, for simplicity. In this subspace the Hamiltonian of the system is equal to the adjacency matrix of the underlying graph, $\mathcal{H}=A(\mathcal{G})$, provided that the spin-spin interactions are homogeneous. Deviations due to engineering errors in the interactions are also examined later on. The network is prepared in the state

$$|\Psi_0(\mathbf{j})\rangle \equiv |j\rangle \coloneqq |0_1 0_2 \cdots 1_j \cdots 0_N\rangle, \tag{2}$$

where only spin j is excited. The propagation of an arbitrary state $\alpha|0_j\rangle+\beta|1_j\rangle$, where $|\alpha|^2+|\beta|^2=1$, is equivalent to the propagation of the state $|1_j\rangle$ (since the $+1$ eigenstate of Z_j, $|0_j\rangle$, does not evolve under $\mathcal{H}$). The aim is to transfer the excitation from j to $N/2+j$, that is, to the vertex that is diametrically opposite from j across the ring—hence we initially consider that N is even. The state transfer is quantified by the fidelity

$$F(t) \coloneqq |\langle\Psi_0(\mathbf{N}/2+\mathbf{j})|\exp(-i\mathcal{H}t)|\Psi_0(\mathbf{j})\rangle|. \tag{3}$$

Perfect state transfer is achieved at a certain time t_0 if and only if $F(t_0)=1$.

III. QUANTUM STATE TRANSFER AND CONNECTIVITY

We now ask, "How is the fidelity of quantum state transfer influenced by the connectivity of a network?" The connectivity $C(\mathcal{G})$ is defined here as the number of edges that are incident on a vertex, counting only within the half disk defined by that vertex and the opposite one (i.e., it is half the degree of the graph). The extreme cases are those of a ring ($C=1$) and a fully connected network ($C=N/2$), but in general we have $C=1,2,\ldots,N/2$ (see Fig. 1). Before we analyze this question analytically, we calculate numerically the fidelity $F(t)$ of Eq. (3) for $t\in[0,\Delta t]$, given the number of spins N and the connectivity C. The maximum fidelity $\max(F_{\Delta t})$ is then determined for the interval Δt. It is assumed that $\omega_k=0$ and $J_{k,l}=1$ in the Hamiltonian of Eq. (1). Therefore the Hamiltonian of the network is equal to the adjacency matrix of the underlying graph structure. Under these conditions it is observed in Fig. 2(a) that the fidelity is a nonmonotonic and rather complicated function of the connectivity. However, it displays remarkable behavior for $C=N/2-1$, which corresponds to a $2k$-cross polytope graph (CPG) with $N=4k$ spins, where k is a positive integer. In this case, PST is

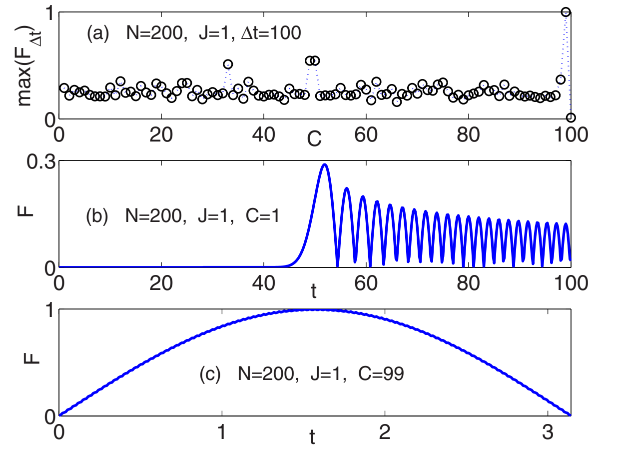

FIG. 2. (Color online) (a) Maximum fidelity in the interval $[0,\Delta t=100]$ against connectivity for a network of size $N=200$ and homogeneous interactions. (b) Fidelity against time for the simple ring network. (c) Fidelity against time for the 100-cross polytope graph network.

achieved at $t_0=\pi/2+n\pi$, i.e., $F(\pi/2+n\pi)=1$, where $n\geqslant 0$ is an integer. For $k=1$ we recover the known result [17] for a ring with $N=4$.

In Ref. [16] it was shown that circulant graphs of odd order *do not allow* perfect state transfer (so our choice of even N is justified) and, moreover, it was left as an open question whether there exist circulant graphs of even order with $N>4$ that support PST. Our results show that such graphs do indeed exist: the $2k$ CPG is isomorphic to the circulant graph $\mathrm{Ci}_{4k}(1,2,\ldots,2k-1)$. In these networks every spin interacts with every other spin, except for one [e.g., see Fig. 1(b) for an example]. The appropriate choice of Δt is made by comparing trial values with the occurrence time of the first peak in the evolution of the fidelity for a spin ring [this evolution is shown in Fig. 2(b)]. In Fig. 2(c) we show the evolution of the fidelity for a network with connectivity $C=N/2-1=99$. It is seen that the fidelity becomes equal to 1 at $t_0=\pi/2$.

IV. ANALYTICAL RESULTS

In this section we analyze the perfect (for $N=4k$) or near-perfect state transfer (for $N=4k+2$) in the configurations introduced previously. The Hamiltonian H_{CPG} of a cross polytope graph is that of Eq. (1) with $\omega_k=0$ and $J_{k,l}=(1-\delta_{l,N/2+k})(1-\delta_{k,l})=J_{l,k}$. The Hamiltonian of a fully connected network, denoted as H_{fc}, is that of Eq. (1) with $J_{k,l}=1-\delta_{k,l}=J_{l,k}$. The Hamiltonian where only opposite pairs are connected, denoted as H_{pair}, has $J_{k,l}=\delta_{l,N/2+k}=J_{l,k}$ for all $k=1,\ldots,N/2$.

We start by noting that we have

$$H_{\mathrm{CPG}}=H_{\mathrm{fc}}-H_{\mathrm{pair}}, \quad [H_{\mathrm{fc}},H_{\mathrm{pair}}]=0. \tag{4}$$

Therefore, the evolution operator is

$$e^{-iH_{\mathrm{CPG}}t}=e^{iH_{\mathrm{pair}}t}e^{-iH_{\mathrm{fc}}t}. \tag{5}$$

Due to the fact that

$$H_{\text{pair}} = \sum_{k=1}^{N/2} \left(|k\rangle\left\langle \frac{N}{2}+k\right| + \left|\frac{N}{2}+k\right\rangle\langle k| \right) \tag{6}$$

only couples opposite sites, its dynamics is very simple. It is straightforward to obtain

$$e^{iH_{\text{pair}}t} = \mathbb{1}\cos t + i\sum_{k=1}^{N/2} \left(|k\rangle\left\langle \frac{N}{2}+k\right| + \left|\frac{N}{2}+k\right\rangle\langle k| \right)\sin t.$$

In particular, after a time $t=\pi/2+n\pi$, a starting state $|k\rangle$ will have been transformed to $(-1)^n i|N/2+k\rangle$. Finally, to determine the dynamics of H_{CPG} we need to consider H_{fc}. The latter can be expressed as

$$H_{\text{fc}} = N|+\rangle\langle+| - \mathbb{1}, \tag{7}$$

where

$$|+\rangle = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle. \tag{8}$$

We have

$$e^{-iH_{\text{fc}}t} = [|+\rangle\langle+|e^{-iNt} + (\mathbb{1}-|+\rangle\langle+|)]e^{it}.$$

Therefore, a state $|k\rangle$ is mapped onto itself, up to a global phase, under $e^{-iH_{\text{fc}}t}$ when $Nt=2\pi k$ with $k\in\mathbb{N}$.

As a consequence, the dynamics due to H_{CPG} allows for PST if both $Nt=2\pi k$ and $t=\pi/2$ are satisfied for the same t. This implies the condition

$$N = 4k \tag{9}$$

and explains the possibility of PST in $2k$-cross polytopes. For $N=4k+2$ the analysis above immediately applies and shows that we do not have PST at $t=\pi/2$.

More generally, we can find the transfer fidelity for $t=\pi/2$. Starting with $|k\rangle$ and using $e^{iH_{\text{pair}}t}|+\rangle=e^{it}|+\rangle$, we find at $t=\pi/2$ the state

$$e^{iH_{\text{pair}}t}e^{-iH_{\text{fc}}t}|+\rangle = -\left[\frac{1}{\sqrt{N}}|+\rangle e^{-iN\pi/2} + \left(\left|\frac{N}{2}+k\right\rangle - \frac{1}{\sqrt{N}}|+\rangle\right)\right].$$

Then the fidelity $|\langle N/2+k|e^{iH_{\text{pair}}t}e^{-iH_{\text{fc}}t}|k\rangle|^2$ is

$$F = 1 - \frac{2}{N}\left(1-\frac{1}{N}\right)\left(1-\cos\frac{N\pi}{2}\right). \tag{10}$$

For $N=4k$ we recover $F=1$, while for $N=4k+2$ we find that $F=(1-2/N)^2$. Therefore, as $N\to\infty$, the fidelity approaches 1 and we obtain almost PST.

V. QUANTUM BABINET PRINCIPLE

These results provide a clear insight into the basic mechanisms that facilitate PST in these systems. The key realization is that a fully connected network in which some couplings $J_{k,l}$ are removed can behave similarly to an initially unconnected network which is supplemented with the very same $J_{k,l}$ links. This result is in fact reminiscent of the Babinet principle of classical optics [18], which is illustrated in Fig. 3. In our context of state transfer through connected networks, the situation is similar in the sense that

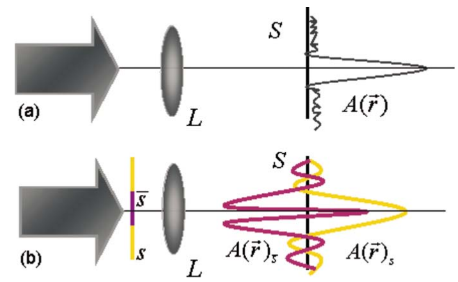

FIG. 3. (Color online) Illustration of Babinet's principle in an optical setup with Fraunhofer conditions. (a) An unobstructed plane wave is focused by a lens L and produces a diffraction pattern of amplitude $A(\vec{r})$ on the screen S. (b) Diffraction patterns resulting from complementary screens s and $\bar{s}$, whose opaque and transparent areas are swapped. At any point downstream from s and $\bar{s}$, the sum of the two diffracted amplitudes, $A_s(\vec{r})+A_{\bar{s}}(\vec{r})$, equals the amplitude diffracted from the unobstructed plane wave, $A(\vec{r})$. Away from the central spike, this amplitude is zero and therefore $A_s(\vec{r})=-A_{\bar{s}}(\vec{r})$, which leads to Babinet's prediction of identical diffracted light fields for complementary apertures. Complementary apertures play the role of complementary graphs describing quantum spin networks. An increase in $\vec{r}$ corresponds to increasing the number of nodes N.

$$e^{-iH_{\text{CPG}}t}e^{-iH_{\text{pair}}t} = e^{-iH_{\text{fc}}t}$$

because H_{CPG} and H_{pair} commute and also $e^{-iH_{\text{fc}}t}$ equals the identity at specific times t (in the optical setting this is the situation when all incident light emerges unaffected). Of course, in the quantum setting we have the added problem that $e^{-iH_{\text{CPG}}t}e^{-iH_{\text{pair}}t} \neq e^{-i(H_{\text{CPG}}+H_{\text{pair}})t}$ in general.

The analog of the Babinet theorem does hold, however, for much more general settings than just that of commuting H_{CPG} and H_{pair}. Indeed, as before, let us assume that

$$H_f = N\mathbb{P}, \quad \mathbb{P} = |+\rangle\langle+|. \tag{11}$$

For a sequence of H_N that satisfies

$$\|\mathbb{P}H_N(\mathbb{1}-\mathbb{P}) + (\mathbb{1}-\mathbb{P})H_N\mathbb{P}\| = O\left(\frac{1}{\sqrt{N}}\right), \tag{12}$$

we compare the dynamics of H_N and $H_f-H_N\equiv H_c$ in the limit of large N. The following argument is not fully rigorous in that it does not provide detailed error estimates, but these may be provided in a more detailed analysis.

Let us consider the dynamics under H_c in an interaction picture with respect to H_f when this Hamiltonian becomes time dependent,

$$H_{c,I}(t) = e^{iH_ft}(H_c - H_f)e^{-iH_ft} = -e^{iH_ft}H_Ne^{-iH_ft}.$$

The corresponding time-evolution operator from t_1 until t_2 in the interaction picture will be denoted by $U_I(t_1\to t_2)$. Now we note that $H_{c,I}(t)$ may contain rapidly oscillating terms (those coupling the subspace defined by $\mathbb{P}$ to the subspace defined by $\mathbb{1}-\mathbb{P}$) thanks to the action of e^{iH_ft}. These rapidly oscillating terms may be neglected for large N, leading to a correction of order $1/N$ in the dynamics. Hence, we find

$$H_{c,I}(t) \cong -(\mathbb{1}-\mathbb{P})H_N(\mathbb{1}-\mathbb{P}) - \mathbb{P}H_N\mathbb{P} = -H + \mathbb{P}H_N(\mathbb{1}-\mathbb{P}) + (\mathbb{1}-\mathbb{P})H_N\mathbb{P}. \quad (13)$$

As we had assumed earlier that $\|\mathbb{P}H_N(\mathbb{1}-\mathbb{P})+(\mathbb{1}-\mathbb{P})H_N\mathbb{P}\|$ is of order $1/\sqrt{N}$, we find that $H_{c,I}(t)$ is well approximated by $-H_N$ up to corrections that decrease with increasing N. Hence $U_I(0\rightarrow t)\approx e^{iH_N t}$, and we find that

$$e^{-iH_c t} = e^{-iH_f t}U_I(0 \rightarrow t) \cong e^{-iH_f t}e^{iH_N t}. \quad (14)$$

Now we consider the transfer fidelity from state $|k\rangle$ to $|N/2+k\rangle$, as an example. The amplitude $\langle N/2+k|e^{-iH_c t}|k\rangle$, using $e^{-iH_c t}=e^{-iH_f t}e^{iH_N t}$, is found to be equal to

$$\left\langle \frac{N}{2}+k|e^{iH_N t}|k\right\rangle + \left\langle \frac{N}{2}+k|+\right\rangle\langle +|e^{iH_N t}|k\rangle(e^{iNt}-1) \cong \left\langle \frac{N}{2}+k|e^{iH_N t}|k\right\rangle,$$

where the difference decreases with increasing N. Therefore, the transition amplitudes according to the dynamics under $-H_N$ and H_c are asymptotically (in N) equal. Note that for a real Hamiltonian H_N we have $(\langle N/2+k|e^{iH_N t}|k\rangle)^*=\langle N/2+k|e^{-iH_N t}|k\rangle$, so that

$$\left|\left\langle \frac{N}{2}+k|e^{-iH_c t}|k\right\rangle\right| \cong \left|\left\langle \frac{N}{2}+k|e^{-iH_N t}|k\right\rangle\right|, \quad (15)$$

again with an error that decreases with increasing N. This is the quantum Babinet principle.

VI. INFLUENCE OF DISORDER

We provide here a brief analysis of realistic engineering errors in the interactions of a $2k$ CPG network in order to assess the robustness of a possible experimental implementation. We take into account two types of errors: (i) disorder in the interactions, and (ii) random breaking of interactions. For case (i) we assume that if p and q are interacting then the interaction strength can take any value in the interval $[1-\delta,1+\delta]$, with equal probability. The amount of disorder is thus quantified by $\delta\in[0,1]$. In case (ii), some interactions are randomly broken, that is, J_{pq} vanishes for a fixed number of pairs (p,q). The number of broken interactions is $B\in[0,1)$, given as a ratio to the total number of interactions in the network. The main results of Monte-Carlo simulations on $N/2$ CPG networks with N=40, 80, 120, 200, 400 spins, are as follows. For type-(i) errors we find that disorder up to $\delta=0.02$ allows for almost PST in smaller networks ($N<100$). In particular, the maximum fidelity F is greater than 0.99, on average, with a worst-case value of 0.98 in the case of $N=40$; while for $N>100$ the average maximum F is over 0.95 for disorder that is less than 2%. For type-(ii) errors we find that the random breaking of very few bonds, so that $B<0.001$, still allows for very high-quality state transfer, where the maximum F is larger than 0.95, on average. However, the value of the worst-case fidelity peak fluctuates considerably on individual cases, depending on the positions of the broken bonds.

In this connection, the usefulness of the quantum Babinet principle can be illustrated in the case of transport of excitations through noisy networks, a setting that has recently been introduced independently in [19,20]. Initially all population resides in a given site and we evaluate how much population may be transferred asymptotically to a selected target site. To this end, we let the target site be attached to a sink to which the population is transferred irreversibly. We want to analyze whether the presence of local dephasing can assist the excitation transfer. If the sink is attached to site $N/2+1$, then the Babinet principle implies that the evolution is that of a system where only the opposite two sites are coupled, and we recover a situation for which it was proven in [20] that no dephasing enhanced transport is possible [21].

VII. SUMMARY AND DISCUSSION

We have shown that PST is achieved in a network of size N, whose structure is that of a $(N/2)$-cross polytope graph, if N is a multiple of 4. If N is even, but not a multiple of 4, then almost PST is achieved for larger networks of this kind, so that F approaches 1 for $N\rightarrow\infty$. These results can be interpreted in terms of a quantum Babinet principle, which establishes the conditions required for having *complementary* graphs leading to the same fidelity of state transfer, in analogy with the classical situation of obtaining identical diffraction patterns from complementary screens. As shown in various examples, invoking Babinet's principle alone can simplify the analysis of the performance of connected networks and therefore become a useful tool in tackling a variety of problems in quantum-information theory.

Note added. Recently we became aware of the closely related work of Ref. [22], in which a similar result is established using different methodology. We would like to thank S. Severini for useful correspondence.

ACKNOWLEDGMENTS

This work was supported by the EU via Integrated Projects QAP and SCALA, STREP actions CORNER and HIP, and the EPSRC through the QIP-IRC. D.I.T acknowledges the EPSRC for financial support (Grant No. EP/D065305/1). I.D.V acknowledges support from Ministerio de Educacion y Ciencia, Spain. We thank Neil Oxtoby for careful reading of the manuscript.

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