Wigner-transform phase-space densities of a two-dimensional harmonically confined charged quantum gas subjected to a magnetic field

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(Received 29 July 2008; published 10 November 2008)

Closed form analytical expressions are obtained for the Wigner transform of the Bloch density matrix and for the Wigner phase-space density of a two-dimensional harmonically trapped charged quantum gas in a uniform magnetic field of arbitrary strength, at zero and nonzero temperatures. An exact analytic expression is also obtained for the autocorrelation function. The strong magnetic field case, where only few Landau levels are occupied, is also examined, and useful approximate expressions for the spatial and momentum densities are given.

DOI: 10.1103/PhysRevA.78.053614 PACS number(s): 03.75.Ss, 05.30.Fk, 73.21.La

I. INTRODUCTION

Considerable interest has been shown in the study of the properties of the so-called low-dimensional systems. The advances in nanotechnology allows nowdays the realization of quasi-two-dimensional systems such as quantum dots [1,2]. In a different context, the experimental achievement of trapped ultracold atom gases allows us to study quantum-mechanical effects of quantum statistics in such gases [3]. The abovementioned physical systems have originated a great volume of theoretical work in order to understand such a fascinating world in a reduced physical space [4]. In this context, using the canonical Bloch density matrix as a tool, exact analytical expressions have been obtained for the particle and the kinetic energy densities in spatial coordinates at zero and nonzero temperatures [5]. Very recently, this method has been generalized to take into account the effect of a uniform perpendicular magnetic field on a confined charged two-dimensional quantum gas [6]. In the present work, we are interested in obtaining exact analytical expressions for the Wigner transforms of both, the canonical Bloch density matrix and the first-order density matrix. Our interest in the Wigner transform is based on the fact that it provides a useful tool to study various properties of many-body systems [7]. In addition to the fact that it allows a reformulation of quantum mechanics in terms of classical concepts, and it is also used to generate semiclassical approximations [8,9], the Wigner transform may help to gain a better understanding of the properties of the system. It is also very interesting because with the recent progress of the experimental techniques, phase space densities can be nowdays measured for certain quantum systems [10].

The canonical Bloch density matrix is defined as $C(\boldsymbol{r},\boldsymbol{r}',\beta)=\Sigma_j\phi_j(\boldsymbol{r})\phi_j^*(\boldsymbol{r}')\exp(-\beta\epsilon_j)$, where $\phi_j(\boldsymbol{r})$ and ϵ_j are eigenfunctions and eigenvalues of a one particle Hamiltonian H associated to the system. Here, β is to be interpreted as a mathematical variable, which in general, is taken to be complex, and not necessarily the inverse temperature. The Bloch density matrix is of particular interest since its knowledge enables the first-order density matrix $\rho(\boldsymbol{r},\boldsymbol{r}')$ to be found, through the inverse Laplace transform [11]. In fact at $T=0$, the first-order density matrix, for a given Fermi energy λ, is given by

$$\rho(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{C(\boldsymbol{r},\boldsymbol{r}',\beta)}{\beta} e^{\beta\lambda}. \tag{1}$$

The system we are going to study is a harmonically confined charged atom gas in a two-dimensional x-y plane subjected to a perpendicular homogeneous magnetic field $\boldsymbol{B}=B\boldsymbol{k}$, taken along the z axis. The one particle Hamiltonian is then given by

$$H = \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{1}{2}m^*\omega_0^2\boldsymbol{r}^2 \tag{2}$$

with $\boldsymbol{r}^2=x^2+y^2$, $\boldsymbol{A}=\frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r})$ is the vector potential, m^* and $-e$ are, respectively, the effective mass and the charge of the particle and ω_0 is the oscillation frequency of the confining potential. For the Hamiltonian under study, a closed analytical expression was obtained long time ago for the corresponding Bloch density [12]. Here we rewrite it in the following useful form:

$$C(\boldsymbol{r},\boldsymbol{r}',\beta) = \frac{m^*\Omega/2\pi\hbar}{\sinh\beta\hbar\Omega}\exp\left\{\frac{-2m^*\Omega/\hbar}{\sinh\beta\hbar\Omega}\left[\boldsymbol{R}^2\sinh\frac{\beta\hbar\Omega_-}{2}\times\sinh\frac{\beta\hbar\Omega_+}{2} + \frac{\boldsymbol{s}^2}{4}\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2} + i\frac{(\boldsymbol{R}\times\boldsymbol{s})\cdot\boldsymbol{k}}{2}\sinh\beta\hbar\omega_L\right]\right\}, \tag{3}$$

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where $\boldsymbol{R}=(\boldsymbol{r}+\boldsymbol{r}')/2$ and $\boldsymbol{s}=\boldsymbol{r}-\boldsymbol{r}'$ are, respectively, the center-of-mass and relative coordinates and

$$\omega_L = \frac{eB}{2m^*c}, \quad \Omega = \sqrt{\omega_0^2+\omega_L^2}, \quad \Omega_\pm = \Omega \pm \omega_L. \tag{4}$$

ω_L is the Larmor frequency and $\Omega_\pm$ are two frequencies that correspond to excitations in the center-of-mass motion–the so-called "Kohn modes" [13]. Note that, the Hamiltonian in Eq. (2) has the same partition function, $Z=1/[4\sinh(\beta\hbar\Omega_+/2)\sinh(\beta\hbar\Omega_-/2)]$, as that an anisotropic two dimensional harmonic oscillator with frequencies Ω_- and Ω_+.

The rest of the paper is organized as follows. In the next section, we calculate the Wigner transform of the Bloch density and alternative useful analytical forms for such Wigner transform are also derived. The Wigner phase space density matrix is calculated at zero and nonzero temperatures in Sec. III, showing some interesting plots of it. In Sec. IV, we derive a closed analytical form, in terms of Laguerre polynomials, for the so called autocorrelation function. The high magnetic field strength case is examined in Sec. IV. In the last section, a summary and outlook are given.

II. THE WIGNER TRANSFORM OF THE BLOCH DENSITY MATRIX

In the following we shall calculate the Wigner transform of the Bloch density matrix given in Eq. (3). The Wigner transform of an arbitrary one particle operator A, defined by its matrix elements in spatial coordinates $\boldsymbol{A}(\boldsymbol{r}+\frac{s}{2},\boldsymbol{r}-\frac{s}{2})$, is the following function $\boldsymbol{A}_W$ of the phase space variables $\boldsymbol{r}$ and $\boldsymbol{p}$ [14]:

$$A_W(\boldsymbol{r},\boldsymbol{p}) = \int_{\mathbb{R}^2} A(\boldsymbol{r}+\boldsymbol{s}/2,\boldsymbol{r}-\boldsymbol{s}/2)e^{-i\boldsymbol{p}\cdot\boldsymbol{s}/\hbar}d\boldsymbol{s}, \tag{5}$$

where its inverse transform is

$$A(\boldsymbol{r}+\boldsymbol{s}/2,\boldsymbol{r}-\boldsymbol{s}/2) = \int_{\mathbb{R}^2} \frac{d\boldsymbol{p}}{(2\pi\hbar)^2} A_W(\boldsymbol{r},\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{s}/\hbar}. \tag{6}$$

According to Eq. (6), the local part of the operator $\boldsymbol{A}$ can be computed as

$$A(\boldsymbol{r},\boldsymbol{r}) \equiv A(\boldsymbol{r}) = \int_{\mathbb{R}^2} \frac{d\boldsymbol{p}}{(2\pi\hbar)^2} A_W(\boldsymbol{r},\boldsymbol{p}). \tag{7}$$

We can now calculate, by making use of Eq. (5), the Wigner transform of the Bloch density matrix (3). Let us take the Wigner transform of $C(\boldsymbol{r}+\frac{s}{2},\boldsymbol{r}-\frac{s}{2},\beta)$ and call it $C_W(\boldsymbol{r},\boldsymbol{p},\beta)$, so that

$$\begin{aligned} C_W(\boldsymbol{r},\boldsymbol{p},\beta) = {} & \frac{m^*\Omega/2\pi\hbar}{\sinh\beta\hbar\Omega}\exp\left[-\frac{2m^*\Omega}{\hbar\sinh(\beta\hbar\Omega)} \times\sinh\frac{\beta\hbar\Omega_-}{2}\sinh\frac{\beta\hbar\Omega_+}{2}\boldsymbol{r}^2\right] \\ & \times\int_{\mathbb{R}^2}\exp\left[-\frac{m^*\Omega}{2\hbar\sinh(\beta\hbar\Omega)}\times\left(\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2}\right)\boldsymbol{s}^2 \right. \\ & \left. -i\left(\frac{m^*\Omega\sinh\beta\hbar\omega_L}{\hbar\sinh(\beta\hbar\Omega)}(\boldsymbol{k}\times\boldsymbol{r})+\frac{\boldsymbol{p}}{\hbar}\right)\cdot\boldsymbol{s}\right]d\boldsymbol{s}. \end{aligned} \tag{8}$$

The above two-dimensional integral can be easily evaluated by using the well known identity

$$\int_{\mathbb{R}^2} d\boldsymbol{s}\, e^{-a s^2 - i\boldsymbol{b}\cdot\boldsymbol{s}} = \frac{\pi}{a}e^{-\boldsymbol{b}^2/(4a)} \tag{9}$$

to obtain the result

$$C_W(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{e^{-f(\beta)\boldsymbol{r}^2}e^{-g(\beta)[\boldsymbol{p}+u(\beta)(\boldsymbol{k}\times\boldsymbol{r})]^2}}{\cosh\frac{\beta\hbar\Omega_+}{2}\cosh\frac{\beta\hbar\Omega_-}{2}}, \tag{10}$$

where for notational simplicity we have introduced the following functions of β

$$f(\beta) = \frac{2m^*\Omega\sinh\frac{\beta\hbar\Omega_+}{2}\sinh\frac{\beta\hbar\Omega_-}{2}}{\hbar\sinh(\beta\hbar\Omega)},$$

$$g(\beta) = \frac{\sinh(\beta\hbar\Omega)}{2m^*\hbar\Omega\cosh\frac{\beta\hbar\Omega_+}{2}\cosh\frac{\beta\hbar\Omega_-}{2}},$$

$$u(\beta) = \frac{m^*\Omega\sinh\beta\hbar\omega_L}{\sinh(\beta\hbar\Omega)}. \tag{11}$$

It can be easily checked that when the magnetic field is absent, so that $\omega_L=0$ then $\Omega_+=\Omega_-=\omega_0$ and $\Omega=\omega_0$, Eq. (10) yields to the correct Wigner transform for a harmonic oscillator in two dimensions, that is [8]

$$C_W^{B=0}(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{\exp\left[-\frac{2\tanh\frac{\beta\hbar\omega_0}{2}}{\hbar\omega_0}\left(\frac{\boldsymbol{p}^2}{2m^*}+\frac{m^*\omega_0^2}{2}\boldsymbol{r}^2\right)\right]}{\cosh^2\frac{\beta\hbar\omega_0}{2}} \tag{12}$$

For the case of an unconfined system subjected to a magnetic field, i.e. $\omega_0=0$, then $\Omega=\omega_L$, $\Omega_-=0$, $\Omega_+=2\omega_L$, and Eq. (10) reduces to

$$C_W^{\omega_0=0}(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{\exp\left[-\frac{\tanh\beta\hbar\omega_L}{2m^*\hbar\omega_L}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2\right]}{\cosh\beta\hbar\omega_L}, \tag{13}$$

which is the correct expression of the Wigner transform [15].

In the following, we present two alternative analytical forms of the result in Eq. (10), which can be rewritten as

$$C_W(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{1}{\cosh\frac{\beta\hbar\Omega_+}{2}\cosh\frac{\beta\hbar\Omega_-}{2}}\exp[-G(\boldsymbol{r},\boldsymbol{p},\beta)] \tag{14}$$

with

$$G(\boldsymbol{r},\boldsymbol{p},\beta) = (f+gu^2)\boldsymbol{r}^2 + g\boldsymbol{p}^2 + 2guL_z \tag{15}$$

and L_z is the component of the orbital angular momentum along the z axis. We show in Appendix A that $G(\boldsymbol{r},\boldsymbol{p},\beta)$ takes the following simple form:

$$G(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{H_0+\Omega L_z}{\hbar\Omega}\tanh\frac{\beta\hbar\Omega_+}{2} + \frac{H_0-\Omega L_z}{\hbar\Omega}\tanh\frac{\beta\hbar\Omega_-}{2}. \tag{16}$$

Substitution of this result into Eq. (14), leads to the factorized analytical form

$$C_W(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{\exp\left[-\frac{\tanh\frac{\beta\hbar\Omega_+}{2}}{\hbar\Omega}(H_0+\Omega L_z)\right]}{\cosh\frac{\beta\hbar\Omega_+}{2}} \times \frac{\exp\left[-\frac{\tanh\frac{\beta\hbar\Omega_-}{2}}{\hbar\Omega}(H_0-\Omega L_z)\right]}{\cosh\frac{\beta\hbar\Omega_-}{2}}, \tag{17}$$

where

$$H_0 = \frac{\boldsymbol{p}^2}{2m^*} + \frac{m^*\Omega^2}{2}\boldsymbol{r}^2. \tag{18}$$

Let us now obtain a third closed expression for Wigner transform of the Bloch density. For that purpose, we use the following expansion in terms of Laguerre polynomials [16]:

$$\frac{\exp(-x\tanh y)}{\cosh y} = 2e^{-x}\sum_{n=0}^{\infty}(-1)^n L_n(2x)\exp\left[-2y\left(n+\frac{1}{2}\right)\right] \tag{19}$$

for $x=(H_0 \pm \Omega L_z)/\hbar\Omega$ and $y=\beta\hbar\Omega_\pm/2$, Eq. (17) becomes

$$C_W(\boldsymbol{r},\boldsymbol{p},\beta) = 4\exp\left[-\frac{2H_0}{\hbar\Omega}\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} \times L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right)L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right) \times\exp(-\beta E_{n,m}), \tag{20}$$

where

$$E_{n,m} = \hbar\Omega_+(n+1/2) + \hbar\Omega_-(m+1/2) \tag{21}$$

are the eigenvalues of the Hamiltonian (2). To our knowledge, the results given in Eqs. (10), (17), and (20) are new and seem not to have been reported before in the literature.

III. QUANTUM WIGNER PHASE SPACE DISTRIBUTION AT ZERO AND NONZERO TEMPERATURES

A. Phase space distribution at zero temperature

Having established in the previous section various analytical forms for the Wigner transform of the Bloch density, we shall now calculate analytically the expression for the quantum Wigner phase space distribution or Wigner transform density of the first-order density matrix $\rho(\boldsymbol{r},\boldsymbol{r}')$. Let $\rho_W(\boldsymbol{r},\boldsymbol{p})$ denote such density, defined as

$$\rho_W(\boldsymbol{r},\boldsymbol{p}) = \int_{\mathbb{R}^2}\rho\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2},\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)e^{-i\boldsymbol{p}\cdot\boldsymbol{s}/\hbar}d\boldsymbol{s}. \tag{22}$$

The above distribution can also be obtained through the use of the Wigner phase space version of Eq. (1), that is,

$$\rho_W(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\beta\frac{C_W(\boldsymbol{r},\boldsymbol{p},\beta)}{\beta}e^{\beta\lambda}. \tag{23}$$

Inserting Eq. (20) into Eq. (23), and performing the inverse Laplace transform [17]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\beta}{\beta}e^{\beta(\lambda-E_{n,m})} = \Theta(\lambda-E_{n,m}), \tag{24}$$

where Θ is the Heaviside step function, we find

$$\rho_W(\boldsymbol{r},\boldsymbol{p}) = 4e^{-2H_0/\hbar\Omega}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right) \times L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right)\Theta(\lambda-E_{n,m}). \tag{25}$$

Due to the presence of the step function, the quantum numbers n,m are restricted to $\hbar\Omega_+(n+1/2)+\hbar\Omega_-(m+1/2)<\lambda$. The highest allowed value for n, N_+, is given by

$$N_+ = \mathrm{Int}\left[\frac{\lambda}{\hbar\Omega_+} - \frac{\Omega}{\Omega_+}\right], \tag{26}$$

where $\mathrm{Int}(x)$ denotes the integer part of $x>0$. For a given allowed value of n, the maximum allowed value of m, N_-, is

$$N_- = \mathrm{Int}\left[\frac{\lambda}{\hbar\Omega_-} - \frac{\Omega_+}{\Omega_-}n - \frac{\Omega}{\Omega_-}\right]. \tag{27}$$

Therefore, the density distribution in Eq. (25), can be rewritten as

$$\rho_W(\boldsymbol{r},\boldsymbol{p}) = 4e^{-2H_0/\hbar\Omega}\sum_{n=0}^{N_+}\sum_{m=0}^{N_-}(-1)^{n+m}L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right) \times L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right). \tag{28}$$

Notice that the above density depends not only on the moduli

$|\boldsymbol{r}|$ and $|\boldsymbol{p}|$, but also on the relative angle θ between $\boldsymbol{r}$ and $\boldsymbol{p}$, i.e., $\rho_W(\boldsymbol{r},\boldsymbol{p})=\rho_W(r,p,\theta)$. For $N=20$ particles, Fermi energy $\lambda=6.35\hbar\omega_L$ and $\omega_0/\omega_L=1$, we display in Figs. 1(a)–1(d) this Wigner phase space density for $\theta=0,\pi/6,\pi/3,\pi/2$, respectively.

In the absence of a magnetic field, the system becomes of a pure harmonic oscillator with frequency ω_0 and thus all the Ω are all equal to ω_0, we show in Appendix B, that the above density has indeed the correct limit, given by [18]

$$\rho_W^{B=0}(\boldsymbol{r},\boldsymbol{p}) = 4e^{-2H_0/\hbar\omega_0}\sum_{p=0}^{M}(-1)^p L_p^1\left(\frac{4H_0}{\hbar\omega_0}\right), \qquad (29)$$

where L_n^1 is the generalized Laguerre polynomial of order one and the quantum number M is related to the Fermi energy by $\lambda=\hbar\omega_0(M+1)$.

In what follows, we shall deduce from Eq. (25) a result which will greatly simplify us the treatment when we will deal, in the next subsection, with the finite temperature case. Let $\phi_{n,m}(\boldsymbol{r})$ denote the eigenfunction of the Hamiltonian (2) with eigenvalues $E_{n,m}$ given in Eq. (21). In terms of the single particle wave functions, the first-order density matrix is

$$\rho\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2},\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right) = \sum_{m,n=0}^{\infty}\phi_{n,m}\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\phi_{n,m}^*\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)\Theta(\lambda-E_{n,m}), \qquad (30)$$

where λ is the Fermi energy. Taking the Wigner transform of Eq. (30), we get

$$\rho_W(\boldsymbol{r},\boldsymbol{p}) = \sum_{m,n=0}^{\infty}\mathcal{W}\left[\phi_{n,m}\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\phi_{n,m}^*\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)\right]\Theta(\lambda-E_{n,m}). \qquad (31)$$

Here the symbol $\mathcal{W}$ stands for Wigner transform. Comparing this result with Eq. (25), we deduce that

$$\mathcal{W}\left[\phi_{n,m}\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\phi_{n,m}^*\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)\right] = 4e^{-2H_0/\hbar\Omega}(-1)^{n+m} \times L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right) \times L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right). \qquad (32)$$

Thus, we have found the Wigner transform of the product $\phi_{n,m}(\boldsymbol{r}+\boldsymbol{s}/2)\phi_{n,m}^*(\boldsymbol{r}-\boldsymbol{s}/2)$ without the explicit use of the single particle wave functions. As stated before this result will immediately be used in the following subsection.

B. Wigner phase space distribution at nonzero temperatures

Here, we shall generalize the result obtained in Eq. (25), valid for $T=0$, to nonzero temperatures. We start with the definition of the first-order density matrix at temperature T, $\rho(\boldsymbol{r}+\boldsymbol{s}/2,\boldsymbol{r}-\boldsymbol{s}/2,T)$, in terms of the normalized single particle wave functions $\phi_{n,m}$, which reads for fermions

$$\rho^F\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2},\boldsymbol{r}-\frac{\boldsymbol{s}}{2},T\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\phi_{n,m}\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\phi_{n,m}^*\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)+1}, \qquad (33)$$

where $[\exp(\frac{E_{n,m}-\mu}{k_BT})+1]^{-1}$ is the Fermi distribution function for the level energy $E_{n,m}$, k_B is Boltzmann's constant and μ the chemical potential. Taking the Wigner transform of both sides in Eq. (33) and using obvious notations, we get

$$\rho_W^F(\boldsymbol{r},\boldsymbol{p},T) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathcal{W}[\phi_{n,m}(\boldsymbol{r}+\boldsymbol{s}/2)\phi_{n,m}^*(\boldsymbol{r}-\boldsymbol{s}/2)]}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)+1}, \qquad (34)$$

where we have used the fact that, the Fermi distribution is not affected by the Wigner transformation. Substituting Eq. (32) into Eq. (34), one arrives at

$$\rho_W^F(\boldsymbol{r},\boldsymbol{p},T) = 4e^{-2H_0/\hbar\Omega}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right) \times L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right)\frac{1}{e^{(E_{n,m}-\mu)/k_BT}+1}. \qquad (35)$$

Since, in the $T\to 0$ limit the Fermi distribution function becomes the step function, with $\lambda=\mu(T=0)$ as the Fermi energy, that is,

$$\frac{1}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)+1} \to \Theta(\lambda-E_{n,m}), \qquad (36)$$

the result (35) reduces to the correct zero temperature limit given in Eq. (25). As can be seen in Eq. (35), the Fermi distribution function enters in a simple way in the expression of the phase space distribution. This suggests to examine a similar situation for the case of bosons. In this case, the first-order density matrix in spatial coordinates at temperature T is

$$\rho^B\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2},\boldsymbol{r}-\frac{\boldsymbol{s}}{2},T\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\phi_{n,m}\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\phi_{n,m}^*\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)-1}, \qquad (37)$$

where we have included the Bose distribution function. Following the same derivation as done for Fermions, one immediately gets for the phase space density of bosons at finite temperature, the result

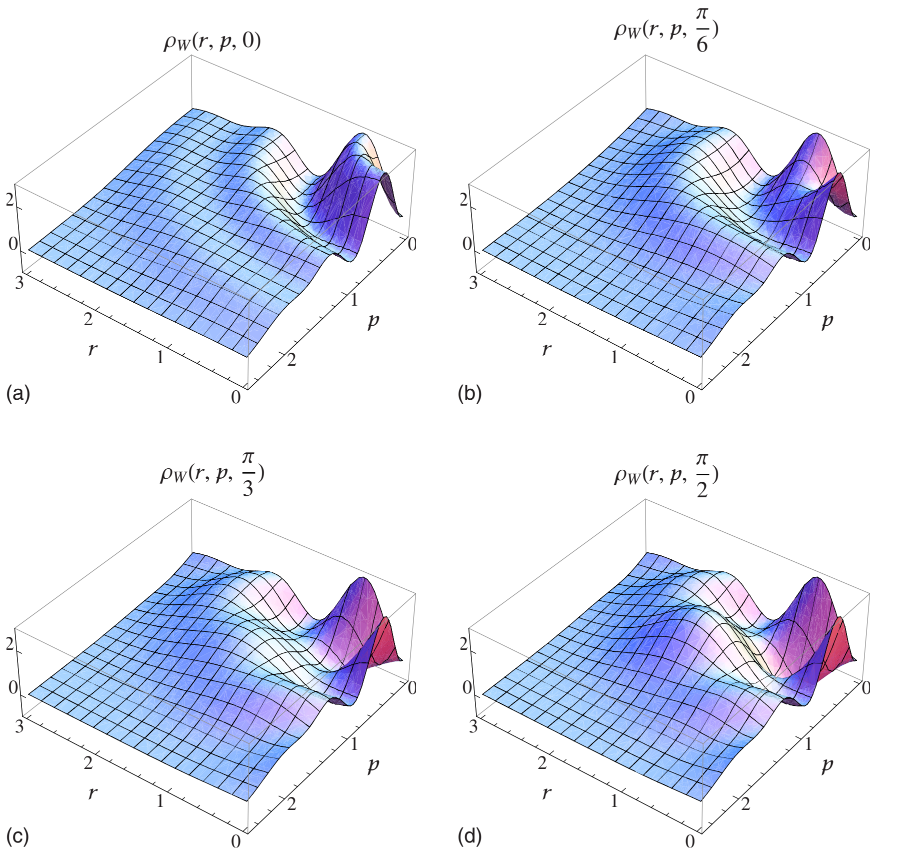

FIG. 1. (Color online) (a)–(d) correspond to plots of the Wigner phase space density $\rho_W(\boldsymbol{r},\boldsymbol{p})=\rho_W(r,p,\theta)$ for $N=20$ particles at $\theta=0,\pi/6,\pi/3,\pi/2$, respectively. We have chosen parameters $\omega_0/\omega_L=1$ with Fermi energy $\lambda=6.35\hbar\omega_L$. Lengths are plotted in units of the magnetic length $l=\sqrt{\hbar c/eB}$ for r and in units of l^{-1} for the momentum p.

$$\rho_W^B(\boldsymbol{r},\boldsymbol{p},T)=4e^{-2H_0/\hbar\Omega}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}L_n\left(\frac{2(H_0+\Omega L_z)}{\hbar\Omega}\right)\times L_m\left(\frac{2(H_0-\Omega L_z)}{\hbar\Omega}\right)\frac{1}{e^{(E_{n,m}-\mu)/k_BT}-1}. \tag{38}$$

This last equation may constitute a useful starting point to study thermodynamical properties in phase space of charged Bose gas, in particular at low temperatures.

IV. THE AUTOCORRELATION FUNCTION

The autocorrelation function, also called the reciprocal form factor [19], is known to provide information on the off-diagonal part of the density matrix $\rho(\boldsymbol{r},\boldsymbol{r}')$ and is defined in spatial coordinates as

$$B(\boldsymbol{s})=\int_{\mathbb{R}^2}\exp\left(-\frac{i\boldsymbol{p}\cdot\boldsymbol{s}}{\hbar}\right)n(\boldsymbol{p})d\boldsymbol{p} \tag{39}$$

with $\boldsymbol{s}=\boldsymbol{r}-\boldsymbol{r}'$ and $n(\boldsymbol{p})$ is the density profile in momentum space. The latter is defined by a similar relation as in Eq. (30), where one has to convert the normalized spatial wave functions $\phi_{n,m}$ into their analogs $\tilde{\phi}_{n,m}(\boldsymbol{p})$ in momentum space, that is,

$$n(\boldsymbol{p})=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\phi}_{n,m}(\boldsymbol{p})\tilde{\phi}^{*}_{n,m}(\boldsymbol{p})\Theta(\lambda-E_{n,m}). \tag{40}$$

On the other hand, the momentum density $n(\boldsymbol{p})$ can also be obtained through the Wigner phase space distribution

$$n(\boldsymbol{p})=\int_{\mathbb{R}^2}\rho_W(\boldsymbol{r},\boldsymbol{p})\frac{d\boldsymbol{r}}{(2\pi\hbar)^2} \tag{41}$$

and is normalized to the total particle number N of the system

$$\int_{\mathbb{R}^2}n(\boldsymbol{p})d\boldsymbol{p}=N. \tag{42}$$

Therefore, it follows from Eq. (39), that $B(\boldsymbol{0})=N$. In the following we shall derive a closed analytical result for $B(\boldsymbol{s})$. To do so, we first insert Eq. (23) into Eq. (41), to obtain

$$n(\boldsymbol{p})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\beta\frac{e^{\beta\lambda}}{\beta}\int_{\mathbb{R}^2}\frac{d\boldsymbol{r}}{(2\pi\hbar)^2}C_W(\boldsymbol{r},\boldsymbol{p},\beta) \tag{43}$$

to carry out the $\boldsymbol{r}$ integration, we use the analytical form of $C_W(\boldsymbol{r},\boldsymbol{p},\beta)$ given in Eq. (10) and we rewrite it as follows:

$$C_W(\boldsymbol{r},\boldsymbol{p},\beta)=\frac{1}{\cosh\frac{\beta\hbar\Omega_+}{2}\cosh\frac{\beta\hbar\Omega_-}{2}}\exp\left[-\frac{fg}{f+gu^2}\boldsymbol{p}^2\right]\times\exp\left[-(f+gu^2)\left(\boldsymbol{r}+\frac{gu}{f+gu^2}(\boldsymbol{p}\times\boldsymbol{k})\right)^2\right], \tag{44}$$

where we have used $(\boldsymbol{p}\times\boldsymbol{k})^2=\boldsymbol{p}^2$, since $\boldsymbol{p}$ is a planar vector. The above result can now be inserted into Eq. (43), to obtain

$$n(\boldsymbol{p}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{\exp(\beta\lambda)}{(2\pi\hbar)^2\beta}\left[\frac{1}{\cosh\frac{\beta\hbar\Omega_+}{2}\cosh\frac{\beta\hbar\Omega_-}{2}}\frac{\pi}{(f+gu^2)}\exp\left(-\frac{fg}{f+gu^2}\boldsymbol{p}^2\right)\right]$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{\exp(\beta\lambda)}{(2\pi\hbar)^2\beta}\left[\frac{2\pi\hbar}{m^*\Omega\sinh(\beta\hbar\Omega)}\exp\left(-\frac{2\sinh\frac{\beta\hbar\Omega_+}{2}\sinh\frac{\beta\hbar\Omega_-}{2}}{\hbar m^*\Omega\sinh(\beta\hbar\Omega)}\boldsymbol{p}^2\right)\right], \quad (45)$$

where we have used Eq. (11). Putting this result with this present form into Eq. (39) and using Eq. (9), one then finds

$$B(\boldsymbol{s}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{\exp(\beta\lambda)}{2\pi\hbar m^*\Omega\beta\sinh(\beta\hbar\Omega)}\left[\int_{\mathbb{R}^2} d\boldsymbol{p}\exp\left(-\frac{2\sinh\frac{\beta\hbar\Omega_+}{2}\sinh\frac{\beta\hbar\Omega_-}{2}}{\hbar m^*\Omega\sinh(\beta\hbar\Omega)}\boldsymbol{p}^2 - \frac{i\boldsymbol{p}\cdot\boldsymbol{s}}{\hbar}\right)\right]$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{\exp(\beta\lambda)}{\beta}\left[\frac{1}{4\sinh\frac{\beta\hbar\Omega_-}{2}\sinh\frac{\beta\hbar\Omega_+}{2}}\exp\left(-\frac{m^*\Omega}{8\hbar}\frac{\sinh(\beta\hbar\Omega)}{\sinh\frac{\beta\hbar\Omega_+}{2}\sinh\frac{\beta\hbar\Omega_-}{2}}\boldsymbol{s}^2\right)\right]. \quad (46)$$

Remember that, $\Omega = (\Omega_+ + \Omega_-)/2$, so that

$$\frac{\sinh(\beta\hbar\Omega)}{\sinh\frac{\beta\hbar\Omega_+}{2}\sinh\frac{\beta\hbar\Omega_-}{2}} = \coth\frac{\beta\hbar\Omega_+}{2} + \coth\frac{\beta\hbar\Omega_-}{2} \quad (47)$$

plugging this result into the exponential of Eq. (46), to get

$$B(\boldsymbol{s}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\beta \frac{e^{\beta\lambda}}{\beta}\left[\frac{\exp\left[-\frac{m^*\Omega}{8\hbar}\left(\coth\frac{\beta\hbar\Omega+}{2}\right)s^2\right]}{2\sinh\frac{\beta\hbar\Omega_+}{2}} \times \frac{\exp\left[-\frac{m^*\Omega}{8\hbar}\coth\left(\frac{\beta\hbar\Omega_-}{2}\right)s^2\right]}{2\sinh\frac{\beta\hbar\Omega_-}{2}}\right]. \quad (48)$$

At this level, we can carry out explicitly the inverse Laplace transform by first using the following expansion in terms of Laguerre polynomials [16]

$$\frac{\exp\left[-x\coth\left(\frac{\beta\hbar\Omega_\pm}{2}\right)\right]}{\sinh\left(\frac{\beta\hbar\Omega_\pm}{2}\right)} = 2e^{-x}\sum_{n=0}^{\infty}L_n(2x)e^{-\beta\hbar\Omega_\pm(n+1/2)} \quad (49)$$

and followed by Eq. (24), one then finds

$$B(\boldsymbol{s}) = e^{-m^*\Omega/4\hbar s^2}\sum_{m,n=0}^{\infty}L_n\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)L_m\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)\Theta(\lambda - E_{n,m}). \quad (50)$$

Notice that an interesting feature of the above autocorrelation function is that it is expressed in terms of Laguerre polynomials with same arguments, note also that it is isotropic in spatial coordinates, i.e., depends only on the length $|\boldsymbol{s}|$ of the vector $\boldsymbol{s}$. Setting $\boldsymbol{s}=0$, one obtains $B(\boldsymbol{0})=N$, as it is required. In Fig. 2 we display the $T=0$ spatial dependence of the autocorrelation function for $N=20$ particles, choosing the following values of the parameters $\omega_0/\omega_L=1$ and Fermi energy $\lambda=6.35\hbar\omega_L$.

Finally, let us come to the finite temperature expression of $B(\boldsymbol{s})$. In a similar way as was done in the previous section, one can immediately write down its expression. For Fermions, this is simply achieved by replacing the $\Theta(\lambda - E_{n,m})$ in Eq. (50) by the Fermi function

$$B(\boldsymbol{s}) = e^{-(m^*\Omega/4\hbar)s^2}\sum_{m,n=0}^{\infty}\frac{L_n\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)L_m\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)+1}. \quad (51)$$

For bosons, all that is required is the replacement of the Fermi function by the Bose function

$$B(\boldsymbol{s}) = e^{-(m^*\Omega/4\hbar)s^2}\sum_{m,n=0}^{\infty}\frac{L_n\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)L_m\left(\frac{m^*\Omega}{4\hbar}\boldsymbol{s}^2\right)}{\exp\left(\frac{E_{n,m}-\mu}{k_BT}\right)-1}. \quad (52)$$

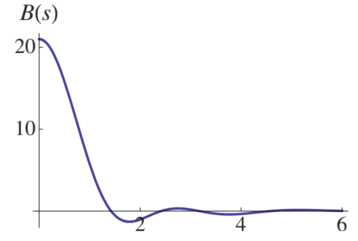

FIG. 2. (Color online) The $T=0$ autocorrelation function $B(\boldsymbol{s}) = B(s)$ for $N=20$ particles, with $\omega_0/\omega_L=1$ and Fermi energy $\lambda = 6.35\hbar\omega_L$. Lengths are plotted in units of the magnetic length $l = \sqrt{\hbar c/eB}$.

V. STRONG MAGNETIC FIELD CASE

The strong magnetic field case at low temperature is of particular interest in quantum dots. In this limit only a few Landau levels are occupied and the magnetic field length $l = (\hbar c/eB)^{1/2}$ is small leading to slowly varying confining external harmonic potential on the scale of l. In what follows we shall examine the strong magnetic field (SB) case, where one has $\omega_0/\omega_L \ll 1$ then $\Omega \approx \omega_L$, $\Omega_+ \approx 2\omega_L$ and $\Omega_- \approx \omega_0^2/(2\omega_L)$ which yields, respectively, for the functions given in Eq. (11), to the leading order

$$f(\beta) \approx \beta m^* \omega_0^2/2, \quad g(\beta) \approx \frac{\tanh(\beta\hbar\omega_L)}{2m^*\hbar\omega_L}, \quad u(\beta) \approx m^*\omega_L. \tag{53}$$

Substituting this results into Eq. (10), gives immediately the result for the Wigner transform of the Bloch density

$$C_W^{\mathrm{SB}}(\boldsymbol{r},\boldsymbol{p},\beta) = \frac{e^{-\beta m^*\omega_0^2 r^2/2}}{\cosh \beta\hbar\omega_L} \exp\left[-\frac{\tanh \beta\hbar\omega_L}{2m^*\hbar\omega_L}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2\right]. \tag{54}$$

In this case, the net result we get is a product of the Bloch density of free charged particles in magnetic field [see Eq. (13)] and an exponential factor limiting the spatial distribution. Let us now, calculate the corresponding Wigner phase space density. Inserting Eq. (54) into Eq. (23) and making use of Eqs. (19) and (24), one obtains

$$\rho_W^{\mathrm{SB}}(\boldsymbol{r},\boldsymbol{p}) = 2 \exp\left[-\frac{H_{\mathrm{magn}}}{\hbar\omega_L}\right] \sum_{n=0}^{\infty} (-1)^n L_n\left(\frac{2H_{\mathrm{magn}}}{\hbar\omega_L}\right) \times \Theta\left(\lambda - (2n+1)\hbar\omega_L - \frac{m^*\omega_0^2}{2}\boldsymbol{r}^2\right), \tag{55}$$

where $H_{\mathrm{magn}} = [\boldsymbol{p} + (e/c)\boldsymbol{A}]^2/(2m^*)$ is the Hamiltonian for a particle in the presence of the magnetic field alone. One immediately recognizes in the argument of the Heaviside function the discrete Landau level energies $(2n+1)\hbar\omega_L$. As can be seen, the phase space density in above has a simple analytical form, therefore we can easily obtain, in this high magnetic field limit, the corresponding density matrix $\rho(\boldsymbol{r} + \boldsymbol{s}/2, \boldsymbol{r} - \boldsymbol{s}/2)$ in spatial coordinates. To do so, we make use of the inverse Wigner transformation. According to Eq. (6), one has

$$\rho^{\mathrm{SB}}(\boldsymbol{r} + \boldsymbol{s}/2, \boldsymbol{r} - \boldsymbol{s}/2) = 2\sum_{n=0}^{\infty} (-1)^n \Theta\left(\lambda - (2n+1)\hbar\omega_L - \frac{m^*\omega_0^2}{2}\boldsymbol{r}^2\right) \times \int_{\mathbb{R}^2} \frac{d\boldsymbol{p}}{(2\pi\hbar)^2} e^{+i\boldsymbol{p}\cdot\boldsymbol{s}/\hbar} e^{-H_{\mathrm{magn}}/\hbar\omega_L} L_n\left(\frac{2H_{\mathrm{magn}}}{\hbar\omega_L}\right). \tag{56}$$

The last integral can be carried out as follows. Denoting it by I and using the canonical momentum $\boldsymbol{K} = \boldsymbol{p} + (e/c)\boldsymbol{A}$, one obtains

$$I = e^{-i(e/\hbar c)\boldsymbol{A}\cdot\boldsymbol{s}} \int_{\mathbb{R}^2} \frac{d\boldsymbol{K}}{(2\pi\hbar)^2} e^{i\boldsymbol{K}\cdot\boldsymbol{s}/\hbar} e^{-(\boldsymbol{K}^2/2m^*\hbar\omega_L)} L_n\left(\frac{\boldsymbol{K}^2}{m^*\hbar\omega_L}\right), \tag{57}$$

and changing to the variable $t = K/\sqrt{m^*\hbar\omega_L}$, we get

$$\begin{aligned} I &= \frac{m^*\hbar\omega_L}{(2\pi\hbar)^2} e^{-i(e/\hbar c)\boldsymbol{A}\cdot\boldsymbol{s}} \int_0^{\infty} t e^{-t^2/2} L_n(t^2) dt \times \int_0^{2\pi} d\phi e^{i(\sqrt{m^*\omega_L/\hbar})ts\cos\phi} \\ &= \frac{e^{-i(e/\hbar c)\boldsymbol{A}\cdot\boldsymbol{s}}}{4\pi l^2} \int_0^{\infty} t e^{-t^2/2} L_n(t^2) J_0(\sqrt{m^*\omega_L/\hbar}ts) dt, \end{aligned} \tag{58}$$

where we have made use of the relation [16]

$$\int_0^{2\pi} d\phi e^{ix\cos(\phi)} = 2\pi J_0(x)$$

to get the last line, $J_0(x)$ being the Bessel function. The following relation [16]:

$$\int_0^{\infty} x e^{-x^2/2} L_n(x^2) J_0(xy) dx = (-1)^n e^{-y^2/2} L_n(y^2)$$

helps us to perform the integral in Eq. (58), to find

$$I = \frac{(-1)^n}{2\pi l^2} e^{-i(e/\hbar c)\boldsymbol{A}\cdot\boldsymbol{s}} e^{-(m^*\omega_L/2\hbar)\boldsymbol{s}^2} L_n\left(\frac{m^*\omega_L}{\hbar}\boldsymbol{s}^2\right). \tag{59}$$

Substituting this result into Eq. (56), yields

$$\rho^{\mathrm{SB}}\left(\boldsymbol{r} + \frac{\boldsymbol{s}}{2}, \boldsymbol{r} - \frac{\boldsymbol{s}}{2}\right) = \frac{e^{-i(e/\hbar c)\boldsymbol{A}\cdot\boldsymbol{s}}}{2\pi l^2} e^{-(m^*\omega_L/2\hbar)\boldsymbol{s}^2} \sum_{n=0}^{\infty} L_n\left(\frac{m^*\omega_L}{\hbar}\boldsymbol{s}^2\right) \times \Theta(\lambda - (2n+1)\hbar\omega_L - m^*\omega_0^2\boldsymbol{r}^2/2).$$

The local density is obtained by setting $\boldsymbol{s} = 0$,

$$\rho^{\mathrm{SB}}(\boldsymbol{r}) = \frac{1}{2\pi l^2} \sum_{n=0}^{\infty} \Theta\left(\lambda - (2n+1)\hbar\omega_L - \frac{m^*\omega_0^2}{2}\boldsymbol{r}^2\right). \tag{60}$$

In their study of a two-dimensional electron gas subjected to a magnetic field and partially confined by a harmonic potential, the authors of Ref. [20] obtained a similar result for the spatial density using a different approach [we note that their parabolic potential is taken only in the x direction, i.e., $V(x,y) = m^*\omega_0^2 x^2/2$]. As noticed by these authors, the density contains compressible and incompressible regions.

In order to rewrite Eq. (60) in a more compact form, we use the following identity relating the Heaviside and the integer part functions:

$$\sum_{n=0}^{\infty} \Theta(x - n) = \Theta(x+1)\mathrm{Int}(x+1). \tag{61}$$

Then, $\rho(\mathbf{r})$ becomes

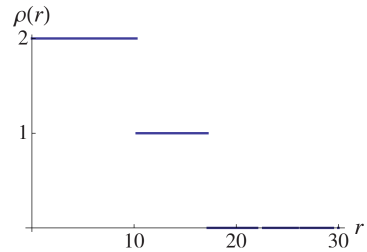

FIG. 3. (Color online) Plot of the spatial density $\rho(r)=\rho(r)$ in units of $(2\pi l^2)^{-1}$ at $T=0$ in a harmonic oscillator potential with $\omega_0/\omega_L=0.2048$ and Fermi energy $\lambda=4.1\hbar\omega_L$, corresponding to $N=200$ particles. Lengths are plotted in units of the magnetic length $l=\sqrt{\hbar c/eB}$.

$$\rho(\boldsymbol{r})=\frac{1}{2\pi l^2}\Theta\left(\frac{\lambda-m^*\omega_0^2\boldsymbol{r}^2/2+\hbar\omega_L}{2\hbar\omega_L}\right)\times\mathrm{Int}\left(\frac{\lambda-m^*\omega_0^2\boldsymbol{r}^2/2+\hbar\omega_L}{2\hbar\omega_L}\right). \quad (62)$$

In Fig. 3 we plot the above zero-temperature spatial density for the case of $N=200$ particles with parameters $\omega_0/\omega_L=0.2048$ and Fermi energy $\lambda=4.1\hbar\omega_L$.

For ultrastrong magnetic field, such that all the particles reside in the lowest Landau level (LLL), Eq. (60) reduces to

$$\rho^{\mathrm{LLL}}(\boldsymbol{r})=\frac{1}{2\pi l^2}\Theta(\lambda-\hbar\omega_L-m^*\omega_0^2\boldsymbol{r}^2/2). \quad (63)$$

Before closing this section, it is intersting to calculate the momentum density or the momentum distribution $n(\boldsymbol{p})$ in the strong magnetic field case. This important distribution was already introduced in the previous section but its calculation has not been fully completed since this density was used there as an intermediate to obtain the autocorrelation function. In a short, we start from its expression given in Eq. (46) and introduce the strong magnetic field approximations we used above, namely, $\sinh(\beta\hbar\Omega_+/2)\approx\sinh\beta\hbar\omega_L$, $\sinh(\beta\hbar\Omega_-/2)\approx\beta\hbar\omega_0^2/(4\omega_L)$, and $\sinh(\beta\hbar\Omega)\approx\sinh\beta\hbar\omega_L$ to get

$$n^{\mathrm{SB}}(\boldsymbol{p})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\beta\frac{\exp(\beta\lambda)}{(2\pi\hbar)m^*\omega_L\beta}\frac{\exp\left(-\beta\frac{\omega_0^2}{2m^*\omega_L^2}\boldsymbol{p}^2\right)}{\sinh\beta\hbar\omega_L}. \quad (64)$$

At this level, it is easy to perform the inverse Laplace transform by using the expansion

$$\frac{1}{\sinh(\beta\hbar\omega_L)}=2\sum_{n=0}^{\infty}\exp[-(2n+1)\beta\hbar\omega_L]$$

with Eq. (24), which results in

$$n^{\mathrm{SB}}(\boldsymbol{p})=\frac{2l^2}{\pi\hbar^2}\sum_{n=0}^{\infty}\Theta\left(\lambda-(2n+1)\hbar\omega_L-\frac{\omega_0^2}{2m^*\omega_L^2}\boldsymbol{p}^2\right). \quad (65)$$

As with the spatial density in Eq. (60), the momentum density exhibits the same structure consisting of a series of wide steps in momentum space. The above results have been obtained for spinless charged particles at $T=0$ and the generalization to finite temperatures can be done without any particular difficulties.

VI. SUMMARY AND OUTLOOK

We have derived some simple exact closed expressions for the Wigner transform of the canonical Bloch density of two-dimensional harmonic oscillator in a uniform magnetic field. We have also obtained exact analytical form for the Wigner phase space density at zero and nonzero temperature. Our results are valid for arbitrary magnetic field strengths and hold for both fermions and bosons. For the system under study, we have found simple and exact analytical expression for the so-called autocorrelation function. The high magnetic field case has been examined. Our investigation in phase space complement the recent works in spatial coordinates. The results we obtained would constitute useful starting point for the study, in phase space, of thermodynamical properties in the field of cold atom gases.

ACKNOWLEDGMENTS

This work has been partially supported by the Spanish Ministerio de Educación y Ciencia (Grant No. MTM2005-09183) and Junta de Castilla y León (Excellence Project No. GR224). K.B. gratefully acknowledges the hospitality and the financial support granted to him during his stay at the Departamento de Física Teórica, Atómica y Óptica, Valladolid (Spain), where part of this work was done.

APPENDIX A

The purpose of this appendix is to derive the expression of the function $G(r,p,\beta)$ given in Eq. (16). First, we evaluate the various functions of β in Eq. (15), namely, $U(\beta)=[f(\beta)+g(\beta)u^2(\beta)]$, $g(\beta)$ and $V(\beta)=2g(\beta)u(\beta)$. Using Eq. (11), we get for $U(\beta)$

$$U(\beta)=\frac{2m^*\Omega\sinh\frac{\beta\hbar\Omega_-}{2}\sinh\frac{\beta\hbar\Omega_+}{2}}{\hbar\sinh(\beta\hbar\Omega)}+\frac{m^*\Omega\sinh^2\beta\hbar\omega_L}{2\hbar\sinh(\beta\hbar\Omega)\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2}}=\frac{m^*\Omega/2\hbar}{\sinh\beta\hbar\Omega}\frac{\sinh(\beta\hbar\Omega_-)\sinh(\beta\hbar\Omega_+)+\sinh^2\beta\hbar\omega_L}{\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2}}. \quad (A1)$$

From $\sinh(\beta\hbar\Omega_-)\sinh(\beta\hbar\Omega_+)=\frac{1}{2}[\cosh(\beta\hbar(\Omega_-+\Omega_+))-\cosh(\beta\hbar(\Omega_--\Omega_+))]$, and $\Omega_-+\Omega_+=2\Omega$, $\Omega_--\Omega_+=-2\omega_L$, we obtain

$$U(\beta)=\frac{m^*\Omega[\cosh 2\beta\hbar\Omega-\cosh\beta\hbar\omega_c+2\sinh^2\beta\hbar\omega_L]}{4\hbar\sinh(\beta\hbar\Omega)\cosh\left(\frac{\beta\hbar\Omega_-}{2}\right)\cosh\frac{\beta\hbar\Omega_+}{2}}$$

$$=\frac{m^*\Omega\sinh\beta\hbar\Omega}{2\hbar\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2}}.$$

If we use again $\Omega_-+\Omega_+=2\Omega$, the above result becomes

$$U(\beta)=\frac{m^*\Omega}{2\hbar}\left(\tanh\frac{\beta\hbar\Omega_+}{2}+\tanh\frac{\beta\hbar\Omega_-}{2}\right). \tag{A2}$$

We now turn to the two remaining functions. For convenience, we rewrite the function $g(\beta)$, given in Eq. (11), as

$$g(\beta)=\frac{1}{2m^*\hbar\Omega}\left(\tanh\frac{\beta\hbar\Omega_+}{2}+\tanh\frac{\beta\hbar\Omega_-}{2}\right). \tag{A3}$$

For the function $V(\beta)$, one simply gets

$$V(\beta)=\frac{\sinh 2\beta\hbar\omega_L}{\hbar\cosh\frac{\beta\hbar\Omega_-}{2}\cosh\frac{\beta\hbar\Omega_+}{2}},$$

and using $\Omega_+-\Omega_-=2\omega_L$, one ends with

$$V(\beta)=\frac{1}{\hbar}\left(\tanh\frac{\beta\hbar\Omega_+}{2}-\tanh\frac{\beta\hbar\Omega_-}{2}\right). \tag{A4}$$

Substituting Eqs. (A1)–(A3) into Eq. (15), simple manipulations yield to the desired result (16).

APPENDIX B

In this appendix we shall show that, in the absence of magnetic field, the phase space density in Eq. (28) reduce to the result in Eq. (29). In this limit Eq. (28) becomes

$$\rho_W^{B=0}(\boldsymbol{r},\boldsymbol{p})=4e^{-2H_0/\hbar\omega_0}\sum_{n=0}^{N_+}\sum_{m=0}^{N_-}(-1)^{n+m}L_n\left(\frac{2(H_0+\omega_0L_z)}{\hbar\omega_0}\right)\times L_m\left(\frac{2(H_0-\omega_0L_z)}{\hbar\omega_0}\right). \tag{B1}$$

Here, $N_+=\text{Int}(\frac{\lambda}{\hbar\omega_0})-1$ and $N_-=\text{Int}(\frac{\lambda}{\hbar\omega_0}-n-1)=N_+-n$. The physical meaning of N_+ is only but the quantum number of the last occupied harmonic oscillator shell, denoted by M in Eq. (29). The Hamiltonian H_0 appearing in Eq. (B1) refers to Eq. (18) but with $\Omega=\omega_0$. Putting $p=n+m$, Eq. (B1) rewrites

$$\rho_W^{B=0}(\boldsymbol{r},\boldsymbol{p})=4e^{-2H_0/\hbar\omega_0}\sum_{p=0}^{M}(-1)^p\sum_{m=0}^{p}L_{p-m}\left(\frac{2(H_0+\omega_0L_z)}{\hbar\omega_0}\right)\times L_m\left(\frac{2(H_0-\omega_0L_z)}{\hbar\omega_0}\right), \tag{B2}$$

and using the identity $\sum_{m=0}^{p}L_{p-m}(x)L_m(y)=L_p^1(x+y)$ (see Ref. [16]), the result in Eq. (29) is then recovered.

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