

## Detection of quantum light in the presence of noise

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Detection of quantum light in the presence of dark counts and background radiation noise is considered. The corresponding positive operator-valued measure is obtained and photocounts statistics of quantum light in the presence of noise is studied.

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Photodetectors play a crucial role in all experimental investigations dealing with quantum optics, fundamentals of quantum physics, and quantum-information processing. These devices are used for measuring the photon number of radiation fields. The theory of photodetection has been developed for both classical [1,2] and quantum light fields [2–6].

Different kinds of losses are a serious problem in the photoelectric detection of quantum light. Presently available technologies enable us to get the detection efficiency near 0.9 and even more [7]. At the same time, attempts to improve it increase the dark counts rate [8]. Besides, in many applications the background radiation noise contributes to the total statistics of photocounts similarly to dark counts [9,10].

In some models (see, e.g., [11]), noise counts, originated from dark counts and background radiation, are described by coupling the radiation field to a single mode of the thermal bath. However, such a simple model has, at least, two serious drawbacks. First, it does not predict Poissonian statistics usually [10] peculiar to noise counts. Second, it does not consider the presence of other modes of the thermal bath. Although some of these modes are not coupled to the mode of the radiation field, they contribute to the total statistics of photocounts. Therefore, a more appropriate model should include a multimode noise.

For the classical fields, a similar problem was considered in Refs. [9,10]. At the same time, quantum radiation demonstrates many nonclassical properties such as sub-Poissonian statistics [12], quadrature squeezing [13], etc. These properties can be described only in the framework of the consistent quantum formalism, which is the subject of this paper.

According to the quantum measurement theory, see, e.g., [14], the process of photodetection is characterized by the positive operator-valued measure (POVM),  $\hat{\Pi}_n$ . For the quantum radiation field characterized by the density operator  $\hat{\rho}$ , the probability distribution  $P_n$  to get  $n$  photocounts is written as

$$P_n = \text{Tr}(\hat{\Pi}_n \hat{\rho}). \quad (1)$$

A well-known result of the photodetection theory [1–6] is the expression for the POVM of a single-mode radiation field and for the detectors with losses,

$$\hat{\Pi}_n = \frac{(\eta \hat{a}^\dagger \hat{a})^n}{n!} e^{-\eta \hat{a}^\dagger \hat{a}}, \quad (2)$$

where  $\eta$  is the efficiency of detection,  $\hat{a}^\dagger$  and  $\hat{a}$  are creation and annihilation operators of the field mode, correspondingly, and  $::$  means normal ordering. Expression (2) plays a crucial role in various investigations dealing with quantum optical measurements; see, e.g., [15–17]. We generalize this expression for the case of detection in the presence of noise counts.

Let us consider the normal ordered (Husimi-Kano) symbol [18] of the POVM,

$$\Pi_n(\alpha) = \langle \alpha | \hat{\Pi}_n | \alpha \rangle, \quad (3)$$

where  $|\alpha\rangle$  is a coherent state. It is obvious that  $\Pi_n(\alpha)$  is a probability to get  $n$  photocounts when the detector is irradiated by a coherent light with an amplitude  $\alpha$ . The operator form of the POVM can be obtained from the explicit form of expression (3) by replacing  $\alpha$  with  $\hat{a}$  and  $\alpha^*$  with  $\hat{a}^\dagger$  under the sign of normal ordering.

As was mentioned above, noise counts for realistic photodetection should be described with a multimode heat bath. Finite detection time enables one to restrict consideration to a discrete set of modes for the radiation field and, consequently, for the thermal bath. The number of thermal modes  $\mu$  can be approximately evaluated as follows:

$$\mu \approx \Delta\omega T, \quad (4)$$

once the bandwidth of heat bath  $\Delta\omega$  and detection time  $T$  is known.

For a sufficiently small detection time, the model with single-mode thermal noise [11] seems meaningful. However, an account of an additional thermal mode may significantly change the statistics of photocounts; see [2,19,20]. We simulate the detection in the presence of noise counts by  $\mu$  modes of the thermal bath coupled to  $\nu$  modes of the radiation field; see the beam-splitter replacement scheme in Fig. 1. The corresponding normal-ordered symbol of the POVM is equal to the probability distribution of photocounts for mixture of the coherent and thermal fields. The analytical expression for the probability distribution of photocounts for the superposition

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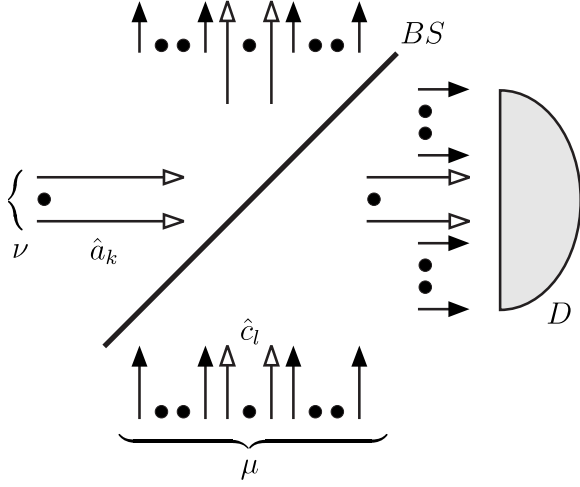


FIG. 1. The beam-splitter model for the detection in the presence of noise.  $D$  is an ideal detector and  $BS$  is a beam splitter with transmission coefficient  $\sqrt{\eta}$ ; the operators  $\hat{a}_k$ ,  $k=1 \dots \nu$  and  $\hat{c}_l$ ,  $l=1 \dots \mu$  describe the modes of the incident radiation field and thermal bath, correspondingly.

of the coherent and thermal light can be obtained under the assumption that all the modes of the thermal bath have an equal mean number of photons,  $\bar{N}_{nc}/\mu$  [21],

$$\begin{aligned} \Pi_n(\{\alpha_k\}; \bar{N}_{nc}, \mu) &= \frac{\left(\frac{\bar{N}_{nc}}{\mu}\right)^n}{\left(1 + \frac{\bar{N}_{nc}}{\mu}\right)^{n+\mu}} \exp\left(-\frac{\eta}{1 + \frac{\bar{N}_{nc}}{\mu}} \sum_{k=1}^{\nu} |\alpha_k|^2\right) \\ &\times L_n^{\mu-1}\left(-\frac{\eta}{\frac{\bar{N}_{nc}}{\mu} \left(1 + \frac{\bar{N}_{nc}}{\mu}\right)} \sum_{k=1}^{\nu} |\alpha_k|^2\right), \quad (5) \end{aligned}$$

where  $\alpha_k$ , for  $k=1, \dots, \nu$ , is the complex coherent amplitude for the  $k$ th mode of the radiation field,  $\bar{N}_{nc}$  is the overall number of thermal photons (noise counts) in the output of the beam splitter, and  $L_n^m(x)$  is the generalized Laguerre polynomial.

For the sake of simplicity, we consider a single-mode radiation field, i.e.,  $\nu=1$ . All the results can be generalized simply for the case of an arbitrary  $\nu$ . In this case the POVM, cf. Eq. (5), is rewritten as

$$\begin{aligned} \Pi_n(\alpha; \bar{N}_{nc}, \mu) &= \frac{\left(\frac{\bar{N}_{nc}}{\mu}\right)^n}{\left(1 + \frac{\bar{N}_{nc}}{\mu}\right)^{n+\mu}} \exp\left(-\frac{\eta}{1 + \frac{\bar{N}_{nc}}{\mu}} |\alpha|^2\right) \\ &\times L_n^{\mu-1}\left(-\frac{\eta}{\frac{\bar{N}_{nc}}{\mu} \left(1 + \frac{\bar{N}_{nc}}{\mu}\right)} |\alpha|^2\right). \quad (6) \end{aligned}$$

In the operator form, the POVM is written as

$$\begin{aligned} \hat{\Pi}_n(\bar{N}_{nc}, \mu) &= \frac{\left(\frac{\bar{N}_{nc}}{\mu}\right)^n}{\left(1 + \frac{\bar{N}_{nc}}{\mu}\right)^{n+\mu}} \exp\left(-\frac{\eta}{1 + \frac{\bar{N}_{nc}}{\mu}} \hat{a}^\dagger \hat{a}\right) \\ &\times L_n^{\mu-1}\left(-\frac{\eta}{\frac{\bar{N}_{nc}}{\mu} \left(1 + \frac{\bar{N}_{nc}}{\mu}\right)} \hat{a}^\dagger \hat{a}\right). \quad (7) \end{aligned}$$

One can prove that in the framework of the considered model, Eqs. (6) and (7) can also be applied for the detection of wideband quantum light even in the case in which several modes of the heat bath are coupled to the radiation field. In this case, the operator  $\hat{a}$  describes the corresponding non-monochromatic mode. Therefore, we conclude that the statistics of photocounts does not depend on the number  $\nu$  of thermal-bath modes coupled to the signal and depends only on the total number  $\mu$  of heat-bath modes.

In most practical implementations of the quantum optical schemes, one seemingly deals with a large number of thermal modes. If, for example, the bandwidth of the heat bath includes the whole optical band, i.e.,  $\Delta\omega \sim 10^{15} \text{ s}^{-1}$ , and the detection time is  $T=10^{-9} \text{ s}$ , from Eq. (4) it follows that the number of thermal modes  $\mu \sim 10^6$ . The limit of Eq. (6) for a large number of noise modes,

$$\Pi_n(\alpha; \bar{N}_{nc}) = \lim_{\mu \rightarrow +\infty} \Pi_n(\alpha; \bar{N}_{nc}, \mu), \quad (8)$$

can be easily obtained as

$$\Pi_n(\alpha; \bar{N}_{nc}) = \frac{(\eta|\alpha|^2 + \bar{N}_{nc})^n}{n!} \exp(-\eta|\alpha|^2 - \bar{N}_{nc}). \quad (9)$$

This equation determines the normal-ordered symbol of the POVM in the presence of noise that can be applied in most practical situations. The corresponding operator form of the POVM is represented as

$$\hat{\Pi}_n(\bar{N}_{nc}) = \frac{(\eta \hat{a}^\dagger \hat{a} + \bar{N}_{nc})^n}{n!} \exp(-\eta \hat{a}^\dagger \hat{a} - \bar{N}_{nc}). \quad (10)$$

Coupling between the noise modes and the radiation field is negligible as soon as

$$\frac{\bar{N}_{nc}}{\mu} \ll 1. \quad (11)$$

Consider the variance of photocounts,  $\overline{\Delta n^2}$ ,

$$\overline{\Delta n^2} = \bar{n} + \eta^2 \langle \Delta \hat{n}^2 \rangle + \frac{\bar{N}_{nc}}{\mu} (2\eta \langle \hat{n} \rangle + \bar{N}_{nc}). \quad (12)$$

In this expression,

$$\bar{n} = \eta \langle \hat{n} \rangle + \bar{N}_{nc} \quad (13)$$

is the mean number of photocounts, which includes the mean number of photons,  $\langle \hat{n} \rangle$ , and that of noise counts,  $\bar{N}_{\text{nc}}$ . This means that noise counts contribute to the shot noise of the detector. The second term in Eq. (12) describes excess noise caused by the stochastic nature of the light, and  $\langle : \Delta \hat{n}^2 : \rangle$  is the normal-ordered dispersion of photon number.

The third term in Eq. (12) corresponds to the excess noise caused by noise counts. This noise disappears when condition (11) is satisfied. Therefore, for the detection times

$$T \approx \frac{\bar{N}_{\text{nc}}}{\Delta \omega}, \quad (14)$$

the statistics of noise counts differs from Poissonian, and the corresponding POVM is described by Eq. (7). This can take place, e.g., for the femtosecond detection times. However, such an interesting case from the point of different applications cannot be implemented with modern technologies. In this case, the contribution of  $\bar{N}_{\text{nc}}^2 / \mu$  into the third term is significant in any situation, and another contribution,  $2\eta \langle \hat{n} \rangle \bar{N}_{\text{nc}} / \mu$ , is significant only in the case in which the field source of noise counts is coupled to the radiation field. For detection times sufficiently greater than that defined by Eq. (14), noise counts obey the Poissonian statistics and contribute only to the shot noise of the detector, cf. Eq. (13). The corresponding POVM is described by Eq. (10).

As an example, consider the light with sub-Poissonian statistics of photocounts [12]. This property can be characterized by the Mandel parameter [2,22], which is defined as the ratio of the excess-noise and shot-noise variations. The Mandel parameter in the case of Poissonian statistics of noise counts,

$$Q = \eta \frac{\langle : \Delta \hat{n}^2 : \rangle}{\langle \hat{n} \rangle + \frac{\bar{N}_{\text{nc}}}{\eta}}, \quad (15)$$

is a monotonic function of  $\bar{N}_{\text{nc}} / \eta$ . For large values of  $\bar{N}_{\text{nc}}$ , the Mandel parameter slowly tends to zero. Otherwise, in the case of the detection time comparable with that given by Eq. (14), the Mandel parameter as a function of  $\bar{N}_{\text{nc}}$ ,

$$Q = \eta \frac{\langle : \Delta \hat{n}^2 : \rangle + \frac{1}{\mu} \frac{\bar{N}_{\text{nc}}}{\eta} \left( 2\langle \hat{n} \rangle + \frac{\bar{N}_{\text{nc}}}{\eta} \right)}{\langle \hat{n} \rangle + \frac{\bar{N}_{\text{nc}}}{\eta}}, \quad (16)$$

has a threshold value,

$$\bar{N}_{\text{nc}} = \eta \sqrt{\langle \hat{n} \rangle^2 - \mu \langle : \Delta \hat{n}^2 : \rangle} - \eta \langle \hat{n} \rangle, \quad (17)$$

starting from which the Mandel parameter is positive and the corresponding statistics is super-Poissonian.

In the ideal photodetection, the presence of  $n$  photons is always converted to the  $n$  photocounts. In other words, for the Fock-number state,  $|n\rangle$ , the probability to get  $n$  photocounts is always equal to 1. In the case of noisy detection,

the presence of  $n$  photons may result in a different number of photocounts,  $m$ . The probability to get  $m$  photocounts under the condition that  $n$  photons are present is

$$P_{m|n} = \langle n | \hat{\Pi}_m | n \rangle. \quad (18)$$

The POVM is expanded into series,

$$\hat{\Pi}_m = \sum_{n=0}^{+\infty} P_{m|n} |n\rangle \langle n|, \quad (19)$$

and the probability to get  $m$  photocounts,  $P_m$ , is

$$P_m = \sum_{n=0}^{+\infty} P_{m|n} p_n, \quad (20)$$

where

$$p_n = \langle n | \hat{\rho} | n \rangle \quad (21)$$

is the noiseless statistics of photocounts, i.e., the probability that  $n$  photons are present.

For the case of Poissonian statistics of noise counts, the conditional probability, Eq. (18), is given by

$$P_{m|n}(\eta, \bar{N}_{\text{nc}}) = e^{-\bar{N}_{\text{nc}}} \bar{N}_{\text{nc}}^{m-n} \eta^n \mathcal{L}_n^{m-n} \left( \frac{\bar{N}_{\text{nc}}(1-\eta)}{\eta} \right) \quad (22)$$

for  $m > n$  and

$$P_{m|n}(\eta, \bar{N}_{\text{nc}}) = e^{-\bar{N}_{\text{nc}}} (1-\eta)^{n-m} \eta^m \mathcal{L}_m^{n-m} \left( \frac{\bar{N}_{\text{nc}}(1-\eta)}{\eta} \right) \quad (23)$$

for  $m \leq n$ . We will consider two important limiting forms of these expressions.

The first limit corresponds to the detectors without noise counts,  $\bar{N}_{\text{nc}} = 0$ ,

$$P_{m|n}(\eta, \bar{N}_{\text{nc}} = 0) = 0 \quad (24)$$

for  $m > n$  and

$$P_{m|n}(\eta, \bar{N}_{\text{nc}} = 0) = \binom{n}{m} \eta^m (1-\eta)^{n-m} \quad (25)$$

for  $m \leq n$ . Since in this case Eq. (20) presents the binomial transform, it can be analytically inverted,

$$p_n = \sum_{m=0}^n \binom{n}{m} \frac{1}{\eta^n} \left( 1 - \frac{1}{\eta} \right)^{n-m} P_m, \quad (26)$$

by replacing  $\eta$  with  $1/\eta$  [17].

Another limit corresponds to the detectors with noise counts, and with unit efficiency,  $\eta=1$ ,

$$P_{m|n}(\eta=1, \bar{N}_{\text{nc}}) = e^{-\bar{N}_{\text{nc}}} \frac{\bar{N}_{\text{nc}}^{m-n}}{(m-n)!} \quad (27)$$

for  $m \geq n$  and

$$P_{m|n}(\eta=1, \bar{N}_{\text{nc}}) = 0 \quad (28)$$

for  $m < n$ . This is the shifted Poisson distribution. For this case, Eq. (20) can also be analytically inverted,

$$p_n = e^{\bar{N}_{\text{nc}}} \sum_{m=n}^{+\infty} \frac{(-\bar{N}_{\text{nc}})^{m-n}}{(m-n)!} P_m. \quad (29)$$

Equations (26) and (29) express in an explicit form the noiseless statistics in terms of the noisy statistics of photo-counts for two special cases once the characteristics of noise are known.

As has been mentioned above, the POVM has the form of Eq. (7) for the sufficiently small detection times defined by Eq. (14). In this case, the conditional probability, Eq. (18), is given by

$$P_{m|n}(\eta, \bar{N}_{\text{nc}}, \mu) = \frac{\left(\frac{\bar{N}_{\text{nc}}}{\mu}\right)^m \left(1 + \frac{\bar{N}_{\text{nc}}}{\mu} - \eta\right)^n}{\left(1 + \frac{\bar{N}_{\text{nc}}}{\mu}\right)^{n+m+\mu}} \binom{m+\mu-1}{m} \times {}_2F_1\left(-n, -m; \mu; \frac{\eta}{\frac{\bar{N}_{\text{nc}}}{\mu} \left(1 + \frac{\bar{N}_{\text{nc}}}{\mu} - \eta\right)}\right), \quad (30)$$

where  ${}_2F_1(m, n; k; z)$  is the hypergeometric function. It can be checked by direct calculations that for  $\mu \gg \bar{N}_{\text{nc}}$  this equation is transformed into Eqs. (22) and (23).

In conclusion, we note that the appearance of dark counts and background radiation noise can be ascribed to a heat bath of harmonic oscillators—thermal noise modes of different nature; each of them is either coupled or uncoupled to the signal. For the realistic situation of a large number of noise modes, noise counts are not coupled to the signal field and obey the Poissonian statistics. Their contribution is totally accounted for as the shot noise. In this case, nonclassicality of quantum-light statistics disappears slowly as the total number of noise counts increases.

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