Interference of Bose-Einstein condensates: Quantum nonlocal effects

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Quantum systems in Fock states do not have a phase. When two or more Bose-Einstein condensates are sent into interferometers, they nevertheless acquire a relative phase under the effect of quantum measurements. The usual explanation relies on spontaneous symmetry breaking, where phases are ascribed to all condensates and treated as unknown classical quantities. However, this image is not always sufficient: when all particles are measured, quantum mechanics predicts probabilities that are sometimes in contradiction with it, as illustrated by quantum violations of local realism. In this Rapid communication, we show that interferometers can be used to demonstrate a large variety of violations with an arbitrarily large number of particles. With two independent condensates, we find violations of the Bell-Clauser-Horne-Shimony-Holt inequalities, as well as *N*-body Hardy impossibilities. With three condensates, we obtain Greenberger-Horne-Zeilinger-type contradictions.

Gaseous Bose-Einstein condensates (BECs) can be used as sources to perform experiments with atomic interferometers [1]. With a single condensate, the interference effects depend on the difference of the accumulated phases along the arms of the interferometer. With two or more BECs, their relative phase introduces more physics into the problem. The usual view is that, when spontaneous symmetry breaking takes place at the Bose-Einstein transition, each condensate acquires a phase, with a completely random value. The outcome of a given experiment can then be obtained by assuming the existence of this initial classical phase; for an ensemble of realizations, an average over all of its possible values is necessary. Spectacular experiments with alkali-metal atoms originating from two independent BECs have confirmed this view [2]. Long before, Anderson [3] had proposed a thought experiment raising the famous question "Do superfluids that have never seen each other have a well-defined relative phase?" The question is not trivial since, in quantum mechanics, the Bose-Einstein condensates of super-fluids are naturally described by Fock states, for which the phase is completely undetermined. Nevertheless, various authors [4,5] have shown that repeated quantum measurements of the relative phase of two Fock states make a well-defined value emerge spontaneously with a random value [6,7]. Then, considering that the phase appears under the effect of spontaneous symmetry breaking when the BECs are formed, or later, under the effect of measurements, seems to be only as a matter of taste.

But a closer examination of the problem shows that this is not always true [8]: situations do exist where the two points of view are not equivalent, and even where the predictions of quantum mechanics for an ensemble of measurements are at variance with those obtained from an average over a phase [7]. This is not so surprising after all: the idea of a preexisting phase is very similar to the notion of "elements of reality" [9] introduced by Einstein, Podolsky, and Rosen [10]—for a double Fock state, the relative phase is nothing but what is often called a "hidden variable"—and we know that this idea combined with locality leads to the Bell theorem [11] and to contradictions with quantum mechanics. It is then natural that the notion of classical phase should also lead to

Bell-type inequalities and to similar contradictions.

Such contradictions were indeed predicted in [7], but in conditions that seemed difficult to reach experimentally: precise spin measurements in *N* separate regions of space were required (*N* is the total number of particles), and the numbers of results at each end of the experiment were fixed. Here we consider more realistic situations where spinless particles enter interferometers, and where the only requirement is to measure in which arm they leave them; this is accessible by laser atomic fluorescence (repeated measurements are possible in a quantum nondemolition scheme). Moreover, the number of results in each region may fluctuate freely. We study various situations involving two or three BECs, used as sources for interferometers, and show that quantum mechanics predicts violations of the Bell-Clauser-Horne-Shimony-Holt (BCHSH) inequalities [12], of the Greenberger-Horne-Zeilinger (GHZ) contradictions [13,14] as well as of the Hardy impossibilities [15,16]. Fock-state condensates appear as remarkably versatile, able to create violations that usually require elaborate entangled wave functions, and produce interesting *N*-body violations.

We first study an interferometer with a double Fock state representing the sources, as shown in Fig. 1: a condensate containing N_α particles reaches a semireflecting plate and is
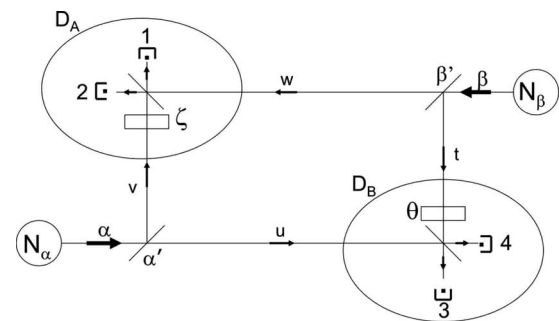

FIG. 1. Two independent condensates (populations N_α and N_β) are split into two coherent components, and then enter interferometers in two remote places D_A and D_B; the quantum results strongly violate the BCHSH local realist inequalities.

split into two coherent components u and v; similarly, another condensate containing N_β particles reaches another semireflecting plate and is split into two components w and t. We assume that, in two remote regions of space D_A and D_B, two experimenters Alice and Bob make measurements with semireflecting plates, recombining components v and w for the former, u and t for the latter; before the plates, they insert devices providing a phase shift, ζ for Alice, θ for Bob. We denote by m_1 and m_2 the numbers of particles that Alice detects in outputs 1 and 2, respectively, and by m_3 and m_4 the similar quantities for Bob. Reference [17] gives a study of a two-particle Bell inequality with this interferometer. We now calculate the probability $P(m_1, m_2, m_3, m_4)$ of such events.

The destruction operators $a_1 \cdots a_4$ associated with the output modes can be written in terms of the mode operators at the sources a_α, a_β, $a_{\alpha'}$, and $a_{\beta'}$ by tracing back from the detectors to the sources, with a phase shift of $\pi/2$ at each reflection, ζ or θ at the shifters, and $1/\sqrt{2}$ at each beam splitter. This gives the projections of the two different source modes onto each detector mode:

$$a_1 = \frac{1}{2}(ie^{i\zeta}a_\alpha + ia_\beta),$$

$$a_2 = \frac{1}{2}(-e^{i\zeta}a_\alpha + a_\beta),$$

$$a_3 = \frac{1}{2}(ia_\alpha + ie^{i\theta}a_\beta),$$

$$a_4 = \frac{1}{2}(a_\alpha - e^{i\theta}a_\beta), \tag{1}$$

where we have eliminated $a_{\alpha'}$ and $a_{\beta'}$, which do not contribute; in short, we write these equations as $a_i = u_{i\alpha}a_\alpha + u_{i\beta}a_\beta$. The source state is

$$|\Phi\rangle = \frac{1}{\sqrt{N_\alpha! N_\beta!}} a_\alpha^{\dagger N_\alpha} a_\beta^{\dagger N_\beta} |0\rangle, \tag{2}$$

where $a_\alpha^\dagger$ and $a_\beta^\dagger$ are creation operators and $|0\rangle$ is the vacuum. The amplitude for the system crossing all beam splitters with $m_1 \cdots m_4$ particles at the detectors is

$$C_{m_1,\ldots,m_4} = \langle m_1, m_2, m_3, m_4 | \Phi \rangle = \langle 0 | \frac{a_1^{m_1} \cdots a_4^{m_4}}{\sqrt{m_1! \cdots m_4!}} \frac{a_\alpha^{\dagger N_\alpha} a_\beta^{\dagger N_\beta}}{\sqrt{N_\alpha! N_\beta!}} |0\rangle. \tag{3}$$

We substitute (1) into this expression, and make binomial expansions of the sums $(u_{i\alpha}a_\alpha + u_{i\beta}a_\beta)^{m_i}$ to find

$$C_{m_1,\ldots,m_4} = \frac{1}{\sqrt{N_\alpha! N_\beta!}\sqrt{m_1! \cdots m_4!}}$$

$$\times \left(\prod_{i=1}^{4} \sum \frac{m_i!}{p_{\alpha i}! p_{\beta i}!} (u_{i\alpha})^{p_{\alpha i}}(u_{i\beta})^{p_{\beta i}} \right)$$

$$\times \langle 0 | a_\alpha^{p_{\alpha 1}+\cdots+p_{\alpha 4}} a_\beta^{p_{\beta 1}+\cdots+p_{\beta 4}} a_\alpha^{\dagger N_\alpha} a_\beta^{\dagger N_\beta} |0\rangle \tag{4}$$

where the Σ is a summation over $p_{\alpha i}$ and $p_{\beta i}$ with $p_{\alpha i}+p_{\beta i}$

$= m_i$. The evaluation of the expectation value of the operator product gives $N_\alpha! N_\beta! \delta_{N_\alpha, p_{\alpha 1}+\cdots+p_{\alpha 4}} \delta_{N_\beta, p_{\beta 1}+\cdots+p_{\beta 4}}$, which can be included by inserting the relations $\delta_{N_\gamma, p_{\alpha 1}+\cdots+p_{\alpha 4}} = \int_{-\pi}^{\pi} (d\lambda_\gamma/2\pi)e^{i(p_{\gamma 1}+\cdots+p_{\gamma 4}-N_\gamma)\lambda_\gamma}$ with $\gamma=\alpha,\beta$. If the total number of measurements $M=\Sigma_i m_i$ is equal to the total number of particles $N=N_\alpha+N_\beta$, the probability of obtaining the result (m_1, m_2, m_3, m_4) is

$$\mathcal{P}(m_1, m_2, m_3, m_4) = |C_{m_1,\ldots,m_4}|^2 = \frac{N_\alpha! N_\beta!}{m_1! \cdots m_4!} \int d\tau \int d\tau'$$

$$\times e^{-i[N_\alpha(\lambda_\alpha-\lambda_\alpha')+N_\beta(\lambda_\beta-\lambda_\beta')]} \prod_{i=1}^{4} [\Omega_i'^* \Omega_i]^{m_i} \tag{5}$$

with $d\tau$ representing integration over λ_α and λ_β and $d\tau'$ over the λ''s, and $\Omega_i(\lambda_a, \lambda_b) = (u_{i\alpha}e^{i\lambda_\alpha} + u_{i\beta}e^{i\lambda_\beta}), \Omega_i' = \Omega_i(\lambda_a', \lambda_b')$. This expression simplifies with the integration variables $\lambda_\pm = [(\lambda_\alpha+\lambda_\alpha') \pm (\lambda_\beta+\lambda_\beta')]/2$, $\Lambda_\pm = [(\lambda_\alpha-\lambda_\alpha') \pm (\lambda_\beta-\lambda_\beta')]/2$, since two integrations then become trivial and disappear. Using Λ_- parity, we then obtain

$$\mathcal{P}(m_1, m_2, m_3, m_4) \sim \int_{-\pi}^{\pi} \frac{d\Lambda_-}{2\pi} \cos(N_\alpha - N_\beta)\Lambda_- \int_{-\pi}^{\pi} \frac{d\lambda_-}{2\pi} \prod_{i=1}^{4}$$

$$\times [\cos \Lambda_- + \eta_i \cos(\lambda_- - \varphi_i)]^{m_i}, \tag{6}$$

where $\eta_1 = \eta_3 = 1$; $\eta_2 = \eta_4 = -1$; $\varphi_1 = \varphi_2 = -\varsigma$; $\varphi_3 = \varphi_4 = \theta$.

When $N=2$ it is easy to show that $\mathcal{P}(0, 1, 0, 1)$ is equal to $\frac{1}{4}\cos^2[(\varsigma+\theta)/2]$, in agreement with Ref. [17]. For any N, we recover the same form of the probability as for two interfering spinor condensates [7]; detectors 1 and 3 in Fig. 1 correspond to results $\eta=+1$ for spin measurements, 2 and 4 to $\eta=-1$ results. Nevertheless, instead of assuming that Alice and Bob measure a fixed number of spins; here the number of particles they detect can fluctuate freely, which changes the averages.

If in (6) we set $\Lambda_-=0$, we obtain the predictions of a preexisting phase λ_-, with a product of phase-dependent local probabilities $\frac{1}{2}[1 + \eta_i \cos(\lambda_- - \varphi_i)]$ summed over all possible values of λ_-. But, when Λ_- varies, negative "probabilities" appear in the integrand, introducing limitations to the notion of classical phase and the possibility of violations of local realism. This can happen only if all particles are measured: if $M < N$ particles are detected, summing over the unobserved results amounts to setting $\eta_i=0$ in the corresponding factors of (6), so that a factor $(\cos \Lambda_-)^{N-M}$ appears in the formula, peaked at $\Lambda_-=0$. Quantum violations then disappear, while the notion of relative phase reappears.

Counting $\eta=+1$ and -1 values as above, we can define their product at Alice's location as a quantity $\mathcal{A} = \pm 1$, and at Bob's location as their product as $\mathcal{B} = \pm 1$; we then have two functions to which the BCHSH theorem can be applied. The quantum average of their product is

$$\langle \mathcal{AB} \rangle = \sum_{m_1 \cdots m_4} (-1)^{m_2+m_4} \mathcal{P}(m_1, m_2, m_3, m_4). \tag{7}$$

The m_i sums can be done, leading to an exponential of a sum of three terms that can be reexpanded in three series. If M
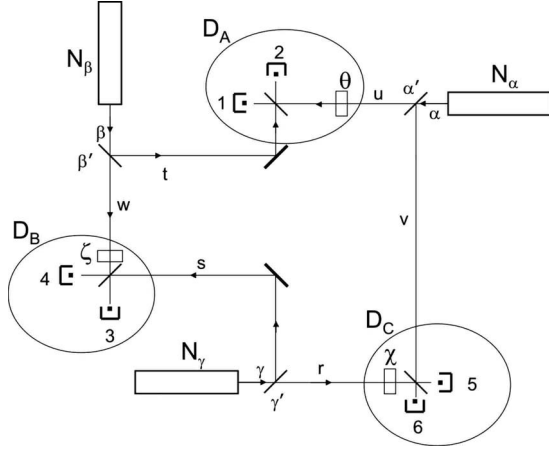
FIG. 2. Each of three condensates is split in two parts, which then enter interferometers in three remote places D_A, D_B, and D_C; the quantum results show GHZ contradictions with local realism.

FIG. 3. An interferometer with particle sources α and β, beam splitters BS, and mirrors M. In both detection regions, the detectors at D_i may be replaced by D_i', placed before the beam splitters. For appropriate path lengths and reflectivities of the beam splitters, quantum mechanics predicts the existence of events that are forbidden by local realism.

$=N$, the four τ integrals are easy, resulting in Kronecker δ's that collapse the sums to a single term:

$$\langle \mathcal{AB} \rangle = \left[\cos\left(\frac{\zeta + \theta}{2} \right) \right]^N \delta_{N_\alpha, N_\beta}. \qquad (8)$$

Now, the CHSH inequality tells us that

$$\langle \mathcal{AB} \rangle + \langle \mathcal{AB}' \rangle + \langle \mathcal{A'B} \rangle - \langle \mathcal{A'B'} \rangle \leq 2, \qquad (9)$$

where letters with and without primes imply measurements at differing angles. Alice's measurement angle is taken for convenience as $\phi_a = \varsigma/2$ and Bob's as $\phi_b = -\theta/2$. We define $E(\phi_a - \phi_b) = \cos^N(\phi_a - \phi_b)$, set $\phi_a - \phi_b = \phi_b - \phi_{a'} = \phi_{b'} - \phi_a = \xi$ and $\phi_{b'} - \phi_{a'} = 3\xi$, and maximize $Q = 3E(\xi) - E(3\xi)$ to find the greatest violation of the inequality for each N. For $N=2$ we find $Q_{\max} = 2.41$ in agreement with Ref. [17]; for $N=4$, $Q_{\max} = 2.36$; and for $N \to \infty$, $Q_{\max} \to 2.32$. The system continues to violate local realism for arbitrarily large condensates. Note that every source particle must be measured, otherwise no violation is found [7].

Consider next the arrangement of Fig. 2 with three Fock condensate sources and three detector pairs; it will allow GHZ contradictions. A similar device was discussed in Ref. [18], but with only one particle per source; Ref. [14] also considered measuring spinless particles in an interferometer. We proceed as above to find a probability $P(m_1 \cdots m_6)$.

For obtaining GHZ contradictions, we consider the case $N_\alpha = N_\beta = N_\gamma = N/3$ and the events where all three detectors receive the same number of particles. Because we are considering a limited number of cases, the normalization is now different, and we must compute

$$\mathcal{N} \equiv \sum_{m_1 \cdots m_6} \delta_{m_1+m_2, N/3} \delta_{m_3+m_4, N/3} \times \delta_{m_5+m_6, N/3} P(m_1 \cdots m_6). \qquad (10)$$

In order to limit the sum over the m_i's to these cases, we introduce three more integrals over ρ_A, ρ_B, and ρ_C of the form $\delta_{m_1+m_2, N/3} = \int_{-\pi}^{\pi} (d\rho_A/2\pi) e^{i(m_1+m_2-N/3)\rho_A}$. To find the average $\langle \mathcal{ABC} \rangle$ for measurements done by Alice, Bob, and Car-
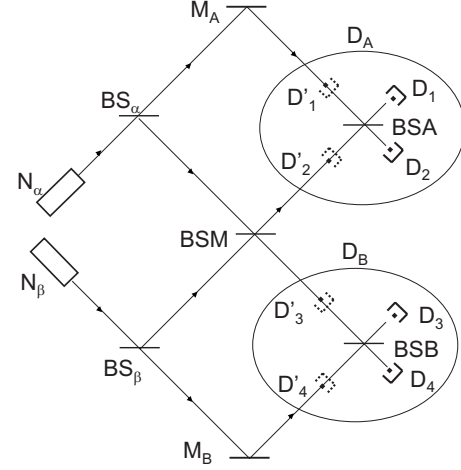
ole we introduce $(-1)^{m_2+m_4+m_6}$ into the sums over the m_i and perform them in the same way. Dividing by the normalization gives us

$$\langle \mathcal{ABC} \rangle = \frac{\sum_q \left(\dfrac{N/3!}{(N/3-q)! q!} \right)^3 e^{i(\varsigma + \theta + \chi)(N/3-2q)}}{\sum_q \left(\dfrac{N/3!}{(N/3-q)! q!} \right)^3}. \qquad (11)$$

In the usual GHZ case $N=3$, we find $\langle \mathcal{ABC} \rangle = \cos(\varsigma + \theta + \chi)$, and perfect correlations if the sum of angles is π. Local realism then gives $\langle \mathcal{ABC} \rangle = A(\varsigma) B(\theta) C(\chi)$ and

$$A(\pi/2) B(\pi/2) C(0) = -1,$$

$$A(\pi/2) B(0) C(\pi/2) = -1,$$

$$A(0) B(\pi/2) C(\pi/2) = -1. \qquad (12)$$

But then we must have $A(0) B(0) C(0) = -1$, while quantum mechanics gives $+1$. For larger N we also obtain contradictions, for instance when $N=9$, where

$$\langle \mathcal{ABC} \rangle = \frac{1}{28} [27 \cos(\varsigma + \theta + \chi) + \cos 3(\varsigma + \theta + \chi)]. \qquad (13)$$

Since both cosines change sign when the angles increase by π, the above argument remains unchanged and, again, leads to complete sign contradiction. Actually, any time $N/3$ is odd, we get a similar result for arbitrary N.

Hardy impossibilities are treated by use of the interferometer shown in Fig. 3 based on the one discussed in Ref. [15] for $N=2$. The heart of the system is the beam splitter at the center; due to Bose interference it has the property that, if an equal number of particles approaches each side, then an even number emerges from each side. The detection beam splitters BSA and BSB are set to have a transmission probability of

1/3 and the path differences are such that, by destructive interference, no particle reaches D_2 if only source N_α is used; similarly, no particle reaches D_3 if N_β alone is used. Alice can use either the detectors $D_{1,2}$ after her beam splitter, or $D'_{1,2}$ before; Bob can choose either $D_{3,4}$, or $D'_{3,4}$. This gives four arrangements of experiments DD, DD', $D'D$, or $D'D'$, with probability amplitudes $C_{XY}(m_1,m_2;m_3,m_4)$, where XY is any of these four arrangements and the m values are the numbers of particles detected at each counter.

Assume for instance that $N=6$ and that both experimenters observe three particles. The probability amplitude $C_{D'D'}(0,3;3,0)$ vanishes because of the beam splitter rule. The destructive interference effect at BSA and BSB lead to $C_{DD'}(0,3;1,2)=C_{DD'}(0,3;2,1)=C_{DD'}(0,3;0,3)=0$; but $C_{DD'}(0,3;3,0)\neq 0$. Thus, if Alice observes three particles at D_2, when Bob uses the primed detectors he observes with certainty three particles at D'_3; similarly, if Bob has seen three particles in D_3, in the $D'D$ configuration Alice must see three in D'_2.

If both do unprimed experiments, we find $C_{DD}(0,3;3,0)=1/216$, which shows that events exist where 3 particles are detected at both detectors D_2 and D_3. In any of these events, if Bob had at the last instant changed to the primed detectors, he would surely have obtained three particles in D'_3, because

of the certainty mentioned above; if Alice had changed detectors instead of Bob, she would have obtained three particles in D'_2. Now, had both changed their minds after the emission and chosen the primed arrangement, local realism implies that they would have found three particles in each D'_2 and D'_3: such events must exist. But its quantum probability is exactly zero, in complete contradiction. This argument can be generalized to all cases of odd $N/2$ emitted from each source.

In conclusion, we think that the answer to Anderson's question is that "in most cases, this view leads to correct quantum predictions, but not always." It is sufficient for instance in the situations described in [2]; but when all particles are measured, quantum mechanics sometimes predicts probabilities that cannot be explained in terms of a pre-existing phase, and reveal a more fundamental quantum character of this physical quantity. This creates the possibility of significant $N-$ body violations of local realism with the use of Fock-state condensates.

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