Problem with the single-particle description and the spectra of intrinsic modes of degenerate boson-fermion systems

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The problem with the construction of the Gross-Pitaevskii (GP) equation and related wave functions in a medium is associated with the necessity to begin with a description of the configuration space and proceed to a description of the physical space. We show that the balance equations of the number of particles and momentum immediately follow from the multiparticle Schrödinger equation. From the obtained set of balance equations, the equation for the wave function in the medium coincides in form with the GP equation, where we can restrict ourselves only by the first-order terms of the interaction radius of the stress tensor of bosons, for dilute gases. As a generalization of the GP equation, we made allowance for the contribution of the third-order terms of the interaction radius to the stress tensor of bosons. For a system of particles that comprises an ultracold mixture of bosons and fermions, a two-kind quantum hydrodynamics is constructed for the third-order terms of the interaction radius. The spectrum of eigenmodes involves additional information on the interparticle interaction as a correction to the Bogoliubov spectrum.

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I. INTRODUCTION

Progress in the development of the experimental technique [1] of magnetic and optical confinement of atomic-molecular systems in traps, the wide possibilities for varying the cooling and excitation modes of captured particles, and the experimental realization of the Bose-Einstein condensate (BEC) [2–4] stimulated the development of theoretical methods of investigation of physical processes in Bose systems in external fields. In this connection, the Bogoliubov method [5] for microscopic description of the ground state of the homogeneous Bose gas with a weak interaction was further developed. For example, in [6–8], methods for taking correct account of correlations of short- and long-range ordering are developed. However, these methods are applicable for the homogeneous Bose gas only. Generalization of these methods for the equilibrium and nonequilibrium cases in the context of the Hartree-Fock-Bogoliubov variational scheme was obtained in [9], as usual, by varying the energy functional. Theoretical field methods, which were used for the first time for the low-density Bose gas by Belyaev [10] and Hugenholtz and Pines [11], were further developed and generalized for the case of finite temperatures [12,13]. In the context of the above-mentioned and other methods (see, for example, the reviews [14,15]), under the condition that the system of particles is diluted and the interaction is relatively weak, the Gross-Pitaevskii (GP) equation [16–18] can be obtained. This equation is an effective tool for investigation of physical properties of Bose-Einstein condensates in traps. Note that this equation for a system of N bosons with $3N$ degrees of freedom is written for the function of three variables $\Phi(\mathbf{r},t)$, which is usually called [9] "the ordering parameter." The authors of [19] suggested the generalization of the GP equation for the case of arbitrary interaction intensity, which is in agreement with the limiting cases of weak [14] and strong [20] interaction. In [21–23], generalization of the GP equation is considered, allowing for the three-particle interaction. In [24–26] equations for Bose-Einstein spinor condensates were discussed. However, the Schrödinger description of a quantum system in $3N$-dimensional space with the use of a set of field functions of various tensor dimensionalities has developed into quantum hydrodynamics [27–29] which therefore is a natural method for the field description of non-steady-state processes in systems of interacting particles. The fact that the time-dependent GP equation in the limit of the dilute gas can be obtained from the multiparticle Schrödinger equation is investigated in detail in [30–32]. The equation of quantum hydrodynamics for one particle in an external field was obtained by Madelung in 1926 [33], and for one particle with spin by Takabayasi [34]. The general equations of quantum hydrodynamics for many particles were obtained in [27,28]. As for boson systems, in order to formulate the hydrodynamic method itself, the problem of obtaining the total quantum stress tensor should be resolved and the conditions of its existence should be clarified. Here, we will restrict ourselves to the development of the quantum stress tensor and show that under the standard assumptions this tensor can be transformed so that the momentum balance equations for bosons will coincide with the analogous Gross-Pitaevskii equation. For the momentum balance equation thus obtained, we can construct the equivalent nonlinear Schrödinger equation (for the wave function in the medium), which coincides with the GP equation. The stress tensor is symmetric and consists of three parts, namely, the tensor of stresses caused by the thermal motion of bosons, the stress tensor itself conditioned by the interaction of particles only,

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and the purely quantum term associated with the extension of the particles in terms of quantum mechanics. When proceeding to the GP equation, we should restrict ourselves to only the first term of the series for the interaction tensor, which can represent the stress tensor in the general case. In this case, the stress tensor will be diagonal. Irrespective of the selection of the orthonormal system of single-particle wave functions, which are involved in the formula for the stress tensor, the coupling constant equals $-\int d\mathbf{r}\, \mathbf{r} dU(r)/d\mathbf{r}$, which equals $-\lambda \int d\mathbf{r}\, U(r)$ for the nonuniform degrees λ of the potentials. It is evident that the coupling constant depends on the degree of homogeneity λ. For potentials such that the limiting values of $r^3 U$ at zero and infinity turn to zero, the integral determining the coupling constant can be calculated by parts and will coincide with the GP result.

For sufficiently wide traps, we can select plane waves as the orthonormal system of single-particle wave functions. In this case, the equation for the wave function in the medium $\Phi(\mathbf{r},t)$ will coincide with the GP equation. For narrow traps, we can select the single-particle wave function of the ground state of the harmonic oscillator. In this case, the equation for $\Phi(\mathbf{r},t)$ will also have the same form as the GP equation.

In experiments on Bose-Einstein condensates (BEC) of atomic gases located in traps, an atomic gas of bosons is mixed with fermionic atoms. Examples of such mixtures are ^{7}Li-^{6}Li [35,36], ^{23}Na-^{6}Li [37], and ^{87}Rb-^{40}K [38]. In certain works [39–44], physical processes in boson-fermion systems were analyzed based on model equations for $\Phi_b(\mathbf{r},t), \Phi_f(\mathbf{r},t)$ which have the form of modified GP equations. Using theory based on the notions of the mean field, experiments for ultracold rarified ^{87}Rb-^{40}K mixtures were analyzed in [44]. Modified GP equations for the mixture are used in [39,42] for numerical analysis of the collapse in such systems. In this connection, it is of interest to give the derivation of GP-type equations for boson-fermion systems based on the quantum-hydrodynamic momentum balance equations. It is shown below that the equations obtained in this manner agree qualitatively with the modified GP equations. Derivation of the equation for bosons and boson-fermion systems based on quantum hydrodynamics and their comparison with the above-mentioned works is performed, allowing for the first term of expansion of the stress tensor of the interaction radius. Below, we find equations for the wave functions in the medium in the third order of the interaction radius. These equations are integro-differential equations and can be considered as generalizations of the GP equation. Based on these equations, we investigate below the intrinsic modes in the boson-fermion system and establish the dispersion law of such modes.

II. SET OF QUANTUM HYDRODYNAMIC EQUATIONS

Let us consider a system of N Bose particles with a short-range potential. The Hamiltonian of the system under consideration has the form

$$\hat{H} = \sum_i \frac{1}{2m_i} \hat{p}_i^\alpha \hat{p}_{\alpha i} + \sum_i V_{\text{ext}}(\mathbf{r}_i,t) + \frac{1}{2} \sum_{i,j,i\neq j} U(|\mathbf{r}_i - \mathbf{r}_j|), \tag{1}$$

where $\hat{p}_i^\alpha = -\imath\hbar\nabla_i^\alpha$ is the momentum operator of the ith particle, m_i is the mass of the ith particle, and $U_{ij} = U(|\mathbf{r}_i - \mathbf{r}_j|)$ is the interaction potential.

The concentration of particles in the vicinity of the point $\mathbf{r}$ of the physical space is determined as the operator $\Sigma_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)$ averaged over the quantum-mechanical states,

$$n(\mathbf{r},t) = \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i)\psi^\dagger(R,t)\psi(R,t), \tag{2}$$

where $dR = \Pi_{i=1}^N d\mathbf{r}_i$. Differentiating this function over time and using the Schrödinger equation with the Hamiltonian (1), we derive the continuity equation, in which the vector of current density appears in the form

$$j^\alpha(\mathbf{r},t) = \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i)\frac{1}{2m_i}[(\hat{p}_i^\alpha\psi)^\dagger(R,t)\psi(R,t) + \psi^\dagger(R,t) \times (\hat{p}_i^\alpha\psi)(R,t)]. \tag{3}$$

The momentum balance equation for the system of particles under consideration is obtainable similarly, i.e., via differentiation of the current density (3) and application of the Schrödinger equation [27,45]. As a result, we obtain

$$\partial_t j^\alpha(\mathbf{r},t) + \frac{1}{m}\partial_\beta \Pi^{\alpha\beta}(\mathbf{r},t) = -\frac{1}{m}\int d\mathbf{r}'[\nabla^\alpha U(\mathbf{r},\mathbf{r}')]n_2(\mathbf{r},\mathbf{r}',t) - \frac{1}{m} n(\mathbf{r},t)\nabla^\alpha V_{\text{ext}}(\mathbf{r},t). \tag{4}$$

In the momentum balance equation, $\Pi^{\alpha\beta}(\mathbf{r},t)$ is the quantum tensor of the density of the momentum flux. This tensor has the form

$$\Pi^{\alpha\beta}(\mathbf{r},t) = \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i)\frac{1}{4m_i}[\psi^\dagger(R,t)(\hat{p}_i^\alpha \hat{p}_i^\beta\psi)(R,t) + (\hat{p}_i^\alpha\psi)^\dagger(R,t)(\hat{p}_i^\beta\psi)(R,t) + \text{c.c.}]. \tag{5}$$

The interaction between the particles in Eq. (4) is expressed through the two-particle probability density $n_2(\mathbf{r},\mathbf{r}',t)$ normalized over $N(N-1)$ and having the form

$$n_2(\mathbf{r},\mathbf{r}',t) = \int dR \sum_{i,j,i\neq j} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r}' - \mathbf{r}_j)\psi^\dagger(R,t)\psi(R,t). \tag{6}$$

Let us represent the first term in the right-hand side of Eq. (4), i.e., the density of the interaction force of the particles, in the form

$$-\frac{1}{2}\int dR \sum_{i,j.i\neq j} [\delta(\mathbf{r} - \mathbf{r}_i) - \delta(\mathbf{r} - \mathbf{r}_j)][\nabla_i^\alpha U(\mathbf{r}_{ij})]\psi^\dagger(R,t)\psi(R,t), \tag{7}$$

which is possible by virtue of the symmetry (antisymmetry) of the wave function, and let us proceed in Eq. (7) to variables of the center of gravity and variables of the relative distance of the particles:

$$\mathbf{R}_{ij} = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j), \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \tag{8}$$

Since the interaction forces between the particles rapidly decrease at distances of the order of the interaction radius, small $|r_{ij}^{\alpha}|$ give the main contribution to the integral (7). Therefore, in expression (7), we can replace the multipliers of the interaction potential by their expansion in series of r_{ij}^{α}. We have reached the conclusion that the density of the interaction force for bosons with a short-range interaction potential can be represented in the form of the divergence of the tensor field $\partial_\beta \sigma^{\alpha\beta}(\mathbf{r},t)$. Here, $\sigma^{\alpha\beta}(\mathbf{r},t)$ is the quantum stress tensor conditioned by the occurrence of interparticle interaction. The divergence of this tensor is represented by the formula

$$\partial_\beta \sigma^{\alpha\beta}(\mathbf{r},t) = \int dR \sum_{i,j,i\neq j} \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{k}\sum_{m=1}^{n} [\partial_{\beta_1}\cdots\partial_{\beta_{2l+1}}\delta(\mathbf{r}-\mathbf{R}_{ij})](-1)^{2l+p+m+1}$$
$$\times\left(\frac{1}{2}\right)^{2l+k+n+1}\frac{1}{(2l+1)!}\frac{1}{k!}\frac{1}{n!}C_k^p C_n^m \mathbf{r}_{ij}^{\alpha}\mathbf{r}_{ij}^{\beta_1}\cdots\mathbf{r}_{ij}^{\beta_{2l+1}}\mathbf{r}_{ij}^{\gamma_1}\cdots\mathbf{r}_{ij}^{\gamma_k}\mathbf{r}_{ij}^{\delta_1}\cdots\mathbf{r}_{ij}^{\delta_n}$$
$$\times\frac{1}{|\mathbf{r}_{ij}|}\frac{\partial U(|\mathbf{r}_{ij}|)}{\partial|\mathbf{r}_{ij}|}[\partial_1^{\gamma_1}\cdots\partial_1^{\gamma_{k-p}}\partial_2^{\gamma_{k-p+1}}\cdots\partial_2^{\gamma_k}\psi^*(\mathbf{R}_{ij},\mathbf{R}_{ij},R_{N-2})]\times[\partial_1^{\delta_1}\cdots\partial_1^{\delta_{n-m}}\partial_2^{\delta_{n-m+1}}\cdots\partial_2^{\delta_n}\psi(\mathbf{R}_{ij},\mathbf{R}_{ij},R_{N-2})]. \tag{9}$$

The quantity C_k^p is the binomial coefficient, $C_k^p = k!/[p!(k-p)!]$. It is evident from the presented formula that the quantum stress tensor is symmetric with respect to permutation of indices α,β: $\sigma^{\alpha\beta}(\mathbf{r},t)=\sigma^{\beta\alpha}(\mathbf{r},t)$.

Now, the momentum balance equation will take the form

$$\partial_t j^{\alpha}(\mathbf{r},t) + \frac{1}{m}\partial_\beta[\Pi^{\alpha\beta}(\mathbf{r},t) + \sigma^{\alpha\beta}(\mathbf{r},t)] = -\frac{1}{m}n(\mathbf{r},t)\nabla^{\alpha}V_{\text{ext}}(\mathbf{r}). \tag{10}$$

Let us now consider the tensor $\Pi^{\alpha\beta}(\mathbf{r},t)$ and isolate in it the contributions to the momentum flow density for the convective and thermal motions and the purely quantum part. For this purpose, let us introduce the velocities by the formulas

$$\mathbf{v}_i(R,t) = \frac{1}{m_i}\nabla_i S(R,t), \tag{11}$$

where $\mathbf{v}_i(R,t)$ is the velocity of the ith particle, while the $S(R,t)$ phase of the wave function is

$$\psi(R,t) = a(R,t)\exp\left(\frac{\imath S(R,t)}{\hbar}\right).$$

The velocity field $\mathbf{v}(\mathbf{r},t)$ is determined by the formula

$$\mathbf{j}(\mathbf{r},t) = n(\mathbf{r},t)\mathbf{v}(\mathbf{r},t). \tag{12}$$

Thus $\mathbf{u}_i(\mathbf{r},R,t)=\mathbf{v}_i(R,t)-\mathbf{v}(\mathbf{r},t)$ is the quantum analog of the velocity of thermal motion. Isolating the evidently thermal motion of the particles with velocities $\mathbf{u}_i$ and the motion with the velocity $\mathbf{v}(\mathbf{r},t)$ in the continuity and momentum balance equations (10), we come to the following equations:

$$\partial_t n(\mathbf{r},t) + \nabla(n(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)) = 0, \tag{13}$$

$$mn(\mathbf{r},t)(\partial_t + \mathbf{v}\nabla)v^{\alpha}(\mathbf{r},t) + \partial_\beta[p^{\alpha\beta}(\mathbf{r},t) + \sigma^{\alpha\beta}(\mathbf{r},t) + T^{\alpha\beta}(\mathbf{r},t)] = -n(\mathbf{r},t)\nabla^{\alpha}V_{\text{ext}}(\mathbf{r},t). \tag{14}$$

In Eq. (14),

$$p^{\alpha\beta}(\mathbf{r},t) = \int dR\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i)a^2(R,t)m_i u_i^{\alpha}u_i^{\beta}. \tag{15}$$

This tensor tends to zero as the velocities of thermal motion $\mathbf{u}_i$ of the particles go to zero. Therefore, it has the meaning of the kinetic pressure. The temperature $\Theta(\mathbf{r},t)$ of the system of bosons is expressed through this tensor by the formula

$$\Theta(\mathbf{r},t) = \frac{p_{\alpha\beta}(\mathbf{r},t)\delta^{\alpha\beta}}{3n(\mathbf{r},t)}. \tag{16}$$

The tensor

$$T^{\alpha\beta}(\mathbf{r},t) = -\frac{\hbar^2}{2m}\int dR\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i)a^2(R,t)\frac{\partial^2\ln a}{\partial x_{\alpha i}\partial x_{\beta i}} \tag{17}$$

is proportional to $\hbar^2$ and has a purely quantum origin. For a system of numerous noninteracting particles, this tensor is

$$T^{\alpha\beta}(\mathbf{r},t) = -\frac{\hbar^2}{4m}\left(\partial^{\alpha}\partial^{\beta}n(\mathbf{r},t) - \frac{1}{n(\mathbf{r},t)}[\partial^{\alpha}n(\mathbf{r},t)][\partial^{\beta}n(\mathbf{r},t)]\right). \tag{18}$$

Divergence of this tensor is usually presented in the form [14]

$$\partial_\beta T^{\alpha\beta}(\mathbf{r},t) = -\frac{\hbar^2}{2m}n(\mathbf{r},t)\partial_\alpha\frac{\Delta\sqrt{n(\mathbf{r},t)}}{\sqrt{n(\mathbf{r},t)}}. \tag{19}$$

Therefore, it is evident that the total stress tensor $P^{\alpha\beta}(\mathbf{r},t)$ can be represented as the sum of the kinetic pressure tensor, the quantum tensor $T^{\alpha\beta}(\mathbf{r},t)$, and the tensor of stresses $\sigma^{\alpha\beta}(\mathbf{r},t)$, which are determined by interparticle interaction. In this case, the normal stress is determined by the scalar $P=(1/3)P^{\alpha\beta}(\mathbf{r},t)\delta^{\alpha\beta}$, while the tangential or shear stresses are determined by the deviator $P^{\alpha\beta}(\mathbf{r},t)-P\delta^{\alpha\beta}$.

The explicit expression of $P^{\alpha\beta}(\mathbf{r},t)$ through the wave functions of the system of particles is determined by formulas (9), (15), and (17). In problems of the evolution of small perturbations in Bose systems, this tensor can be approximately calculated by perturbation theory. This will allow us to form the closed apparatus for description of such systems.

III. THE GROSS-PITAEVSKII EQUATION

As was mentioned above, the first term of Eq. (9) gives the main contribution to the tensor $\sigma^{\alpha\beta}(\mathbf{r},t)$. Retaining this term only, we have

$$\sigma^{\alpha\beta}(\mathbf{r},t) = -\frac{1}{2}\int dR \sum_{i,j.i\neq j}\delta(\mathbf{r}-\mathbf{R}_{ij})\frac{r_{ij}^{\alpha}r_{ij}^{\beta}}{|\mathbf{r}_{ij}|}\frac{\partial U(\mathbf{r}_{ij})}{\partial|\mathbf{r}_{ij}|} \times \psi^{\dagger}(R,t)\psi(R,t). \quad (20)$$

Using the properties of symmetry of boson wave functions and interparticle interaction potential, formula (20) can be transformed to the form

$$\sigma^{\alpha\beta}(\mathbf{r},t) = -\frac{1}{2}\mathrm{Tr}[n_2(\mathbf{r},\mathbf{r}',t)]\int d\mathbf{r}\frac{r^{\alpha}r^{\beta}}{r}\frac{\partial U(r)}{\partial r}, \quad (21)$$

where

$$\mathrm{Tr}[f(\mathbf{r},\mathbf{r}')] = f(\mathbf{r},\mathbf{r}).$$

For the two-particle probability density, according to definition (6), we have

$$n_2(\mathbf{r},\mathbf{r}',t) = N(N-1)\int dR_{N-2}\langle n_1,n_2,\ldots,|\mathbf{r},\mathbf{r}',R_{N-2},t\rangle \times \langle \mathbf{r},\mathbf{r}',R_{N-2},t|n_1,n_2,\ldots\rangle, \quad (22)$$

where $dR_{N-2}=\Pi_{k=3}^{N}d\mathbf{r}_k$.

Let us use the known formulas for expansion of the wave function $\langle \mathbf{r},\mathbf{r}',R_{N-2},t|n_1,n_2,\ldots\rangle$ [46]. In the case of bosons, we have

$$\begin{aligned}&\langle \mathbf{r},\mathbf{r}',R_{N-2},t|n_1,n_2,\ldots\rangle\\ &= \sum_f \sqrt{\frac{n_f}{N}}\langle \mathbf{r},t|f\rangle\langle \mathbf{r}',R_{N-2},t|n_1,\ldots,(n_f-1),\ldots\rangle\\ &= \sum_f \sum_{f',f'\neq f}\sqrt{\frac{n_f}{N}}\sqrt{\frac{n_{f'}}{N-1}}\langle \mathbf{r},t|f\rangle\langle \mathbf{r}',t|f'\rangle\\ &\times\langle R_{N-2},t|n_1,\ldots,(n_{f'}-1),\ldots,(n_f-1),\ldots\rangle\\ &+\sum_f\sqrt{\frac{n_f(n_f-1)}{N(N-1)}}\langle \mathbf{r},t|f\rangle\langle \mathbf{r}',t|f\rangle\langle R_{N-2},t|n_1,\ldots,\\ &\times(n_f-2),\ldots\rangle. \end{aligned} \quad (23)$$

Here, $\langle \mathbf{r},t|f\rangle=\varphi_f(\mathbf{r},t)$ are the single-particle wave functions.

The first term in formula (23) represents particles situated in two different quantum states, while the second term refers to particles in the same quantum state. Therefore, for particles in the BEC state, it is sufficient to take into account the second term in formula (23). In consideration of a system of bosons with temperature differing from zero, where a certain number of particles is outside the condensate, the first summand of formula (23) gives a contribution both in the case of interaction of excited particles with each other and in the case of their interaction with the particles appearing in the BEC state. In this case, the second term of formula (23) gives the contribution to the interaction both between the particles appearing in the BEC state and between excited particles appearing in the same quantum state.

In this case,

$$\begin{aligned}&\langle n_1,\ldots,(n_{f'}-1),\ldots,(n_f-1),\ldots,|n_1,\ldots,(n_{q'}-1),\ldots,\\ &\quad\times(n_q-1),\ldots\rangle = \delta(f-q)\delta(f'-q')+\delta(f-q')\delta(f'-q),\\ &\langle n_1,\ldots,(n_f-2),\ldots,|n_1,\ldots,(n_q-2),\ldots\rangle = \delta(f-q),\end{aligned} \quad (24)$$

and

$$n_2(\mathbf{r},\mathbf{r}',t) = n(\mathbf{r},t)n(\mathbf{r}',t)+|\rho(\mathbf{r},\mathbf{r}',t)|^2+\sum_g n_g(n_g-1) \times|\varphi_g(\mathbf{r},t)|^2|\varphi_g(\mathbf{r}',t)|^2. \quad (25)$$

Here,

$$n(\mathbf{r},t) = \sum_g n_g\varphi_g^*(\mathbf{r},t)\varphi_g(\mathbf{r},t), \quad (26)$$

$$\rho(\mathbf{r},\mathbf{r}',t) = \sum_g n_g\varphi_g^*(\mathbf{r},t)\varphi_g(\mathbf{r}',t), \quad (27)$$

where $\varphi_g(\mathbf{r},t)$ are arbitrary single-particle wave functions.

The second term of formula (25) represents the part of the stress tensor caused by the exchange interaction. Substituting expression (25) into Eq. (21) for the quantum stress tensor of the boson system, we derive the formula

$$\sigma^{\alpha\beta}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\delta^{\alpha\beta}\left(2n^2(\mathbf{r},t)+\sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2\right), \quad (28)$$

where the following integral is designated through Υ:

$$\Upsilon = \frac{4\pi}{3}\int dr(r)^3\frac{\partial U(r)}{\partial r}. \quad (29)$$

Assuming that the potential satisfies the condition that the quantity $r^3U(r)$ tends to zero as r tends to zero and infinity, we obtain for Υ from Eq. (29), by integration by parts,

$$\Upsilon = -\int d\mathbf{r}\,U(r),$$

which coincides with the result for the interaction constant by Gross and Pitaevskii allowing for the sign in (35).

Taking into account the notations given after formula (23), we obtain the following relation for the quantum tensor of stresses of a system of bosons close to the BEC state:

$$\sigma^{\alpha\beta}_{\mathrm{BEC},n}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\,\delta^{\alpha\beta}\Big(2n_{\mathrm{BEC}}(\mathbf{r},t)n_n(\mathbf{r},t) + 2n_n^2(\mathbf{r},t) + \sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2\Big), \qquad (30)$$

Here, we used the notations $n_{\mathrm{BEC}}(\mathbf{r},t)$ for the concentration of particles situating in the BEC state and $n_n(\mathbf{r},t)$ for the concentration of excited particles. The expression (28) obtained for $\sigma^{\alpha\beta}(\mathbf{r},t)$ in the approximation under consideration should be substituted into the momentum balance equation (14).

The stress tensor, which depends on interparticle interactions, contains the single-particle functions $\varphi_g(\mathbf{r},t)$. By these functions, expansion of the unknown N partial wave function is actually performed. If we neglect the interparticle interaction, the partial problem N will be reduced to the single-particle one, and the set of functions $\varphi_g(\mathbf{r},t)$ will be determined by the single-particle Schrödinger equation. In this case, the momentum balance equation will also not contain interactions and, along with the continuity equation, will determine the single-particle wave function in polar form. Such single-particle functions, which are simultaneously the solutions of the equations of quantum hydrodynamics for the system of noninteracting particles and the single-particle Schrödinger equation, can be used as a first approximation when calculating the stress tensor. The quantum balance equations thus obtained will determine the set of field functions that in turn determine the state of the system in hydrodynamics and, consequently, the single-particle wave function of the system of interacting particles. Such a wave function of spatial coordinates and time will determine the effective Schrödinger equation, which is nonlinear and integro-differential in the general case. The solution of this equation can be used as the second iteration when calculating the stress tensor, etc.

According to this iteration procedure, the basis wave functions $\varphi_g(\mathbf{r},t)$ in formula (28) can be selected as the plane waves

$$\varphi_p(\mathbf{r},t) = \frac{1}{\sqrt{V}}\exp\left(-\frac{\imath}{\hbar}(\varepsilon_p t - \mathbf{p}\mathbf{r})\right), \qquad (31)$$

for sufficiently wide traps, $\varepsilon_p=\mathbf{p}^2/2m$. Assuming that most of the particles of the system are situated in the state of the Bose condensate, the probability of situation of two excited particles in one quantum state is small, and the number of particles is sufficiently large, we will approximately have in this case

$$\sum_g n_g(n_g-1)[\,|\varphi_g(\mathbf{r},t)|^2)^2] = n^2(\mathbf{r},t). \qquad (32)$$

For Bose particles with a weak interaction, which are captured into a sufficiently narrow harmonic trap $V_{\mathrm{ext}}=m\omega_0^2r^2/2$, we can select the wave function of the ground state of the harmonic oscillator $\varphi_g(\mathbf{r},t)$ as $\exp(-\imath\omega t)[m\omega_0/(\pi\hbar)]^{-3/4}\exp[-m\omega_0 r^2/(2\hbar)]$. In this case, formula (32) will also apply.

Using the result (32), we obtain the expression for $\sigma^{\alpha\beta}_{\mathrm{BEC},n}(\mathbf{r},t)$ in the form

$$\sigma^{\alpha\beta}_{\mathrm{BEC},n}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\,\delta^{\alpha\beta}[2n_{\mathrm{BEC}}(\mathbf{r},t)n_n(\mathbf{r},t) + 2n_n^2(\mathbf{r},t) + n_{\mathrm{BEC}}^2(\mathbf{r},t)]. \qquad (33)$$

In the absence of excited particles, the quantum stress tensor takes the form

$$\sigma^{\alpha\beta}_{\mathrm{BEC}}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\,\delta^{\alpha\beta}n^2_{\mathrm{BEC}}(\mathbf{r},t). \qquad (34)$$

The corresponding momentum balance equation of quantum hydrodynamics for the BEC only takes the form

$$mn(\mathbf{r},t)[\partial_t + v_\beta(\mathbf{r},t)\nabla^\beta]v^\alpha(\mathbf{r},t) + \partial_\beta p^{\alpha\beta}(\mathbf{r},t) - \frac{\hbar^2}{2m}\partial_\alpha\frac{\Delta\sqrt{n(\mathbf{r},t)}}{\sqrt{n(\mathbf{r},t)}} - \Upsilon n(\mathbf{r},t)\partial^\alpha n(\mathbf{r},t) = -n(\mathbf{r},t)\nabla^\alpha V_{\mathrm{ext}}(\mathbf{r},t). \qquad (35)$$

For eddy-free motion $\mathbf{v}(\mathbf{r},t)=\boldsymbol{\nabla}\phi(\mathbf{r},t)$ and with the proviso that the kinetic pressure $p^{\alpha\beta}(\mathbf{r},t)=p(\mathbf{r},t)\delta^{\alpha\beta}$ is isotropic and that we have barotropicity,

$$\frac{\boldsymbol{\nabla}p(\mathbf{r},t)}{mn(\mathbf{r},t)} = \boldsymbol{\nabla}\mu(\mathbf{r},t), \qquad (36)$$

where $\mu(\mathbf{r},t)$ is the chemical potential, the momentum balance equation

$$m\partial_t v^\alpha(\mathbf{r},t) + \frac{1}{2}m\partial^\alpha v^2(\mathbf{r},t) + m\partial^\alpha\mu(\mathbf{r},t) - \frac{\hbar^2}{2m}\partial^\alpha\frac{\Delta\sqrt{n(\mathbf{r},t)}}{\sqrt{n(\mathbf{r},t)}} - \Upsilon\partial^\alpha n(\mathbf{r},t) = -\partial^\alpha V_{\mathrm{ext}}(\mathbf{r},t) \qquad (37)$$

has the Cauchy-Lagrangian integral

$$\partial_t\phi(\mathbf{r},t) + \frac{1}{2}v^2(\mathbf{r},t) + \mu(\mathbf{r},t) - \frac{1}{m}\Upsilon n(\mathbf{r},t) - \frac{\hbar^2}{2m^2}\frac{\Delta\sqrt{n(\mathbf{r},t)}}{\sqrt{n(\mathbf{r},t)}} + \frac{1}{m}V_{\mathrm{ext}}(\mathbf{r},t) = \mathrm{const}. \qquad (38)$$

The equations of continuity (13) and momentum pulse (37) can be associated with the equivalent single-particle Schrödinger equation for a certain effective wave function $\Phi(\mathbf{r},t)$. Substituting this function in the form

$$\Phi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp\left(\frac{\imath}{\hbar}m\phi(\mathbf{r},t)\right), \qquad (39)$$

differentiating it with respect to time, and using Eqs. (13) and (38), we obtain the equivalent single-particle Schrödinger equation

$$\imath\hbar\partial_t\Phi(\mathbf{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + \mu(\mathbf{r},t) + V_{\text{ext}}(\mathbf{r},t) - \Upsilon|\Phi(\mathbf{r},t)|^2\right)\Phi(\mathbf{r},t), \qquad (40)$$

which is well known as the Gross-Pitaevskii equation [16,18,14,47]. The wave function $\Phi(\mathbf{r},t)$ is normalized by the condition

$$\int d\mathbf{r}\, \Phi(\mathbf{r},t)^*\Phi(\mathbf{r},t) = N,$$

where N is the number of particles in the system.

The obtained equations (40) and (35) contain the chemical potential and its gradient as separate terms. It is immediately evident from the determination of the chemical potential (36) that it turns to zero at zero temperature. Another distinction of Eq. (40) from the standard GP equation is that, at any choice of $\varphi_g(\mathbf{r},t)$, the coupling constant is determined by the formula $\int d\mathbf{r}\,\mathbf{r}dU(r)/d\mathbf{r}$ and has the form of the Clausius virial. The formulas obtained for the coupling constant, for uniform potentials of the negative first degree of uniformity, and for nonuniform potentials satisfying the condition of approaching zero at the limiting values of r^3U at zero and infinity, coincide with the results presented in [47,15] and other works. For uniform potentials of degree λ,

$$\int d\mathbf{r}\,\mathbf{r}\frac{dU(r)}{d\mathbf{r}} = \lambda\int d\mathbf{r}\, U(r).$$

The interaction of the Bose atoms is presented in the starting Hamiltonian (1) as the interaction of point particles. Therefore, the interaction potential $U(\mathbf{r})$ should be considered as a model, providing repulsion at small finite distances and attraction or repulsion at distances of the order of the interaction radius. This circumstance should be taken into account when calculating the integral of form (29) with specific functions $U(\mathbf{r})$.

IV. STRESS TENSOR IN THE THIRD ORDER OF THE INTERACTION RADIUS

As shown above, in the first order of the interaction radius, the stress tensor is determined by the gradient of the interaction potential and particle distribution $n(\mathbf{r},t)$. It is easily seen that the second term of the expansion $\sigma^{\alpha\beta}(\mathbf{r},t)$ approaches zero. In order to obtain the dependence of the stress tensor on the hydrodynamic velocity and kinetic pressure tensor, the tensor $\sigma^{\alpha\beta}(\mathbf{r},t)$ should be calculated in the third order of the interaction radius. The results of the calculations can be presented in the form

$$\begin{aligned}\sigma^{\alpha\beta}(\mathbf{r},t) = &-\frac{1}{2}\Upsilon\delta^{\alpha\beta}\Big(2n_{\text{BEC}}(\mathbf{r},t)n_n(\mathbf{r},t) + 2n_n^2(\mathbf{r},t) + \sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2\Big) - \frac{1}{48}\Upsilon_2(\delta^{\alpha\beta}\Delta + 2\partial^\alpha\partial^\beta)\Big(2n_{\text{BEC}}(\mathbf{r},t)n_n(\mathbf{r},t)\\ &+ 2n_n^2(\mathbf{r},t) + \sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2\Big) + \frac{1}{\hbar^2}\Upsilon_2\{mn_{\text{BEC}}(\mathbf{r},t)[\delta^{\alpha\beta}\Pi_n^{\gamma\gamma}(\mathbf{r},t) + 2\Pi_n^{\alpha\beta}(\mathbf{r},t)] - \delta^{\alpha\beta}j_n^2(\mathbf{r},t) - 2j_n^\alpha(\mathbf{r},t)j_n^\beta(\mathbf{r},t)\}\\ &-\frac{1}{2}\Upsilon_2 n_{\text{BEC}}(\mathbf{r},t)\text{Tr}[(\delta^{\alpha\beta}\partial^\gamma\partial'_\gamma + 2\partial^\alpha\partial^{\beta'})\rho_n(\mathbf{r},\mathbf{r}',t)] - + \frac{1}{\hbar^2}\Upsilon_2\{mn_n(\mathbf{r},t)[\delta^{\alpha\beta}\Pi_{\text{BEC}}^{\gamma\gamma}(\mathbf{r},t) + 2\Pi_{\text{BEC}}^{\alpha\beta}(\mathbf{r},t)] - \delta^{\alpha\beta}j_{\text{BEC}}^2(\mathbf{r},t)\\ &- 2j_{\text{BEC}}^\alpha(\mathbf{r},t)j_{\text{BEC}}^\beta(\mathbf{r},t)\} - -\frac{1}{2}\Upsilon_2 n_n(\mathbf{r},t)\text{Tr}[(\delta^{\alpha\beta}\partial^\gamma\partial'_\gamma + 2\partial^\alpha\partial^{\beta'})\rho_{\text{BEC}}(\mathbf{r},\mathbf{r}',t)] - + \frac{1}{\hbar^2}\Upsilon_2\{mn_n(\mathbf{r},t)[\delta^{\alpha\beta}\Pi_n^{\gamma\gamma}(\mathbf{r},t)\\ &+ 2\Pi_n^{\alpha\beta}(\mathbf{r},t)] - \delta^{\alpha\beta}j_n^2(\mathbf{r},t) - 2j_n^\alpha(\mathbf{r},t)j_n^\beta(\mathbf{r},t)\} - \frac{1}{2}\Upsilon_2 n_n(\mathbf{r},t)\text{Tr}[(\delta^{\alpha\beta}\partial^\gamma\partial'_\gamma + 2\partial^\alpha\partial^{\beta'})\rho_n(\mathbf{r},\mathbf{r}',t)] - \frac{1}{8}\Upsilon_2(\delta^{\alpha\beta}\delta^{\gamma\delta}\\ &+ \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma})\sum_g n_g(n_g-1)\{\varphi_g^*(\mathbf{r},t)\varphi_g^*(\mathbf{r},t)[\varphi_g(\mathbf{r},t)\partial_\gamma\partial_\delta\varphi_g(\mathbf{r},t) - \partial_\gamma\varphi_g(\mathbf{r},t)\partial_\delta\varphi_g(\mathbf{r},t)] + \text{H.c.}\},\end{aligned} \qquad (41)$$

where $\partial'_\alpha = \partial/\partial x'^\alpha$ and

$$\Upsilon_2 \equiv \frac{4\pi}{15}\int dr(r)^5\frac{\partial U(r)}{\partial r}. \qquad (42)$$

In formula (41), we used

$$j^\alpha(\mathbf{r},t) = \frac{1}{2m}\sum_g n_g[\varphi_g^*(\mathbf{r},t)\hat{p}^\alpha\varphi_g(\mathbf{r},t) + \hat{p}^{\alpha*}\varphi_g^*(\mathbf{r},t)\varphi_g(\mathbf{r},t)], \qquad (43)$$

$$\Pi^{\alpha\beta}(\mathbf{r},t) = \frac{1}{4m}\sum_g n_g\{\varphi_g^*(\mathbf{r},t)\hat{p}^\alpha\hat{p}^\beta\varphi_g(\mathbf{r},t) + [\hat{p}^{\alpha*}\varphi_g^*(\mathbf{r},t)]\hat{p}^\beta\varphi_g(\mathbf{r},t) + \text{c.c.}\}. \qquad (44)$$

The quantities $\rho_{\text{BEC}}(\mathbf{r},\mathbf{r}',t)$, $j_{\text{BEC}}^\alpha(\mathbf{r},t)$, $\Pi_{\text{BEC}}^{\alpha\beta}(\mathbf{r},t)$, $\rho_n(\mathbf{r},\mathbf{r}',t)$, $j_n^\alpha(\mathbf{r},t)$, and $\Pi_n^{\alpha\beta}(\mathbf{r},t)$ in Eq. (41) are the density matrix, current vector, and density tensor of the pulse flux of the particles situating in the BEC state and in the excited states, respectively.

The first term in formula (41) is the quantum stress tensor $\sigma^{\alpha\beta}(\mathbf{r},t)$ at first order in the interaction radius. The second term of formula (41), similarly to the first term, involves the contribution to the interaction of bosons situated in the BEC and particles in excited states. In the third and fourth terms of formula (41), the dependence of $\sigma^{\alpha\beta}(\mathbf{r},t)$ on the current $j^{\alpha}(\mathbf{r},t)$, density tensor of the pulse flux $\Pi^{\alpha\beta}(\mathbf{r},t)$, and two-particle density matrix $\rho(\mathbf{r},\mathbf{r}',t)$ appears. These summands give the contribution to the interaction only in the presence of the particles in excited states and turns to zero if all bosons are in the BEC state. The last term in Eq. (41) describes the contribution to the quantum stress tensor from the interaction of particles situated in the same quantum state; specifically, situated in the BEC state.

These formulas are valid for any selection of basis wave functions $\varphi_g(\mathbf{r},t)$. In order to obtain a closed apparatus, we can use the iteration procedure when calculating expression (41). In the absence of particles in excited states, Eq. (41) takes the form

$$\sigma^{\alpha\beta}_{\mathrm{BEC}}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\delta^{\alpha\beta}\sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2 - \frac{1}{48}\Upsilon_2(\delta^{\alpha\beta}\Delta + 2\partial^\alpha\partial^\beta)\sum_g n_g(n_g-1)[|\varphi_g(\mathbf{r},t)|^2]^2 - \frac{1}{8}\Upsilon_2(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma})\sum_g n_g(n_g-1)\{\varphi^*_g(\mathbf{r},t)\varphi^*_g(\mathbf{r},t) \times[\varphi_g(\mathbf{r},t)\partial_\gamma\partial_\delta\varphi_g(\mathbf{r},t) - \partial_\gamma\varphi_g(\mathbf{r},t)\partial_\delta\varphi_g(\mathbf{r},t)] + \mathrm{H.c.}\}. \tag{45}$$

For sufficiently wide traps and weakly inhomogeneous media in the BEC state, we can select plane waves as the basis functions. If the Bose condensate is situated in a narrow harmonic trap, the choice of eigenfunctions of the harmonic oscillator as the approximated single-particle wave functions is more adequate. Since the particles are in the BEC state, let us select the eigenfunction of the ground state of the harmonic oscillator as φ_g. In this case, from Eq. (45), we find the final expression for the stress tensor allowing for third-order terms in the interaction radius. This expression contains Υ and Υ_2 as parameters and is independent of the concentration and its derivatives:

$$\sigma^{\alpha\beta}(\mathbf{r},t) = -\frac{1}{2}\Upsilon\delta^{\alpha\beta}n^2(\mathbf{r},t) - \frac{1}{48}\Upsilon_2(\delta^{\alpha\beta}\Delta + 2\partial^\alpha\partial^\beta)n^2(\mathbf{r},t). \tag{46}$$

V. ULTRACOLD MIXTURE OF BOSONS AND FERMIONS

There are substantial differences between the quantum hydrodynamics of bosons and of fermions. These distinctions can be explicitly followed if we consider the system of fermions and bosons as a two-component liquid and obtain equations of quantum hydrodynamics for it. Such systems of particles were studied, for example, in [44]. The Hamiltonian of this system can be presented in the form

$$\hat{H} = \sum_i\left(\frac{\mathbf{p}_i^2}{2m_i} + V_{\mathrm{ext}}(\mathbf{r}_i,t)\right) + \frac{1}{2}\sum_{i_b,j_b}U_{i_bj_b}(\mathbf{r}_{i_b}-\mathbf{r}_{j_b}) + \frac{1}{2}\sum_{i_f,j_f}U_{i_fj_f}(\mathbf{r}_{i_f}-\mathbf{r}_{j_f}) + \sum_{i_b,j_f}U_{i_bj_f}(\mathbf{r}_{i_b}-\mathbf{r}_{j_f}). \tag{47}$$

In this Hamiltonian, $U_{i_bj_b}$, $U_{i_fj_f}$, and $U_{i_bj_f}$ are the interaction potentials between bosons and bosons, fermions and fermions, and bosons and fermions, respectively. Strictly speaking, all these potentials are different. The index i in formula (47) runs over values from 1 to N, where N is the total number of particles in the system, both bosons and fermions. The indices i_b, j_b, i_f, and j_f enumerate only bosons and fermions, respectively.

In accordance with the Schrödinger equation, the state of this system of particles is specified in configuration space. In physical space, this state is determined by the set of field functions of different tensor dimensionality. To proceed to the physical state, let us use the same method as for the derivation of formulas (2)–(14). As a result, we obtain the set of equations of continuity and momentum balance for bosons and fermions, respectively. The momentum balance equations for the Bose and the Fermi particles have the forms

$$n_b(\mathbf{r},t)[\partial_t + v_{\beta b}(\mathbf{r},t)\nabla^\beta]v^\alpha_b(\mathbf{r},t) - \frac{\hbar^2}{2m_b^2}\partial^\alpha\frac{\Delta\sqrt{n_b(\mathbf{r},t)}}{\sqrt{n_b(\mathbf{r},t)}} + \frac{1}{m_b}\partial_\beta[p_b^{\alpha\beta}(\mathbf{r},t) + \sigma^{\alpha\beta}_{bb}(\mathbf{r},t)] = \frac{1}{m_b}F^\alpha_{bf}(\mathbf{r},t) - \frac{1}{m_b}n_b(\mathbf{r},t)\nabla^\alpha V_{\mathrm{ext}}(\mathbf{r},t), \tag{48}$$

$$n_f(\mathbf{r},t)[\partial_t + v_{\beta f}(\mathbf{r},t)\nabla^\beta]v^\alpha_f(\mathbf{r},t) - \frac{\hbar^2}{2m_f^2}\partial^\alpha\frac{\Delta\sqrt{n_f(\mathbf{r},t)}}{\sqrt{n_f(\mathbf{r},t)}} + \frac{1}{m_f}\partial_\beta[p_f^{\alpha\beta}(\mathbf{r},t) + \sigma^{\alpha\beta}_{ff}(\mathbf{r},t)] = \frac{1}{m_f}F^\alpha_{fb}(\mathbf{r},t) - \frac{1}{m_f}n_f(\mathbf{r},t)\nabla^\alpha V_{\mathrm{ext}}(\mathbf{r},t). \tag{49}$$

Here, the stress tensor $\sigma^{\alpha\beta}_{bb}(\mathbf{r},t)$, which characterizes the boson-boson interactions in Eq. (48), is determined by the general formula (41) and its limiting case (46).

The two-particle function $n_2(\mathbf{r},\mathbf{r}',t)$ in Eq. (21) is determined by the general formula (22) for bosons and fermions. However, in the formula for fermions, instead of expression (23), which is valid for bosons, the following formula should be used [46]:

$$\langle\mathbf{r},\mathbf{r}',R_{N-2},t|n_1,n_2\ldots\rangle = \sum_f\sum_{f'<f}\sqrt{\frac{n_f}{N}}\sqrt{\frac{n_{f'}}{N-1}}(-1)^{\sum_{f'\leq g<f}n_g}(\langle\mathbf{r},t|f\rangle\langle\mathbf{r}',t|f'\rangle - \langle\mathbf{r}',t|f\rangle\langle\mathbf{r},t|f'\rangle)\times\langle R_{N-2},t|n_1,\ldots,(n_{f'}-1),\ldots,\times(n_f-1),\ldots\rangle, \tag{50}$$

$$\langle n_1,\ldots,(n_{f'}-1),\ldots,(n_f-1),\ldots,|n_1,\ldots, \times(n_{q'}-1),\ldots,(n_q-1),\ldots\rangle=\delta(f-q)\delta(f'-q') - \delta(f-q')\delta(f'-q). \quad (51)$$

Therefore, for the stress tensor of fermions with short-range interaction potential U_{ff}, to the third approximation in the interaction radius, we obtain the following expression:

$$\sigma_{ff}^{\alpha\beta}(\mathbf{r},t)=\frac{1}{2}\Upsilon_{2ff}\Bigg(\delta^{\alpha\beta}\Bigg(n_f(\mathbf{r},t)\sum_g n_g|\nabla\varphi_g(\mathbf{r},t)|^2 - \left|\sum_g n_g\varphi_g^*(\mathbf{r},t)\nabla\varphi_g(\mathbf{r},t)\right|^2\Bigg) + \Bigg(n_f(\mathbf{r},t)\sum_g n_g[\partial^\alpha\varphi_g^*(\mathbf{r},t)]\partial^\beta\varphi_g(\mathbf{r},t) - \sum_g n_g\varphi_g^*(\mathbf{r},t)[\partial^\alpha\varphi_g(\mathbf{r},t)] \times\sum_{g'} n_{g'}[\partial^\beta\varphi_{g'}^*(\mathbf{r},t)]\varphi_{g'}(\mathbf{r},t)+\text{c.c.}\Bigg)\Bigg). \quad (52)$$

Here, $\varphi_{g'}(\mathbf{r},t)$ are the arbitrary single-particle wave functions, and $\Upsilon_{2ff}=(4\pi/15)\int dr(r)^5\partial U_{ff}(r)/\partial r$.

We can see that, similarly to the case of bosons, the stress tensor $\sigma_{ff}^{\alpha\beta}(\mathbf{r},t)$ is symmetric. The main distinction of the fermionic stress tensor from the bosonic one is in the absence of first-order terms in the interaction radius in the fermionic tensor. Let us find the expression for the stress tensor of ultracold fermions, accepting plane waves as the first iteration. In this case, from formula (43), we obtain $j^\alpha(\mathbf{r},t)=n(\mathbf{r},t)v^\alpha(\mathbf{r},t)=0$.

The quantity $\Sigma_g n_g\varphi_g^*(\mathbf{r},t)[\partial^\alpha\varphi_g(\mathbf{r},t)]$ entering expression (52), similarly to j^α, equals zero. From formula (44), we obtain the result that $\Pi^{\alpha\beta}(\mathbf{r},t)$ is not equal to zero. Separating the convective part of the density tensor of the momentum flux, we have

$$\Pi^{\alpha\beta}(\mathbf{r},t)=n(\mathbf{r},t)v^\alpha(\mathbf{r},t)v^\beta(\mathbf{r},t)+p^{\alpha\beta}(\mathbf{r},t)+T^{\alpha\beta}(\mathbf{r},t).$$

By virtue of the equality $v^\alpha(\mathbf{r},t)=0$, the tensor $\Pi^{\alpha\beta}(\mathbf{r},t)$ equals the sum $p^{\alpha\beta}(\mathbf{r},t)+T^{\alpha\beta}(\mathbf{r},t)$. The quantity $\Sigma_g n_g[\partial^\alpha\varphi_g^*(\mathbf{r},t)]\partial^\beta\varphi_g(\mathbf{r},t)$, in the approximation of plane waves for ultracold fermions situated inside the Fermi sphere, equals $m/\hbar^2\Pi^{\alpha\beta}$. Assuming that the kinetic pressure tensor is diagonal, $p^{\alpha\beta}=p_{Fe}\delta^{\alpha\beta}$, and that $p_{Fe}=1/5(3\pi^2)^{2/3}\hbar^2n_f^{5/3}/m_f$ [48], we derive the expression for the stress tensor in the form

$$\sigma_{ff}^{\alpha\beta}(\mathbf{r},t)=\frac{m_f}{2\hbar^2}\Upsilon_{2ff}\Bigg(\delta^{\alpha\beta}(3\pi^2)^{2/3}\frac{\hbar^2}{m_f}n_f^{8/3}(\mathbf{r},t) + \delta^{\alpha\beta}n_f(\mathbf{r},t)T^{\gamma\gamma}(\mathbf{r},t)+2n_f(\mathbf{r},t)T^{\alpha\beta}(\mathbf{r},t)\Bigg). \quad (53)$$

Let us now address the interaction of bosons and fermions. When calculating the force field $F_{bf}^\alpha(\mathbf{r},t)$ that characterizes this interaction, we proceed to new variables of the center of gravity and relative motion, and then replace the functions characterizing the particle distribution by their approximate values in the region of the radius of the boson-fermion interaction. Since now these particles are of different kinds, the even expansion terms of the δ function, which were mutually reduced for the systems of identical particles, will give the first nonzero expansion term and, for this reason, the boson-fermion interaction is unrepresentable in the form of the divergence of a certain tensor. By calculation, we obtain the formula accurate to third-order terms in the interaction radius:

$$F_{bf}^\alpha(\mathbf{r},t)=\Upsilon_{bf}n_b(\mathbf{r},t)\partial^\alpha n_f(\mathbf{r},t)+\frac{1}{2}\Upsilon_{2bf}n_b(\mathbf{r},t)\partial^\alpha\Delta n_f(\mathbf{r},t). \quad (54)$$

Here, as above, Υ_{bf} and Υ_{2bf} are determined by the integrals $\Upsilon_{bf}\equiv(4\pi/3)\int dr(r)^3\partial U_{bf}(r)/\partial r$, $\Upsilon_{2bf}\equiv(4\pi/15)\int dr(r)^5\partial U_{bf}(r)/\partial r$. The expression for the force field $F_{fb}^\alpha(\mathbf{r},t)$ and momentum balance equation for the fermions can be obtained from Eq. (54) via substitution of the index f for b, and vice versa.

From the expressions obtained for the stress tensors (46) and (53), and vector force field (54), an important consequence follows: in the field representation, the short-range forces approach zero simultaneously with equality to zero of concentration gradients, so that for the uniform equilibrium distributions, the resultants of the forces equal zero. The appearance of concentration gradients in the external field causes the appearance of stresses (46), (53), and force (54), and is the physical mechanism causing static equilibrium of the boson-fermion system of particles in an external field.

In the general case, the effective single-particle Schrödinger equation is not an integral of the Cauchy-Lagrangian type, Eqs. (48) and (49). However, for the systems and processes for which the pressure tensor $p^{\alpha\beta}$ can be considered as diagonal and the velocity field can be considered as eddy-free, and we can restrict ourselves to the first order in the interaction radius in the stress tensor, integrals of the specified type exist:

$$\imath\hbar\partial_t\Phi_b(\mathbf{r},t)=\Bigg(-\frac{\hbar^2\nabla^2}{2m_b}+V_{\text{ext}}(\mathbf{r},t)+\mu_b(\mathbf{r},t)-\Upsilon_{bb}n_b(\mathbf{r},t) - \Upsilon_{bf}n_f(\mathbf{r},t)\Bigg)\Phi_b(\mathbf{r},t), \quad (55)$$

$$\imath\hbar\partial_t\Phi_f(\mathbf{r},t)=\Bigg(-\frac{\hbar^2\nabla^2}{2m_f}+V_{\text{ext}}(\mathbf{r},t)+(3\pi^2)^{2/3}\frac{\hbar^2}{2m_f}n_f^{2/3}(\mathbf{r},t) - \Upsilon_{bf}n_b(\mathbf{r},t)\Bigg)\Phi_f(\mathbf{r},t). \quad (56)$$

It is evident that the equation for bosons under the assumptions made differs from the Gross-Pitaevskii equation (40) only in the term representing the interaction of bosons with fermions. As for Eq. (56) for fermions, it is desirable to compare this equation with the results of studies of other authors. Let us rewrite Eq. (56) in the form

$$\imath\hbar\partial_t\Phi_f(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{\chi m_f}+V_{\text{ext}}(\mathbf{r},t)+\mu_f(\mathbf{r},t)+g_{bf}n_b(\mathbf{r},t)\right)\Phi_f(\mathbf{r},t). \quad (57)$$

From this, it is evident that in our work the mass factor is $\chi=2$, while the chemical potential appears as the local Fermi pressure $\mu_f(\mathbf{r},t)=(3\pi^2)^{2/3}(\hbar^2/2m_f)n_f^{2/3}(\mathbf{r},t)$ and $g_{bf}=-\Upsilon_{bf}$. For comparison, note that in [39], in consideration of the boson-fermion mixture for the mass factor, a value of 2 is also used, while for the chemical potential of fermions situated in the harmonic trap, the expression $\mu_f=\hbar\omega_F(6\lambda N_F)^{1/3}$ obtained in the study [49] is used. Here, ω_F is the trap parameter, λ the trap asymmetry parameter, and N_F the number of fermions in the trap. In [40,50–52], taking into account the Weizsäcker kinetic energy, the mass factor is $\chi=6$, while the chemical potential has the form $\mu_f(\mathbf{r},t)=\hbar^2(6\pi^2)^{2/3}|n|^{2/3}(\mathbf{r},t)/2m$. In [53–55], the equation for the single-particle wave function of fermions entering the boson-fermion mixture has the form (57), where $\chi=2$, $\mu_f=0$. However, in [55], the force constants g_{BB} and g_{BF} have the forms $g_{BB}=8\pi\hbar a_{BB}R_B^{-1}$ and $g_{BF}=4\pi\hbar m_F a_{BF}(1+m_F)^{-1}R_B^{-1}$, where $R_B=(\hbar/M_B\Omega_B)^{1/2}$, M_B, and M_F are the mass of bosons and fermions, respectively, $m_F=M_F/M_B$, and Ω_B is the transverse trapped frequency of the bosons. In [56], for the description of the properties of the boson-fermion mixture, a set of equations for single-particle wave functions is obtained. The feature of this set of equations is that the fermions are described by a set of N_F single-particle wave functions, where N_F is the number of fermions in the system. By this method, the equations for the wave functions of fermions are similar to Eq. (57) with the values $\chi=2$, $\mu_f=0$. The equations obtained agree qualitatively with the modified GP equations, to within the accuracy of these corrections.

Therefore, based on the general equations of quantum hydrodynamics, we obtained the GP equation and more general equations containing the additional parameter Υ_2 at higher spatial third-order derivatives with respect to concentration. In this case, the velocity field of the Bose liquid was considered as eddy-free, i.e., $\text{rot}\mathbf{v}_b(\mathbf{r},t)=\mathbf{0}$, and at zero temperature, $\boldsymbol{\nabla}p_b(\mathbf{r},t)/n_b(\mathbf{r},t)=\boldsymbol{\nabla}\mu_b(\mathbf{r},t)$. For fermions, the temperature was also assumed to be equal to zero, and consequently the pressure was assumed to be isotropic and equal to the Fermi pressure. These equations have the form, for bosons,

$$m_bn_b(\mathbf{r},t)\partial_t v_b^\alpha(\mathbf{r},t)+\frac{1}{2}m_bn_b(\mathbf{r},t)\nabla^\alpha v_b^2(\mathbf{r},t)+\nabla^\alpha\mu_b(\mathbf{r},t)-\frac{\hbar^2}{2m_b}n_b(\mathbf{r},t)\partial^\alpha\frac{\Delta\sqrt{n_b(\mathbf{r},t)}}{\sqrt{n_b(\mathbf{r},t)}}-\frac{1}{2}\Upsilon_{bb}\partial_\alpha n_b^2(\mathbf{r},t)-\frac{1}{16}\Upsilon_{2bb}\partial_\alpha\Delta n_b^2(\mathbf{r},t)=\Upsilon_{bf}n_b(\mathbf{r},t)\partial^\alpha n_f(\mathbf{r},t)+\frac{1}{2}\Upsilon_{2bf}n_b(\mathbf{r},t)\partial^\alpha\Delta n_f(\mathbf{r},t)-n_b(\mathbf{r},t)\nabla^\alpha V_{\text{ext}}(\mathbf{r},t), \quad (58)$$

and for fermions,

$$m_fn_f(\mathbf{r},t)[\partial_t+v_{\beta f}(\mathbf{r},t)\nabla^\beta]v_f^\alpha(\mathbf{r},t)-\frac{\hbar^2}{2m_f}n_f(\mathbf{r},t)\partial^\alpha\frac{\Delta\sqrt{n_f(\mathbf{r},t)}}{\sqrt{n_f(\mathbf{r},t)}}+\frac{1}{8}\Upsilon_{2ff}[\partial^\alpha n_f(\mathbf{r},t)]\Delta n_f(\mathbf{r},t)+\frac{1}{4}\Upsilon_{2ff}[\partial^\alpha\partial^\beta n_f(\mathbf{r},t)]\partial_\beta n_f(\mathbf{r},t)-\frac{3}{8}\Upsilon_{2ff}n_f(\mathbf{r},t)\partial^\alpha\Delta n_f(\mathbf{r},t)+\frac{1}{2}(3\pi^2)^{2/3}\Upsilon_{2ff}\partial^\alpha n_f^{8/3}(\mathbf{r},t)+\frac{\hbar^2}{5m_f}(3\pi^2)^{2/3}\partial^\alpha n_f^{5/3}(\mathbf{r},t)=\Upsilon_{bf}n_f(\mathbf{r},t)\partial^\alpha n_b(\mathbf{r},t)+\frac{1}{2}\Upsilon_{2bf}n_f(\mathbf{r},t)\partial^\alpha\Delta n_b(\mathbf{r},t)-n_f(\mathbf{r},t)\nabla^\alpha V_{\text{ext}}(\mathbf{r},t). \quad (59)$$

In this approximation, the corresponding effective wave functions satisfy the equations

$$\imath\hbar\partial_t\Phi_b(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m_b}+V_{\text{ext}}(\mathbf{r},t)+\mu_b(\mathbf{r},t)-\Upsilon_{bb}n_b(\mathbf{r},t)-\Upsilon_{bf}n_f(\mathbf{r},t)-\frac{1}{2}\Upsilon_{2bf}\Delta n_f(\mathbf{r},t)-\frac{1}{16}\Upsilon_{2bb}\int_{\mathbf{r}_0}^{\mathbf{r}}\frac{1}{n_b(\mathbf{r},t)}d[\Delta n_b^2(\mathbf{r},t)]\right)\Phi_b(\mathbf{r},t) \quad (60)$$

(this equation is the generalization of the GP equation and takes into account the third-order terms in the interaction radius) and

$$\imath\hbar\partial_t\Phi_f(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m_f}+V_{\text{ext}}(\mathbf{r},t)+(3\pi^2)^{2/3}\frac{\hbar^2}{2m_f}n_f^{2/3}(\mathbf{r},t)-\Upsilon_{bf}n_b(\mathbf{r},t)-\frac{1}{2}\Upsilon_{2bf}\Delta n_b(\mathbf{r},t)+\frac{4}{5}(3\pi^2)^{2/3}\Upsilon_{2ff}n_f^{5/3}(\mathbf{r},t)-\frac{3}{8}\Upsilon_{2ff}\Delta n_f(\mathbf{r},t)+\frac{1}{8}\Upsilon_{2ff}\int_{\mathbf{r}_0}^{\mathbf{r}}\frac{\Delta n_f(\mathbf{r},t)}{n_f(\mathbf{r},t)}dn_f(\mathbf{r},t)+\frac{1}{8}\Upsilon_{2ff}\int_{\mathbf{r}_0}^{\mathbf{r}}\frac{d[(\nabla n_f(\mathbf{r},t))]^2}{n_f(\mathbf{r},t)}\right)\Phi_f(\mathbf{r},t). \quad (61)$$

Equation (61) differs from the above-given Eq. (56) in that the third-order terms in the interaction radius are taken into account in Eq. (61).

VI. DISPERSION OF EIGENWAVES

The dependence of the energy of elemental excitation on the momentum for a uniform system of bosons in the state of the Bose condensate is described by the Bogoliubov formula [5,14], which also follows from the Gross-Pitaevskii equation. In [57] the low-energy excitations of a dilute atomic Bose gas confined in a harmonic trap interacting with repulsive forces was investigated. In [58,59], collective oscillations of a Bose-Fermi mixture are considered. In [60,32] measurements of the excitation spectrum and the static structure factor of a Bose-Einstein condensate were reported and analyzed.

Therefore, let us consider the eigenmodes which can propagate along the larger symmetry axis of a cigar-shaped magnetic trap based on found above Eqs. (58) and (59). In the approximation of linear concentration and velocity field, we obtain the dispersion equation for elemental excitations in the form

$$\omega_{1,2}^2=\frac{1}{2}\Bigg[\omega_{0b}^2+\omega_{0f}^2 \pm\sqrt{(\omega_{0b}^2-\omega_{0f}^2)^2+\frac{4n_{0f}n_{0b}}{m_f m_b}k^4\left(\Upsilon_{bf}-\frac{1}{2}\Upsilon_{2bf}k^2\right)^2}\Bigg], \tag{62}$$

$$\omega_{0b}^2=\left(\frac{\hbar^2}{4m_b^2}+\frac{n_{0b}\Upsilon_{2bb}}{8m_b}\right)k^4-\frac{\Upsilon_{bb}n_{0b}}{m_b}k^2, \tag{63}$$

$$\omega_{0f}^2=\left(\frac{\hbar^2}{4m^2}+\frac{3n_{of}\Upsilon_{2ff}}{8m_f}\right)k^4+\frac{(3\pi^2)^{2/3}}{3}\left(\frac{\hbar^2}{m_f^2}n_{0f}^{2/3}+\frac{4\Upsilon_{2ff}}{m_f}n_{0f}^{5/3}\right)k^2. \tag{64}$$

In the limit of large moments $\hbar k$, the dependence $\omega(k)$ is parabolic. From the explicit expression for the coefficient Υ_2, Eq. (42), it is evident that, if the attractive forces act between the particles of the system, this coefficient is positive. Consequently, the curve $\omega=\omega(k)$ for particles interacting via attraction is higher than the curve corresponding to the Bogoliubov spectrum.

It is evident that, for the uniform system of bosons ^{4}He in formula (63), the coefficient of k^4 varies qualitatively. In addition to the kinematic contribution proportional to $\hbar^2$, a dynamic contribution Υ_{2bb} that depends on the concentration of the Bose particles appears. Using the known values of physical characteristics of superfluid helium ^{4}He at atmospheric pressure, we can evaluate the numerical value of the concentration-dependent term Υ_{2bb} and compare it with the value of $\hbar^2/(4m_b^2)$. To evaluate the quantity Υ_{2bb}, let us find its relation to Υ_{bb}; in turn, Υ_{bb} can be expressed through the scattering potential. As the model short-range potentials, let us consider the interaction potential of "rigid spheres" $U(r)=U_0=\text{const}$ at $r\leq r_0$ and $U(r)=0$ at $r>r_0$, as well as the Yukawa potential $U(r)=U_0\exp(-r/r_0)/r$.

Calculating the integrals (29) and (42) with these potentials, we can see that the approximate equality $\Upsilon_{2bb}\simeq r_0^{-2}\Upsilon_{bb}$ is valid. For a potential of the type of rigid spheres $\Upsilon_{2bb}=(1/5)r_0^{-2}\Upsilon_{bb}$. For the Yukawa potential $\Upsilon_{2bb}=4r_0^{-2}\Upsilon_{bb}$. To evaluate Υ_{bb}, note that, for the potentials under consideration, is following the condition satisfied: $r^3U(r)\to 0$ as r tends to zero or to infinity. Then $\Upsilon_{bb}=-\int d\mathbf{r}\,U(r)=-g_{bb}=-4\pi\hbar^2a_{bb}/m_b$. The scattering amplitude a_{bb} can be evaluated based on the experimental data. The quantity r_0 has the order of the atomic radius and scattering amplitude.

Dispersion in the system of fermions, formula (64), is actually reduced to a linear dependence $\omega(k)$ in the region of long wavelengths. The interparticle interaction is involved in this formula by means of the coefficient Υ_{2ff}. It is evident from formulas (63) and (64) that the coefficients of k^4 in dispersion relations for bosons and fermions have the same functional form.

As for the boson-fermion mixture, it is evident from the dispersion relations (62) that the interaction between particles of various types leads to the presence of two hybrid branches of wave dispersion, which can be excited in this medium. Evaluation of the quantity Υ_{2bf} can be performed through the quantity Υ_{bf}, similarly to the evaluation of Υ_{2bb}.

Let us also note that the immediate measurement of the dispersion curves (62)–(64) can be used for the direct experimental determination of the parameters Υ_{bb}, Υ_{2bb}, Υ_{2ff}, Υ_{bf}, and Υ_{2bf} immediately associated with the potentials of interparticle interaction.

VII. CONCLUSIONS

The problem of construction of the GP equation and related wave functions in a medium is associated with the necessity of proceeding from the description in configuration space to the description in physical space. We showed that the balance equations for the number of particles and momentum, which are the local conservation laws, immediately follow from the multiparticle Schrödinger equation. In the case of short-range interaction between the particles, the force field is represented as the divergence of the quantum stress tensor. This conclusion is valid for a single-component system of bosons and a single-component system of fermions. The stress tensor $\sigma^{\alpha\beta}(\mathbf{r},t)$ is represented in the form of series (9) at a sufficiently small interaction radius. The GP equation for Bose systems corresponds to the case where we can restrict ourselves only to the first term of series (9). Under the assumptions made, the set of equations of continuity and momentum balance is not closed. Therefore, to close the set of equations, the method of sequential approximations is additionally used. For a sufficiently dilute system of Bose particles in a harmonic magnetic trap, we can use the eigenfunctions of the harmonic oscillator problem as the first iteration. For a uniform system or for sufficiently wide traps, we can use plane waves in this case. It turns out that in both cases the same equations (32) are valid for the BEC. Therefore, we derive a set of equations of quantum hydrodynamics closed with respect to the concentration $n_b(\mathbf{r},t)$ and the velocity field $v_b(\mathbf{r},t)$ of Bose particles. From the set of balance equations (13) and (37) obtained, the equation for the wave function in the medium follows; this coincides with the GP equation.

As a generalization of the GP equation, we made allowance for the contribution of the third-order terms in the interaction radius to the stress tensor of bosons. This contribution also depends on the basis single-particle wave functions. The specific expressions for the stress tensor in this approximation are obtained for plane waves as basis wave functions as well as for functions of the isotropic harmonic oscillator under the condition that the particles are in the BEC state.

For the system of particles that comprises an ultracold mixture of bosons and fermions, two-type quantum hydrodynamics is constructed.

For the fermionic subsystem, the stress tensor is found accurate to third-order terms in the interaction radius. In this case, due to the antisymmetry of the wave function with respect to permutation of the Fermi particles, the first-order term in the interaction radius for fermions identically approaches zero. The stress tensor for fermions in the third order of the interaction radius is expressed through the concentration and basis single-particle wave functions. To close the set of equations of quantum hydrodynamics, we selected the plane waves as the basis wave functions. It was also assumed that all fermions lie inside the Fermi sphere. The expression for the stress tensor for fermions is found through the Fermi pressure and the quantum Bohm potential $T^{\alpha\beta}(\mathbf{r},t)$, which in turn depend on the fermion concentration and its derivatives.

The boson-fermion interaction is unrepresentable in the form of divergence of the stress tensor. The force field of interaction of bosons and fermions, as shown above, is expressed through the concentrations of bosons and fermions and their derivatives for each selection of basis single-particle wave functions accurate to third order in the interaction radius.

The spectrum of the eigenmodes of boson, fermion, and boson-fermion systems is determined by the dispersion relations (62)–(64). For bosons, the previously obtained relation (63) involves additional information on the interparticle interaction as a correction to the Bogoliubov spectrum. We also established the dependence of the dispersion relation of fermions (64) on the character of their interaction. The dispersion law of the waves in a boson-fermion system, as shown above, presents two hybrid branches (62), which explicitly depend on the character of boson-boson, fermion-fermion, and boson-fermion interactions.

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