

Full quantum reconstruction of vortex states

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We propose a complete tomographic reconstruction of vortex states carrying orbital angular momentum. The scheme determines the angular probability distribution of the state at different times under free evolution. To represent the quantum state, we introduce a *bona fide* Wigner function defined on the discrete cylinder, which is the natural phase space for the pair angle-angular momentum. The feasibility of the proposal is addressed.

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Phase singularities of wave fields were brought to attention in a number of seminal papers [1–3]. They manifest in electron wave packets [4], quantum Hall fluids [5], supermedia [6], ferromagnets [7], Bose-Einstein condensates [8], acoustical waves [9], and light fields [10], to cite some relevant examples. Special attention has been paid to the particular case of vortices, which correspond to helicoidal waves exhibiting a pure screw phase dislocation along the propagation axis—i.e., an azimuthal phase dependence $e^{i\ell\phi}$. This means that ℓ plays the role of a topological charge: the phase changes its value in ℓ cycles of 2π in any closed circuit about this axis, while the amplitude is zero there.

One of the most interesting properties of vortices is that they carry orbital angular momentum (OAM): indeed, the integer ℓ can be seen as the eigenvalue of the OAM operator and its sign defines the helicity or direction of rotation. In fact, OAM can be easily transferred, as demonstrated in a number of recent challenging experiments with optically trapped microparticles [11].

At the microscopic level, quantized vortices are now routinely produced by, e.g., mechanical stirring of ultracold atomic clouds. Remarkable images of large lattices containing hundreds of vortices in an Abrikosov-type triangular configuration have been obtained [12]. However, for complete information about the corresponding state, one needs to go beyond this mere photographic picture and perform a full tomography. Efficient methods of state reconstruction are of the greatest relevance for quantum optics. Since the first theoretical proposals, this discipline has witnessed significant growth [13] and laboratory demonstrations of state tomography are numerous and span a broad range of physical systems. The essence of vortices is their helicoidal structure, while other aspects, such as energy distribution, may be often ignored for many purposes: in this sense, a fully quantum reconstruction of the vortex content is pending.

Any reliable quantum tomographical scheme requires three key ingredients [14]: the availability of a tomographically complete measurement, a suitable representation of the quantum state, and a robust algorithm to invert the experimental data. When these conditions are not met, reconstruction becomes difficult, if not impossible: this has been the case so far for vortex states. The construction of a proper Wigner function for them (or any other quasiprobability distribution in phase space) is still an open question. Although some interesting attempts have been published [15,16], they

seem of difficult application for the problem at hand. The twofold goal of this paper is precisely to fill this long overdue gap. We will provide a simple Wigner function with a clear geometrical meaning and also a universal tomographic reconstruction scheme. One could think of using the standard Wigner function for the transverse coordinate-momentum variables instead. However, this is not the right answer in quantum mechanics: such a representation conveys redundant information, it is impossible to plot, and it hides all the relevant angular information.

We consider rotations by an angle ϕ generated by the OAM operator along the z axis, which for simplicity we shall denote henceforth as \hat{L} . We do not want to enter in a long and sterile discussion about the possible existence of an angle operator. For our purposes here the simplest choice is to use the complex exponential of the angle, $\hat{E}=e^{-i\hat{\phi}}$, which satisfies the commutation relation $[\hat{E}, \hat{L}]=\hat{E}$. The action of \hat{E} on the OAM eigenstates is $\hat{E}|\ell\rangle=|\ell-1\rangle$, and it possesses then a simple implementation by means of a phase mask removing a unit charge [17]. Since the integer ℓ runs from $-\infty$ to $+\infty$, \hat{E} is a unitary operator whose normalized eigenvectors $|\phi\rangle$ describe states with well-defined angular positions. In the representation generated by them, \hat{L} acts as $-i\partial_\phi$ (in units $\hbar=1$). Note in passing that one could intuitively expect a Fourier-like relationship between angle and OAM, which can be expressed in this context as $e^{-i\phi'\hat{L}}|\phi\rangle=|\phi-\phi'\rangle$.

For the standard harmonic oscillator, the phase space is the plane \mathbb{R}^2 . In the case of angle and OAM the phase space is the discrete cylinder $S^1 \times \mathbb{Z}$ (where S^1 is the unit circle). While quasiprobability distributions on the sphere are commonplace in quantum optics, to the best of our knowledge their counterparts on the cylinder have never been used to describe angular variables. Given the key role played by the displacement operators in defining the Wigner function in the plane, we introduce a unitary displacement operator $\hat{D}(\ell, \phi)$ on the discrete cylinder as

$$\hat{D}(\ell, \phi) = e^{i\alpha(\ell, \phi)} \hat{E}^{-\ell} e^{-i\phi\hat{L}}, \quad (1)$$

where $\alpha(\ell, \phi)$ is an undefined phase factor. Apart from 2π periodicity in ϕ , the requirement of unitarity imposes the condition $\alpha(\ell, \phi) + \alpha(-\ell, -\phi) = -\ell\phi$. The displacement op-

erators form a non-Hermitian orthogonal basis on the Hilbert space, in the sense that

$$\text{Tr}[\hat{D}(\ell, \phi)\hat{D}^\dagger(\ell', \phi')] = 2\pi\delta_{\ell\ell'}\delta_{2\pi}(\phi - \phi'), \quad (2)$$

where $\delta_{2\pi}$ represents the periodic delta function (or Dirac comb) of period 2π .

Next, we introduce the Wigner kernel as a kind of double Fourier transform of the displacement operators

$$\hat{w}(\ell, \phi) = \frac{1}{(2\pi)^2} \sum_{\ell' \in \mathbb{Z}} \int_{2\pi} d\phi' \exp[-i(\ell'\phi - \ell\phi')] \hat{D}(\ell', \phi'), \quad (3)$$

where the integral extends to the 2π interval within which the angle is defined. The integral of $\hat{w}(\ell, \phi)$ over the whole phase space yields unity, and we may thus regard $\hat{w}(\ell, \phi)$ as equivalent to the phase-point operators introduced by Wootters [18]. In addition, one can check that

$$\hat{w}(\ell, \phi) = \hat{D}(\ell, \phi)\hat{w}(0,0)\hat{D}^\dagger(\ell, \phi). \quad (4)$$

Since for the plane and the sphere the Wigner kernel can be seen as the transform of the parity by the displacement operators [19], this seems to call for interpreting $\hat{w}(0,0)$ as the parity over our phase space.

We next define the Wigner function of a quantum state described by the density matrix $\hat{\rho}$ as

$$W(\ell, \phi) = \text{Tr}[\hat{\rho}\hat{w}(\ell, \phi)]. \quad (5)$$

Using the previous results for $\hat{w}(\ell, \phi)$, one can show that $W(\ell, \phi)$ fulfills all the properties required for a reasonable interpretation as a quasiprobability distribution. Indeed, due to the hermiticity of the Wigner kernel, $W(\ell, \phi)$ is real. It also provides the proper marginal distributions, and it is covariant, which means that if the state $\hat{\rho}'$ is obtained from $\hat{\rho}$ by a displacement in phase space, $\hat{\rho}' = \hat{D}(\ell', \phi')\hat{\rho}\hat{D}^\dagger(\ell', \phi')$, then the Wigner function follows along rigidly: $W'(\ell, \phi) = W(\ell - \ell', \phi - \phi')$. All these properties are fulfilled independently of the choice of the phase $\alpha(\ell, \phi)$.

Since the displacement operators constitute a basis, we can write the expansion

$$\hat{\rho} = \sum_{\ell \in \mathbb{Z}} \int_{2\pi} d\phi \varrho(\ell, \phi) \hat{D}(\ell, \phi), \quad (6)$$

where $\varrho(\ell, \phi) = \text{Tr}[\hat{\rho}\hat{D}^\dagger(\ell, \phi)]/(2\pi)$. In terms of $\varrho(\ell, \phi)$, the Wigner function has the representation

$$W(\ell, \phi) = \frac{1}{2\pi} \sum_{\ell' \in \mathbb{Z}} \int_{2\pi} d\phi' \varrho(\ell', \phi') e^{i(\ell'\phi - \ell\phi')}. \quad (7)$$

To work out explicit examples, one needs to fix once and for all the function $\alpha(\ell, \phi)$. One natural option is to set $\alpha(\ell, \phi) = -\ell\phi/2$, and then the Wigner kernel (3) reduces to

$$\begin{aligned} \hat{w}(\ell, \phi) &= \frac{1}{2\pi} \sum_{\ell' \in \mathbb{Z}} e^{-2i\ell'\phi} |\ell + \ell'\rangle \langle \ell - \ell'| \\ &+ \frac{1}{2\pi^2} \sum_{\ell', \ell'' \in \mathbb{Z}} \frac{(-1)^{\ell'' - \ell}}{\ell'' - \ell + 1/2} \\ &\times e^{-i(2\ell' + 1)\phi} |\ell'' + \ell' + 1\rangle \langle \ell'' - \ell'|, \end{aligned} \quad (8)$$

which agrees with the kernel derived by Plebański and co-workers [20] in the context of deformation quantization on the cylinder.

For an OAM eigenstate $|\ell_0\rangle$, we obtain

$$W_{|\ell_0\rangle}(\ell, \phi) = \frac{1}{2\pi} \delta_{\ell\ell_0}, \quad (9)$$

which is a quite reasonable Wigner function: it is flat in ϕ , and the integral over the whole phase space equals unity, reflecting the normalization of $|\ell_0\rangle$.

For an angle eigenstate $|\phi_0\rangle$, we get

$$W_{|\phi_0\rangle}(\ell, \phi) = \frac{1}{2\pi} \delta_{2\pi}(\phi - \phi_0). \quad (10)$$

Now, it is flat in the conjugate variable ℓ , and thus, the integral over the whole phase space is not finite, which is a consequence of the fact that the state $|\phi_0\rangle$ is unnormalized.

The coherent states $|\ell_0, \phi_0\rangle$ (parametrized by points on the cylinder) introduced in Ref. [21] (see also Ref. [22] for a detailed discussion of the properties of these relevant states) satisfy

$$\begin{aligned} \langle \ell | \ell_0, \phi_0 \rangle &= \frac{1}{\sqrt{\vartheta_3\left(0 \left| \frac{1}{e} \right.\right)}} e^{-i\ell\phi_0} e^{-(\ell - \ell_0)^2/2}, \\ \langle \phi | \ell_0, \phi_0 \rangle &= \frac{e^{i\ell_0(\phi - \phi_0)}}{\sqrt{\vartheta_3\left(0 \left| \frac{1}{e} \right.\right)}} \vartheta_3\left(\frac{\phi - \theta}{2} \left| \frac{1}{e^2} \right.\right), \end{aligned} \quad (11)$$

where ϑ_3 denotes the third Jacobi theta function. One immediately finds that

$$\varrho(\ell, \phi) = \frac{e^{i\ell(\phi/2 - \phi_0)}}{2\pi\vartheta_3\left(0 \left| \frac{1}{e} \right.\right)} e^{-\ell^2/2 - i\ell_0\phi} \vartheta_3\left(\frac{\phi}{2} + \frac{i\ell}{2} \left| \frac{1}{e} \right.\right), \quad (12)$$

which gives

$$W_{|\ell_0, \phi_0\rangle}(\ell, \phi) = W_{|\ell_0, \phi_0\rangle}^{(+)}(\ell, \phi) + W_{|\ell_0, \phi_0\rangle}^{(-)}(\ell, \phi), \quad (13)$$

with

$$W_{|\ell_0, \phi_0\rangle}^{(+)}(\ell, \phi) = \frac{1}{2\pi\vartheta_3\left(0 \left| \frac{1}{e} \right.\right)} e^{-(\ell - \ell_0)^2} \vartheta_3\left(\phi - \phi_0 \left| \frac{1}{e} \right.\right),$$

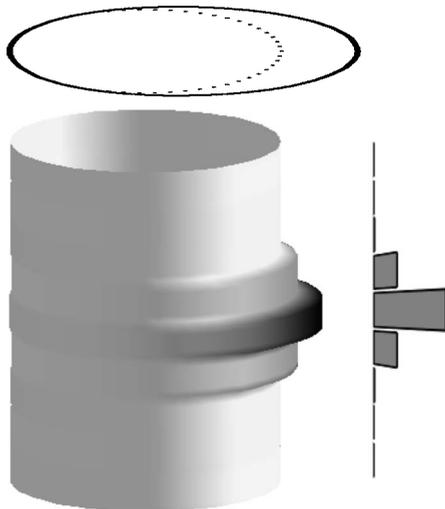


FIG. 1. Plot of the Wigner function for a coherent state with $\ell_0=0$ and $\phi_0=0$. The cylinder extends vertically from $\ell=-4$ to $\ell=+4$. We show the two corresponding marginal distributions.

$$W_{|\ell_0, \phi_0\rangle}^{(-)}(\ell, \phi) = \frac{e^{i(\phi-\phi_0)-1/2}}{2\pi^2 \mathfrak{D}_3(0|\frac{1}{e})} \mathfrak{D}_3\left(\phi - \phi_0 + i/2 \left| \frac{1}{e} \right.\right) \times \sum_{\ell' \in \mathbb{Z}} \frac{(-1)^{\ell' - \ell + \ell_0} e^{-\ell'^2 - \ell'}}{\ell' + \ell_0 - \ell + 1/2}. \quad (14)$$

In spite of the fact that these states may be regarded as a basic set to construct the Wigner kernel, their Wigner function has no simple closed form, but it splits into two different functions with periods π and 2π , respectively.

In Fig. 1 we show the Wigner function for the coherent state $|\ell_0=0, \phi_0=0\rangle$ plotted on the discrete cylinder. We can see a pronounced peak at $\phi=0$ for $\ell=0$ and slightly smaller ones for $\ell=\pm 1$. A closer look at the picture reveals also a remarkable fact: for values close to $\phi=\pm\pi$ and $\ell=\pm 1$, the Wigner function takes negative values. Actually, a numeric analysis suggests the existence of negativities close to $\phi=\pm\pi$ for odd values of ℓ . We also plot the marginals obtained from Eq. (13) by integrating over ϕ or summing over ℓ .

As our last example, we look at the superposition states

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\ell_0\rangle + e^{i\phi_0} |-\ell_0\rangle), \quad (15)$$

which have been proposed for applications in quantum experiments [23]. The analysis can be carried out for more general superpositions, but (15) is enough to display the relevant features. The final result is

$$W_{|\Psi\rangle}(\ell, \phi) = \frac{1}{4\pi}(\delta_{\ell\ell_0} + \delta_{\ell-\ell_0}) + \frac{1}{2\pi} \cos(\phi_0 - 8\phi) \delta_{\ell 0}. \quad (16)$$

The Wigner function presents then three contributions: two flat slices coming from the states $|\ell_0\rangle$ and $|-\ell_0\rangle$ and their interference located at the origin (see Fig. 2).

To complete our theory we also propose a reconstruction scheme for these states. A reconstruction of $\hat{\rho}$ (or, equiva-

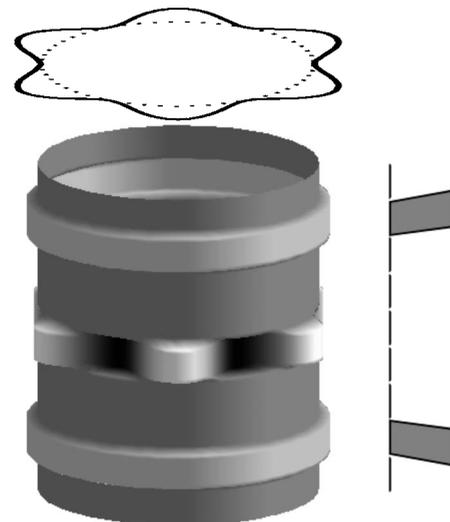


FIG. 2. Plot of the Wigner function (and the corresponding marginals) for a superposition state (15) with $\ell_0=3$ and $\phi_0=\pi$ and its marginal distributions. The cylinder extends vertically from $\ell=-4$ to $\ell=+4$.

lently, of its Wigner function) is tantamount to finding the coefficients $\varrho(\ell, \phi)$. To this end, we need a tomographical measurement that allows us to reconstruct $\varrho(\ell, \phi)$. For $\ell=0$, the coefficients $\varrho(0, \phi)$ read as

$$\varrho(0, \phi) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \langle \ell | \hat{\rho} | \ell \rangle e^{i\ell\phi}, \quad (17)$$

where we have made use of the fact that the undetermined phase $e^{-i\alpha(0, \phi)}$ can be set to 1 for all values of ϕ , for it corresponds to displacements along one of the coordinate axes and no additional phase should be acquired.

For $\ell \neq 0$, we introduce the operator $\hat{U}_t = \exp(-it\hat{L}^2/2)$. If we recall that the generic Hamiltonian of a quantum rotator is $\hat{H} = \hat{L}^2/2I$ (I being the rotational moment of inertia), the operator \hat{U}_t is simply the free propagator of the system (in appropriate units). This quantum rotator Hamiltonian describes a variety of situations, such as molecular rotations [24], single-photon OAM [25], or the azimuthal evolution of optical beams [26]. For all of them, our proposal for $\omega(\phi, t)$ is precisely to measure the angular distribution after a free evolution t —that is,

$$\omega(\phi, t) = \langle \phi(t) | \hat{\rho} | \phi(t) \rangle = \langle \phi | \hat{U}_t^\dagger \hat{\rho} \hat{U}_t | \phi \rangle. \quad (18)$$

In other cases, the scheme also works appropriately provided one can experimentally implement the action of \hat{L}^2 . Using the representations (6) for $\hat{\rho}$ and (1) for \hat{D} , $\varrho(\ell, \phi)$ turns out to be

$$\varrho(\ell, \phi) = \frac{1}{2\pi} e^{-i\alpha(\ell, \phi)} \int_{2\pi} d\phi' e^{-i\ell\phi'} \omega(\phi', \phi/\ell). \quad (19)$$

Plugging these coefficients into Eq. (7), we get the reconstruction of the Wigner function as

$$W(\ell, \phi) = \frac{1}{2\pi} \langle \ell | \hat{Q} | \ell \rangle + \frac{1}{(2\pi)^2} \sum_{\ell' \in \mathbb{Z}} \int_{2\pi} d\phi' d\phi'' e^{-i\alpha(\ell', \phi')} \times e^{i[\ell\phi' - \ell'(\phi - \phi'')]} \omega(\phi'', \phi' / \ell'). \quad (20)$$

Consistently, the reconstruction procedure itself does not depend on the undetermined phase $\alpha(\ell, \phi)$ of the displacement operator, while the Wigner function does have such a dependence. The phase factor acts as a metric coefficient for the mapping from Hilbert space onto the phase space. We recall that for the harmonic oscillator, one can recover the Wigner function via an inverse Radon transform from the quadrature probability distribution [27]. Equation (20) is then analogous for our system.

As a rather simple yet illustrative example, let us note that for the vortex state $|\ell_0\rangle$, \hat{U}_t is diagonal, so the tomograms $\omega(\phi', \phi' / \ell)$ are independent of ϕ and ℓ and all of them equal to $1/(2\pi)$. Performing the integration in Eq. (20) we obtain precisely the Wigner function (9).

Finally, for the feasibility of the proposed scheme, we need the projection onto the eigenstates $|\phi\rangle$. Since these states correspond to the measurement of a continuous variable, such a measurement can be done only approximately. A

good approximation seems to be the projection onto the wedge states, although other experimental schemes are also available [28]. According to Eq. (18), this projection and the determination of t are the two main sources of measurement error. Afterward, the reconstruction errors for each particular experimental implementation should be evaluated following the standard recipes developed in, e.g., Ref. [29].

In summary, we have carried out a full program for the reconstruction of generic vortex states, including a complete phase-space description in terms of a *bona fide* Wigner function. Though the implementation of this scheme may differ depending on the system under consideration, our formulation provides a common theoretical framework on the Hilbert space generated by the action of angle and angular momentum. Experimental demonstrations of the method in terms of optical beams are presently underway in our laboratory and will be reported elsewhere.

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