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Self-focusing of few optical cycle pulses

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Both analytical and numerical investigation of the structural features of ultrashort-laser-pulse self-focusing is presented. The analysis is performed under sufficiently general assumptions concerning the medium dispersion. It is demonstrated that the wave-field self-focusing proceeds with overtaking the steepening of the pulse longitudinal profile and shock-wave formation. As a result, a more complex singularity is formed where an unlimited field increase is followed by wave breaking with a broad power-law pulse spectrum.

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The propagation of high-power laser pulses in a nonlinear medium is accompanied by the well-studied effect of laser beam self-focusing [1–3]. It is evident that as the pulse duration becomes shorter, the amplitude dependence of the group velocity of the wave packet (nonlinear dispersion) becomes more important for the self-action dynamics along with the cubic nonlinearity. It leads to a steepening of the longitudinal profile and formation of a shock wave (wave breaking) [2] even for media without dispersion. For the case of quasimonochromatic radiation, the evaluation of the relative value of the nonlinearity of these two types—i.e., of the corresponding nonlinear members in the equation $S_{ND}/S_K \sim \lambda/L$, where S_K and S_{ND} are the terms responsible for the self-action and steepening of the field, respectively, λ is the wavelength, and L is the pulse duration—shows that the cubic nonlinearity wins in this competition. However, in the process of pulse self-focusing, an increase in the field amplitude leads to a significant decrease in the breaking length and, therefore, to an increase of the influence of the nonlinear dispersion [4–6]. To analyze the problem of the competition of these nonlinearities correctly, we will study the dynamics of the self-action of an ultrashort three-dimensional pulse based on the wave equation, but within the reflection-free approximation and for relatively general assumptions about the media dispersion. It turned out that for this case, one can generalize the methods of studying qualitatively the solutions used in hydrodynamics and in the theory of non-one-dimensional nonlinear Schrödinger equation and make a conclusion about the principal role of nonlinear dispersion in the self-focusing dynamics. The results of analytical study of the formation of shock waves in the self-focusing process are confirmed by numerical simulation.

The wave equation describing the propagation of a packet of electromagnetic waves along the z axis has the form

$$c^2\partial_z^2\boldsymbol{E} + c^2\Delta_\perp\boldsymbol{E} - \partial_t^2\boldsymbol{E} = 4\pi\partial_t^2\boldsymbol{P}, \tag{1}$$

where $\partial_\varsigma = \partial/\partial_\varsigma$ for any ς, $\Delta_\perp = \partial_x^2 + \partial_y^2$, and $\boldsymbol{P}$ is the medium polarization response.

To determine the material equation $\boldsymbol{P}(\boldsymbol{E})$ for the case of intense radiation, we will generalize Lorentz dispersion theory, in which the medium is represented as a set of oscillators. In a simple way it can be done by means of introducing cubic nonlinearity to the oscillator equation. As a result, in the case of the circularly polarized field $\boldsymbol{E} = (\boldsymbol{x}_0 + i\boldsymbol{y}_0)E$, we obtain [2]

$$\partial_\tau^2 P + \omega_0^2 P + \delta|P|^2 P = \chi\omega_0^2 E \tag{2}$$

for the nonlinear response of a medium in the weak-absorption region. Here, ω_0 is the oscillator frequency, δ is the nonlinearity parameter, and χ is the static linear susceptibility of the medium.

For a wave packet with central frequency ω and spectrum width $\Delta\omega$, which is much less than the carrier frequency ω_0 ($\Delta\omega < \omega \ll \omega_0$), one can obtain the following expression for the polarization response of the medium using the perturbation method:

$$\partial_t^2 P = \chi\partial_t^2 E - \chi\omega_0^{-2}\partial_t^4 E - \delta\chi^3\omega_0^{-2}\partial_t^2(|E|^2 E). \tag{3}$$

The linear part of this equation turns out to be insufficient to describe the frequency dependence of the dielectric permittivity in a rather wide frequency range [4,7]. The required generalization can be achieved by using the Kramers-Kronig relation. Following [8] one can obtain (for more details see [9]) $\varepsilon_r(\omega) \simeq \tilde{\varepsilon}_0 + \tilde{b}\omega^2 - \tilde{a}/\omega^2$, where $\tilde{a}$ and $\tilde{b}$ are some coefficients determined by the imaginary part of the dielectric permittivity [9]. In the framework of Eq. (2) the coefficient $\tilde{b}$ can be found as $\tilde{b} = \chi/\omega_0^2$. In essence, it means that one has to supplement Eq. (3) with the term $\tilde{a}E$, which determines the low-frequency dispersion of the medium.

The final equation describing the dynamics of the self-action of the wideband radiation in a medium with focusing nonlinearity ($\chi\delta < 0$) has the following form [7,10]:

$$\partial_{z\tau}^2 u + \partial_\tau^2(|u|^2 u) - b\partial_\tau^4 u = \Delta_\perp u - au. \tag{4}$$

Here, the following new variables have been introduced $z \to z2\omega c\sqrt{1+4\pi\chi}/\tilde{a}$, $r \to rc/\sqrt{\tilde{a}}$, $b = 4\pi\chi\omega^4/(\omega_0^2\tilde{a})$, $\tau = \omega[t - 2z(1+4\pi\chi)\omega/\tilde{a}]$, and $E = u\sqrt{\omega_0^2\tilde{a}/(4\pi\delta\chi^3\omega^2)}$. The parameters a and b determine the low-frequency and high-frequency linear dispersion of the medium. By choosing the parameters $a > 0$, $b > 0$, one can achieve good agreement with the experimental data: e.g., for quartz glass, this agreement is realized for the frequency changes by an order of magnitude [7].

In the considered case of Eq. (4), the corresponding linear law of dispersion for the wave $u \sim \exp(i\omega\tau - ik_z z)$ has the

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form $k_z=-a/\omega+b\omega^3$. Note that one can control the dispersion influence in the wave packet dynamics by changing its central frequency ω. Specifically, for a pulse with frequency $\omega_{bnd}=(a/3b)^{1/4}$, the parameter of the group-velocity dispersion $\mathcal{D}_{\rm GV}=\partial^2 k_z/\partial\omega^2$ becomes to zero. Correspondingly, for the wave fields with frequency $\omega \gg \omega_{bnd}$, the field spectrum is concentrated mainly in the region with the normal group-velocity dispersion $\mathcal{D}_{\rm GV}>0$, and for $\omega \ll \omega_{bnd}$, the anomalous dispersion ($\mathcal{D}_{\rm GV}<0$) is dominant.

In the case of a quasimonochromatic wave packet with the form $u=\psi(z,\boldsymbol{r},\tau)e^{i\omega\tau}$, one can obtain from (4) the generalized nonlinear Schrödinger equation (GNLSE) which contains the usual nonlinear term $|\psi|^2\psi$ and an additional term $i|\psi|^2\partial\psi/\partial\tau$ responsible for the dependence of the wave packet group velocity on the field amplitude (nonlinear dispersion) and for the high-order dispersion.

To study qualitatively the peculiarities of the wave-field self-action, which are described by Eq. (4), we will use the methods similar to those used when analyzing the NSE [3]. The localized solutions of Eq. (4) retain the *quantum number*

$$I=\int\int|\phi_\tau|^2 d\tau\, d\boldsymbol{r}_\perp \tag{5}$$

and the Hamiltonian

$$H=\int\int[|\nabla_\perp\phi|^2-b|\phi_{\tau\tau}|^2-|\phi_\tau|^4/2+a|\phi|^2]d\tau\, d\boldsymbol{r}_\perp. \tag{6}$$

Here $u=\phi_\tau$. Due to the *Hamiltonian nature*, one can obtain the relation

$$I\frac{d^2\langle\rho_\perp^2\rangle}{dz^2}=8H+8\int\int(b|\phi_{\tau\tau}|^2-a|\phi|^2)d\tau\, d\boldsymbol{r}_\perp, \tag{7}$$

which describes the change in the effective transverse size of the wave field $I\langle\rho_\perp^2\rangle=\int\int\boldsymbol{r}_\perp^2|\phi_\tau|^2 d\tau\, d\boldsymbol{r}_\perp$. It is seen that for the dispersion-free medium ($a=b=0$), the right-hand side of Eq. (7) is proportional to the Hamiltonian of the system (6). Therefore, the distributions of the wave field with the negative Hamiltonian collapse in the transverse direction along a finite propagation path. This conclusion stays valid for field distributions with the spectrum localized mainly in the region of the anomalous dispersion of the group velocity ($b\to 0$). In other cases ($b\neq 0$), Eq. (7) shows the possibility of an initial narrowing of the transverse field distribution (a more detailed analysis is presented in [4]).

To study the spatiotemporal dynamics of the wave packet in the pulse, it is convenient to use the transformation

$$u=\mathcal{E}(\zeta,\boldsymbol{\eta},\theta)/\rho(z), \tag{8}$$

which transposes the point of singularity formation to infinity, for the case of pulses with arbitrary duration. Here $\zeta=\int\frac{dz}{\rho^2(z)}$, $\boldsymbol{\eta}=\frac{\boldsymbol{r}}{\rho(z)}$, $\theta=\tau-\frac{\rho_z}{4\rho}r^2$, and the function $\rho(z)$ describes the change in the transverse size of the field. This representation of the solution of Eq. (4) takes into account two processes running in the system: the self-focusing of the packet and the formation of a characteristic U-shaped structure of the field distribution, which is determined by the variable θ. Note that this representation is valid not only for collimated beams [$\rho_z(z=0)=0$], but also for initially focused beams [$\rho_z(z=0)>0$].

Using the transformation (8), we obtain the equation

$$\partial^2_{\zeta\theta}\mathcal{E}+\partial^2_\theta(|\mathcal{E}|^2\mathcal{E})-\rho_{zz}\rho^3\eta^2\partial^2_\theta\mathcal{E}/4+\rho^2(a-b\partial^4_\theta)\mathcal{E}=\Delta_\perp\mathcal{E}. \tag{9}$$

The conversion into the "collapsing" reference system allows one to "segregate" the process of self-focusing in the system and reduce the problem to studying the quasi-unidimensional longitudinal evolution of the pulse. The characteristic transverse scale of the quasiwaveguide structure in new variables is of the order of unity. It is seen that the self-collapse rate $\rho(z)$ determines the level of attenuation of dispersion effects during the collapse [$\rho(z)\to 0$]. Thus, at least in a dispersion-free medium, as well as in a medium with anomalous dispersion of the group velocity, the steepening of the longitudinal profile of the pulse along with the beam self-focusing becomes the key process in the temporal region θ. The case of normal dispersion ($a=0$) requires an additional consideration.

Now, we will find the relationship between the length z_0 of the collapse [$\rho(z_0)=0$] and the length z_B of the wave breaking. The value of z_0 can be estimated from the momentum theorem, Eq. (7). Specifically, near z_0, for the collimated wave field [$\rho(z=0)=\rho_0,\rho_z(z=0)=0$], we have

$$\rho(z)\simeq\sqrt{\langle\rho_\perp^2(z)\rangle}\simeq 2(\sqrt[4]{\rho_0^2|H|/I})(\sqrt{z_0-z}), \tag{10}$$

where the coordinate of the focal point can be also determined via the integrals of the problem $z_0=\rho_0\sqrt{I/(4|H|)}$. As follows from Eq. (9), the evolution of the field in the near-axis region ($\eta\simeq 0$) can be described by the following equation:

$$\partial_\zeta\mathcal{E}+\partial_\theta(|\mathcal{E}|^2\mathcal{E})\simeq 0. \tag{11}$$

The solutions of Eq. (11) have been studied in detail, e.g., in hydrodynamics [11]. The inequality of the velocities of the points in the longitudinal profile leads to a steepening of the rear front of the pulse and the formation of a shock wave in the process of *wave breaking* along the length of ζ_B. One can easily estimate the wave-breaking length $\zeta_B\simeq\tau_0/(3|\mathcal{E}|^2_{\max})$, where $|\mathcal{E}|_{\max}$ is the maximum value of the field in the pulse at $z=0$ (at the entrance to the medium), for the pulses having duration τ_0 from Eq. (11). This corresponds to the distance

$$z_B=z_0\{1-\exp[-4\rho_0\tau_0\sqrt{|H|/I}/(3|\mathcal{E}|^2_{\max})]\}, \tag{12}$$

which is somewhat shorter than the self-focusing length z_0. When finding Eq. (12), we used the relationship between z and ζ in Eq. (8) in the following form: $\zeta=\int_0^z dz/\rho^2(z)=-\sqrt{I/(16\rho_0^2|H|)}\ln(1-z/z_0)$. Note that as $|H|$ decreases and the self-focusing length increases as $z_0\sim|I/H|^{1/2}$, the wave-breaking region retreats from the point of collapse of the transverse field distribution toward the boundary of the nonlinear medium obeying the exponential law. This result ($z_B<z_0$) is in good accordance with the automodel solution of Eq. (4) at $a=b=0$ $|u|=\Phi\{\tau-r^2/[4(z_0-z)]-3\Phi^2/(z_0-z)\}/(z_0-z)$, where Φ is an arbitrary function. Thus, in dispersion-free media ($a=b=0$) and in media with the anomalous group velocity dispersion law ($b=0$), a new type of singularity is

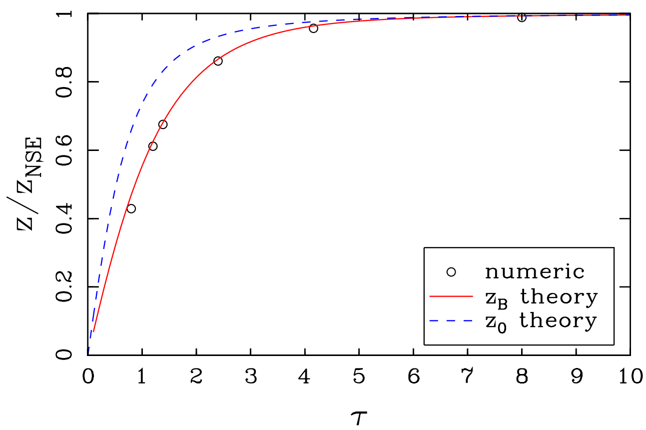

FIG. 1. (Color online) Length of self-focusing z_0 (blue dashed line) calculated by Eq. (7); the red line is the overturn distance calculated by Eq. (12). ○ is the length of the singularity obtained by numerical simulation of Eq. (4) for $a=b=0$.

formed, in which the development of wave breaking near the rear front of the pulse leaves the collapse of the wave field somewhat behind ($z_B < z_0$). The formation of discontinuities (shock waves) is accompanied by the inevitable dissipation of the energy on the front and a consequent attenuation of the wave. In the case of a weak shock wave, the intensity behind the wave-breaking point ($\zeta > \zeta_B$) decreases, obeying the law $|\mathcal{E}|^2 \sim 1/\sqrt{\zeta}$ [11]. However, despite the dissipation in the system, the behavior of the beam width (11) in the region of the nonlinear focus ($z \sim z_0$) stays the same and is determined by formula (10) [10].

To conclude this part, we will estimate the length of self-focusing of the wave field z_0 using Eq. (7) in a dispersion-free medium ($a \equiv b \equiv 0$). For the distribution of the potential of the field having the form $\phi = \mathcal{A}\cosh^{-1}(\tau/\tau_p)\exp[-r_\perp^2/(2\rho_0^2)+i\omega\tau]$, the calculations yield $H=2\pi\mathcal{A}^2\tau_p(1-I/I_{cr})$, where $I_{cr}=140\pi\tau_p^3(1+3\varkappa^2)/(35\varkappa^4+14\varkappa^2+3)$ is the critical energy for the self-focusing of the beam, $I=2\pi\mathcal{A}^2\rho_0^2(1+3\varkappa^2)/(3\tau_p)$, and $\varkappa=\omega\tau_p$. Thus, for a collimated beam, z_0 has the following form:

$$z_0 = \rho_0^2/(2\sqrt{3}\tau_p)\sqrt{1+3\varkappa^2}(I/I_{cr}-1)^{-1/2}. \qquad (13)$$

From here, for the quasimonochromatic pulse $\varkappa \gg 1$ one can find the formula for the length of the singularity formation within the reflection-free approximation, which is well known in the self-focusing theory: $z_0^m=\rho_0^2\omega/(2\sqrt{P/P_{cr}-1})$, where $P=\int|\phi_\tau|^2 d\boldsymbol{r}_\perp = \pi\mathcal{A}^2\omega^2\rho_0^2$ is the beam power and P_{cr} is the critical power of the self-focusing. Figure 1 shows the length z_0 of the singularity formation as a function of the pulse duration τ_p for the following fixed parameters of the wave field: ρ_0, $\mathcal{A}$, and ω. The value of z_0 is normalized to z_0^m. As seen from Fig. 1, the length z_0 decreases as the duration of the electromagnetic radiation decreases. For example, as seen from Fig. 1, for the pulse duration $\tau_p=0.5$ ($\omega=1$), the length of the singularity formation is $z_0 \simeq 0.5 z_0^m$, i.e., the wave-breaking length is half of the collapse length estimated by quasimonochromatic formulas. Summing up, one can say that the peculiar feature of the dynamics of the self-action of wideband radiation in a medium with cubic nonlinearity, as follows from Eq. (13), is connected with the decrease of the length of the self-focusing of the radiation z_0 as the pulse duration τ_p becomes shorter compared with the length of the collapse of the quasimonochromatic wave beam z_0^m for the fixed parameters of the pulse: namely, ρ_0, $\mathcal{A}$, and ω. The reason for this decrease is associated with an additional longitudinal energy flow toward the singularity, which is caused by the inequality of the velocity of the points in the longitudinal profile, as well as with the transverse energy flow from the beam periphery toward its center.

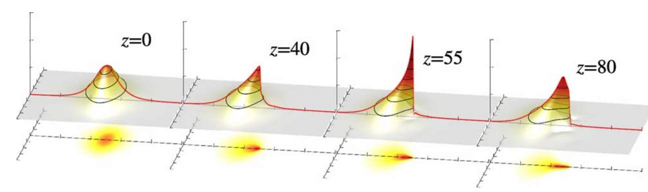

FIG. 2. (Color online) Dynamics of the circularly polarized field $|u(z,\tau,r)|$ in a dispersion-free medium ($a=0,b=0$) for $\gamma=0.04$. Distribution of the field at the boundary of the nonlinear medium $u=0.6e^{i\tau}/\cosh(0.3\tau)\exp(-r^2/100)$.

To illustrate the peculiarities in the dynamics of the self-action of ultrashort pulses, we will use computer simulations of Eq. (4). They confirm that the self-focusing of the radiation and the steepening of the pulse front occur simultaneously [4]. This means that the self-compression of the wave field must be accompanied by a noticeable widening of the spectrum and the dissipation on the shock-wave front. To stabilize the wave breaking (gradient catastrophe) in the numerical investigation, Eq. (4) is supplemented with a diffusion-type term $\gamma\partial^3 u/\partial\tau^3$, as is done conventionally in the theory of shock waves. This term arises naturally when attenuation is taken into account in the oscillator equation (2). This also proved to be sufficient for stabilization of the collapse, which confirms indirectly the anticipated development of the wave-field wave breaking. In the absence of this process, it is known that linear attenuation is insufficient to stabilize the self-focusing. Note that weak enough diffusion does not influence the formation length of the shock wave. The situation is the same as for other shock waves [8]. For illustration, Fig. 2 presents the results of modeling numerically the evolution of the field in a dispersion-free medium ($a=0,b=0$). It is seen that the key process in the dynamics of the self-action of an ultrashort pulse is here the steepening of the longitudinal profile and the formation of a rather sharp transition zone. This is a common scenario for dispersion-free media. Strictly, one can speak about the formation of a shock wave only in media without dispersion and media with anomalous dispersion. In these cases, numerical calculations demonstrate the presence of rather long intervals, in which the field spectrum decreases, obeying the power law. For example, in a dispersion-free medium, the spectrum calculations yield the law $S(\omega)=|\int u(\tau,r=0)e^{i\omega\tau}d\tau| \sim \omega^{-1}$ [see Fig. 3(a)], which is realized along the paths from $z=50$ to $z=75$. It means that there exists a jump in the field, which is a characteristic of a shock wave.

In a medium with anomalous dispersion, which leads to the packet self-compression in the longitudinal direction too, the spectrum decreases faster, specifically, $S(\omega) \sim \omega^{-3/2}$ [see

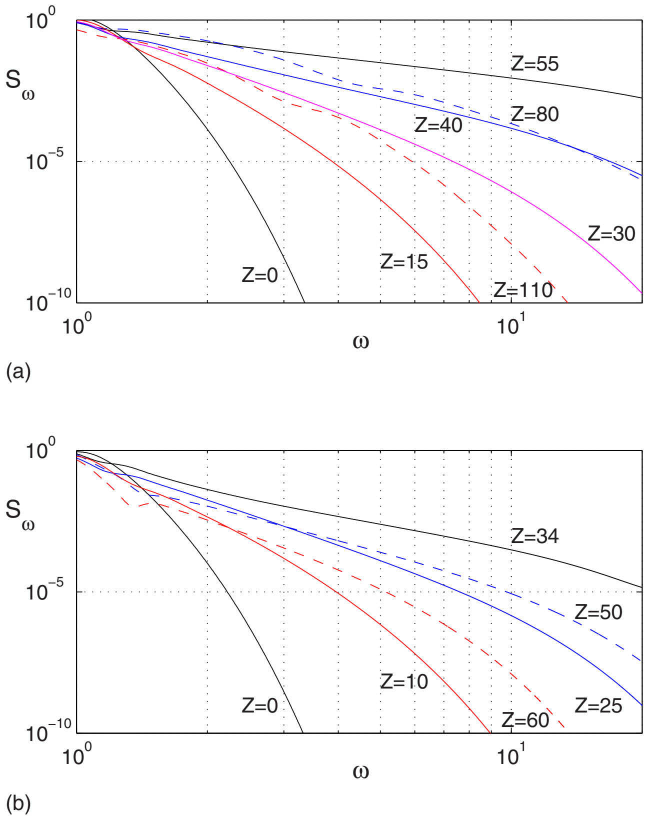

FIG. 3. (Color online) Short-wave part of the spectral power $S^2(\omega>1)$ at the system axis: (a) for $a=0$ and $b=0$; (b) for $a=1$ and $b=0$.

Fig. 3(b)]. The corresponding time dependence $u\sim\sqrt{\tau}$ is characterized by an infinite derivative for $\tau=0$. For the case under consideration it means that the low-frequency dispersion cannot prevent the gradient catastrophe entirely, but it only modifies the field structure in the transitional region. The numerical modeling shows that the spectrum has $\omega^{-3/2}$ law for z in the interval 40–60.

In a medium with normal dispersion, despite the formation of rather steep fronts, the spectrum contains no intervals with the power-law dependence. This can be explained by the fact that in the near-axis region, the self-action dynamics is described by the modified Korteweg–de Vries equation. In this case, the possibility of front steepening stabilization is well known. The structural features of the self-action in this case were studied in [4].

The above generalization of the laser beam self-focusing theory for the case of ultrashort pulses has shown that the account of nonlinear dispersion is of principal importance. New features in the self-action dynamics are determined by the steepening of the longitudinal profile of the pulse and the formation of shock waves in the media without dispersion and with anomalous dispersion. As a result, a new type of singularity is formed, in which the development of the gradient catastrophe near the rear front of the pulse is slightly ahead of the collapse of the wave field. Using numerical simulation of the processes, we demonstrated the consequences of the shock-wave formation in the system: namely, the stabilization of the collapse due to the uncontrolled dissipation on the front and formation of the power-law spectra of the wave field (see Fig. 3), which are characteristic of the wave breaking. Shock-wave formation [on the distance (12)] leads to a decrease of the singularity formation length along with shortening of the pulse duration.

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