

# Universal relations for a strongly interacting Fermi gas near a Feshbach resonance

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(Received 16 June 2008; published 4 November 2008)

A set of universal relations between various properties of any few-body or many-body system consisting of fermions with two spin states and a large but finite scattering length have been derived by Shina Tan. We derive generalizations of the Tan relations for a two-channel model for fermions near a Feshbach resonance that includes a molecular state whose detuning energy controls the scattering length. We use quantum field theory methods, including renormalization and the operator product expansion, to derive these relations. They reduce to the Tan relations as the scattering length is made increasingly large.

DOI: [10.1103/PhysRevA.78.053606](https://doi.org/10.1103/PhysRevA.78.053606)

PACS number(s): 03.75.Ss, 31.15.-p, 34.50.-s, 67.85.Lm

## I. INTRODUCTION

Many-body systems of fermions have long been of great importance in astrophysics, nuclear physics, and solid state physics. The development of trapping and cooling techniques for ultracold atoms has made them important in atomic physics as well. In this case, the strength of the interaction is governed by the two-body scattering length which can be controlled experimentally, adding a new dimension to the problem [1].

If the scattering length  $a$  is much larger than the range of the interactions, the system has universal properties that are determined only by the large scattering length. For sufficiently low number density  $n$ , the universal properties can be calculated using perturbative methods. If  $n|a|^3$  is comparable to 1 or larger, the problem is nonperturbative. In the special case of two equally populated spin states, systematically improvable calculations are possible using Monte Carlo methods [2]. If the populations are not equal, this approach suffers from the fermion sign problem. If there are three or more spin states, the problem is complicated by the Efimov effect [3]. The challenging nature of the general problem makes exact results very valuable. One case in which exact results are known is the unitary limit  $a = \pm\infty$ , where they can be derived by exploiting scale invariance [4] and conformal invariance [5].

Tan has derived a number of universal relations between various properties of an arbitrary system consisting of fermions in two spin states with a large scattering length [6–8]. The Tan relations include the coefficient of the  $1/k^4$  tail in the momentum distribution [6], a decomposition of the energy  $E$  into terms that are insensitive to short distances [6], an expression for the local pair density [6], the rate of change of  $E$  from changes in the scattering length [7], the relation between the pressure and the energy density in a homogeneous system [7], the virial theorem for a system in a harmonic trapping potential [8], and an expression for the inelastic two-body loss rate [9]. These relations all involve a property of the system that Tan called the *integrated contact*

*intensity*. For brevity, we will refer to it simply as the *contact* and denote it by  $C$ . It can be expressed as the integral over space of the *contact density*, which we will denote by  $\mathcal{C}$ . The Tan relations hold for any state of the system: few-body or many-body, homogeneous or in a trapping potential, superfluid or normal, zero or nonzero temperature. Tan derived his relations within the framework of the many-body Schrödinger equation. He used novel methods involving generalized functions to implement the Bethe-Peierls boundary condition associated with the large two-body scattering length.

In Ref. [10], Braaten and Platter derived the Tan relations using a quantum field theory formulation of the problem. They identified the contact density  $\mathcal{C}$  as the expectation value of a local composite operator constructed from the quantum fields. To derive the Tan relations, they used standard methods of *renormalization* together with the *operator product expansion* (OPE). The OPE was invented independently by Ken Wilson [11] and by Leo Kadanoff [12] in 1969. The OPE expresses the product of local operators separated by a short distance as an expansion in local operators with coefficients that are functions of the separation of the operators.

$$\mathcal{O}_A\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right)\mathcal{O}_B\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) = \sum_C C_{A,B}^C(\mathbf{r})\mathcal{O}_C(\mathbf{R}). \quad (1)$$

The sum is, in general, an infinite sum over all possible local operators  $\mathcal{O}_C(\mathbf{R})$ . The functions  $C_{A,B}^C(\mathbf{r})$  are called *Wilson coefficients* or *short-distance coefficients*. Some of the Wilson coefficients can be singular at  $\mathbf{r}=0$ . The OPE is an asymptotic expansion for small  $\mathbf{r}$ : for any power  $N$ , only a finite number of terms on the right side of the OPE have Wilson coefficients that go to zero more slowly than  $r^N$  as  $r \rightarrow 0$  [13].

Wilson proposed the OPE as an alternative framework to the Lagrangian or Hamiltonian formulations of quantum field theory [11]. His motivation was the problem of the strong interactions associated with the nuclear force. Some aspects of the strong interactions had been successfully explained using *current algebra*, which is the algebra satisfied by the currents associated with the symmetries of a quantum field theory. For example, if the charge operators associated with a symmetry satisfy the Lie algebra  $[Q^a, Q^b] = if^{abc}Q^c$ , the asso-

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ciated charge density operators  $\rho^a(\mathbf{R})$  satisfy the equal-time commutation relations

$$\left\{ \rho^a\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right), \rho^b\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) \right\} = if^{abc} \delta^3(\mathbf{r}) \rho^c(\mathbf{R}). \quad (2)$$

This is a special case of the general short-distance expansion in Eq. (1). Wilson observed that in a scale-invariant theory, the dependence of the coefficients  $C_{A,B}^C(\mathbf{r})$  on the separation  $\mathbf{r}$  is determined up to a multiplicative constant by scale invariance and other symmetries. He used this observation to derive nontrivial results for the strong interactions that followed only from the assumption of scale invariance. The quantum field theory that describes the strong interactions was eventually determined to be *quantum chromodynamics* (QCD). Wilson's assumption of scale invariance in the strong interactions was justified by the discovery that the coupling constant of QCD is asymptotically free at short distances [14,15]. QCD and the electroweak gauge theory that together comprise the standard model of elementary particle physics are Lagrangian quantum field theories. Thus the OPE has not proved to be essential as an alternative formulation of the relativistic quantum field theories that describe elementary particles. However, the OPE has become a standard tool for developing systematic approximations to the standard model [16].

Kadanoff used the OPE to understand critical phenomena in condensed matter physics [12]. He showed that the critical exponents that describe the scaling behavior of correlation functions at a critical point can be deduced from the knowledge of which Wilson coefficients are singular as  $\mathbf{r} \rightarrow \mathbf{0}$ . The OPE therefore provides powerful constraints on critical behavior in condensed matter systems [17]. A critical point is usually characterized not only by scale invariance but also by conformal invariance. The long-distance behavior near the critical point can therefore be described by a statistical field theory with conformal symmetry, i.e., a *conformal field theory*. The OPE is one of the basic tools that is used to study conformal field theories. The quantum field theory that describes fermionic atoms with two spin states in the unitary limit is a conformal field theory. The classification of local operators based on conformal symmetry has been exploited by Nishida and Son [18] and by Mehen [19]. However, until Ref. [10], the OPE had not been applied directly to the problem of fermionic atoms with a large scattering length.

The  $T$ -matrix element for  $S$ -wave atom-atom scattering can be written as

$$\mathcal{T}(k) = \frac{4\pi}{m} \frac{1}{k \cot \delta_0(k) - ik}, \quad (3)$$

where  $m$  is the mass of the atoms,  $k$  is the relative wave number, and  $\delta_0(k)$  is the  $S$ -wave phase shift. If the interactions have a finite range, the phase shift has a low energy expansion that is conveniently expressed in the form

$$k \cot \delta_0(k) = -1/a + \frac{1}{2}r_s k^2 + \dots \quad (4)$$

This expansion, which is called the *effective range expansion*, defines the scattering length  $a$  and the effective range

$r_s$ . The Tan relations apply to any system in which the phase shift at the accessible energies can be approximated by the first term in the effective range expansion. We will consider the generalizations of these relations in two *scattering models* that are specified by the  $S$ -wave phase shift.

In the *zero-range model*, the phase shift is given by the leading term in the effective range expansion up to arbitrarily high energy as follows:

$$k \cot \delta_0(k) = -1/a. \quad (5)$$

This model can be formulated as a renormalizable local quantum field theory. Braaten and Platter derived the Tan relations by applying standard renormalization methods and the OPE to the quantum field theory for the zero-range model [10]. One advantage of this derivation is that it can be generalized in a straightforward way to any model that can be formulated as a renormalizable local quantum field theory [20]. One such model is the *resonance model* in which the  $S$ -wave phase shift is given by

$$k \cot \delta_0(k) = -4\pi \left( \lambda - \frac{g^2}{m\nu - k^2} \right)^{-1}, \quad (6)$$

where  $\nu$ ,  $g$ , and  $\lambda$  are real parameters. This model was first formulated as a quantum field theory by Kaplan [21] and by Kokkelmans *et al.* [22]. It is of practical importance in cold atom physics because it provides a natural description of fermions near a Feshbach resonance. An important special case of the resonance model is the *effective range model*, which can be obtained by setting  $\lambda=0$ . The equation for the  $S$ -wave phase shift in Eq. (6) reduces in this limit to the form

$$k \cot \delta_0(k) = -1/a + \frac{1}{2}r_s k^2, \quad (7)$$

where  $a$  is the scattering length and  $r_s < 0$  is the effective range. The effective range must be negative for this phase shift to arise from a local quantum field theory that is renormalizable [23]. The effective range model has been proposed by Petrov as a model for a narrow Feshbach resonance [24].

In this paper, we derive the generalizations of the Tan relations for the resonance model and the effective range model. We begin in Sec. II by enumerating the Tan relations. In Sec. III, we summarize the derivation of these relations in Ref. [10] using the quantum field theory for the zero-range model. In Sec. IV, we describe the formulation of the resonance model as a renormalizable local quantum field theory. The generalizations of the Tan relations in the resonance model are derived in Secs. V–XI. We use the OPE to identify the contact density operator. The generalizations of the Tan relations involve the expectation values of two other local operators besides the contact density operator. In limits in which the scattering length becomes arbitrarily large, the generalized Tan relations of the resonance model must reduce to the original Tan relations. The resulting constraints on matrix elements of local operators are derived in Sec. XII. In Sec. XIII, we derive the generalizations of the Tan relations for the effective range model by exploiting the fact that it is a limiting case of the resonance model. Our results are

summarized in Sec. XIV. The diagrammatic methods that we use to calculate the Wilson coefficients in the OPE are described in the Appendix.

## II. TAN RELATIONS

In a series of papers [6–8], Tan derived some universal relations for states consisting of fermions with two spin states that interact through a large scattering length  $a$  and may be trapped in an external potential  $V(\mathbf{R})$ . We will refer to the fermions as atoms, although this problem also has applications in condensed matter physics and in nuclear physics. We label the two spin states by  $\sigma=1,2$ . The Hamiltonian can be expressed as the sum of a kinetic term  $T$ , an interaction term  $I$ , and an external potential term  $V$ :  $H=T+I+V$ . The Tan relations all involve a quantity  $C$  that we call the *contact*. There are seven distinct Tan relations:

(1) *Tail of the momentum distribution* [6]. The momentum distribution  $\rho_\sigma(\mathbf{k})$  for atoms with spin  $\sigma$  has a power-law tail that decreases at large wave number  $\mathbf{k}$  like  $1/k^4$ .

$$\rho_\sigma(\mathbf{k}) \rightarrow C/k^4. \quad (8)$$

The coefficient  $C$  is the same for both spin states.

(2) *Energy relation* [6]. The energy  $E=\langle H \rangle=\langle T \rangle+\langle I \rangle+\langle V \rangle$  can be decomposed into four terms that are each insensitive to distances much smaller than  $|a|$ .

$$E = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} \left( \rho_\sigma(\mathbf{k}) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \langle V \rangle. \quad (9)$$

The interaction energy  $\langle I \rangle$  has been separated into two terms proportional to  $C$ . One of them has been combined with the kinetic term to make the integral over  $\mathbf{k}$  convergent.

(3) *Local pair density* [6]. If  $N_{\text{pair}}(\mathbf{R}, s)$  is the number of pairs of atoms with spins 1 and 2 in a small ball of volume  $\frac{4}{3}\pi s^3$  centered at the point  $\mathbf{R}$ , the contact density  $\mathcal{C}$  at that point can be expressed as

$$\mathcal{C}(\mathbf{R}) = \lim_{s \rightarrow 0} (4/s^4) N_{\text{pair}}(\mathbf{R}, s). \quad (10)$$

Note that  $N_{\text{pair}}(\mathbf{R}, s)$  scales as  $s^4$  as  $s \rightarrow 0$  instead of  $s^6$ , so the number of pairs with small separation  $s$  is much larger than one might naively expect.

(4) *Adiabatic relation* [7]. The rate of change of the energy  $E=\langle H \rangle$  due to a change in the scattering length is

$$\frac{d}{da} E = \frac{\hbar^2}{4\pi m a^2} C. \quad (11)$$

(5) *Pressure relation* [7,8]. For the homogeneous system with constant external potential, the pressure  $\mathcal{P}$  and the energy density  $\mathcal{E}$  are related by

$$\mathcal{P} = \frac{2}{3} \mathcal{E} + \frac{\hbar^2}{12\pi m a} C, \quad (12)$$

where  $\mathcal{C}$  is the contact density.

(6) *Virial theorem* [8]. The virial theorem for atoms in a harmonic trapping potential in the unitary limit  $a=\pm\infty$  was derived in Ref. [25]. Tan derived the virial theorem for the case of finite  $a$  [8] as follows:

$$E = 2\langle V \rangle - \frac{\hbar^2}{8\pi m a} C. \quad (13)$$

(7) *Inelastic two-body loss rate* [9]. If a pair of atoms has inelastic scattering channels, the scattering length  $a$  has a negative imaginary part. To leading order in  $\text{Im } a$ , the rate at which the atoms are depleted by inelastic collisions is

$$\Gamma \approx \frac{\hbar^2(-\text{Im } a)}{2\pi m |a|^2} C. \quad (14)$$

The adiabatic relation in Eq. (11) can be used as an operational definition of the contact. A simple example in the case  $a>0$  is the shallow diatomic molecule, or *dimer*, formed by atoms with spins 1 and 2. The energy of a dimer at rest is  $E=-\hbar^2/(ma^2)$ . From Eq. (11), the contact for the dimer is

$$C = 8\pi/a. \quad (15)$$

In Ref. [7], Tan used the adiabatic relation in Eq. (11) to determine the contact density for the homogeneous gas consisting of equal populations of atoms with spins 1 and 2 in three limits: the *BCS limit* ( $a \rightarrow 0^-$ ), the *unitary limit* ( $a \rightarrow \pm\infty$ ), and the *BEC limit* ( $a \rightarrow 0^+$ ). If the total number density of atoms is  $n$ , the contact densities in these three limits are

$$\mathcal{C} \rightarrow 4\pi^2 n^2 a^2 \quad \text{as } a \rightarrow 0^-, \quad (16a)$$

$$\rightarrow \frac{2\zeta}{5\pi} (3\pi^2 n)^{4/3} \quad \text{as } a \rightarrow \pm\infty, \quad (16b)$$

$$\rightarrow 4\pi n/a \quad \text{as } a \rightarrow 0^+, \quad (16c)$$

where  $\zeta$  in Eq. (16b) is a universal constant whose value was estimated in Ref. [7] to be approximately 1.

The most promising systems for testing the Tan relations experimentally are cold trapped atoms near a Feshbach resonance. Some of the possibilities have been discussed by Tan in Refs. [6,7]. The momentum distributions  $\rho_\sigma(\mathbf{k})$  can be measured by suddenly turning off the trapping potential and simultaneously using the Feshbach resonance to change the scattering length to 0. The cloud of atoms will expand and the initial momentum distribution can be determined from its density distributions after the expansion. There should be a scaling region in which  $\rho_\sigma(\mathbf{k})$  has the form in Eq. (8) and the contact  $C$  is simply the coefficient. As pointed out by Tan in Ref. [6], the energy relation in Eq. (9) can be used to determine the sum  $\langle T \rangle + \langle I \rangle$  of the kinetic and interaction energies from the momentum distribution. The potential energy  $\langle V \rangle$  can be determined from the density distributions before the expansion. The virial theorem in Eq. (13) is a nontrivial relation between  $\langle T \rangle + \langle I \rangle$ ,  $\langle V \rangle$ , and  $C$  that can be tested experimentally. The adiabatic relation in Eq. (11) can also be tested by using the Feshbach resonance to change the scattering length and then measuring the changes in  $\langle T \rangle + \langle I \rangle$ ,  $\langle V \rangle$ , and  $C$ .

### III. ZERO-RANGE MODEL

In Ref. [10], the Tan relations were rederived using the quantum field theory for the zero-range model. They follow straightforwardly from standard renormalization methods and from the OPE. This analysis revealed that the contact  $C$  can be expressed as the integral over space of the expectation value of a local operator. In this section, we summarize the derivations in Ref. [10].

A quantum field theory that describes atoms with two spin states must have two quantum fields  $\psi_\sigma(\mathbf{r})$ ,  $\sigma=1,2$ . The Hamiltonian for a local quantum field theory can be expressed as the integral over space of a Hamiltonian density:  $H=\int d^3r \mathcal{H}$ . If the atoms are in an external potential  $V(\mathbf{r})$ , the Hamiltonian density is the sum of a kinetic term  $\mathcal{T}$ , an interaction term  $\mathcal{I}$ , and an external potential term  $\mathcal{V}$ :  $\mathcal{H}=\mathcal{T}+\mathcal{I}+\mathcal{V}$ . The simplest quantum field theory that describes atoms with a large scattering length is the *zero-range model*, in which the phase shift has the simple form given in Eq. (5) up to arbitrarily large momentum. For the zero-range model, the three terms in the Hamiltonian density are<sup>1</sup>

$$\mathcal{T} = \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)}, \quad (17a)$$

$$\mathcal{I} = \frac{\lambda_0(\Lambda)}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1^{(\Lambda)}, \quad (17b)$$

$$\mathcal{V} = V(\mathbf{R}) \sum_{\sigma} \psi_{\sigma}^{\dagger} \psi_{\sigma}. \quad (17c)$$

For simplicity, we have set  $\hbar=1$ . The superscripts  $(\Lambda)$  on the operators in Eqs. (17a) and (17b) indicate that their matrix elements are ultraviolet divergent and an ultraviolet cutoff is required to make them well defined. For the ultraviolet cutoff, we impose an upper limit  $|\mathbf{k}| < \Lambda$  on the momenta of virtual particles. In the limit  $\Lambda \rightarrow \infty$ , the Hamiltonian density in Eq. (17) describes atoms with the phase shift given by Eq. (5) if we take the coupling constant to be

$$\lambda_0(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi}. \quad (18)$$

The connected Green's function for the scattering of a pair of atoms with spins 1 and 2 has a well-behaved limit as  $\Lambda \rightarrow \infty$ .

$$\mathcal{A}(E) = \frac{4\pi/m}{-1/a + \sqrt{-mE - i\epsilon}}, \quad (19)$$

where  $E$  is the total energy of the two atoms in their center-of-mass frame. The  $T$ -matrix element for scattering of a pair of atoms with momenta  $+\mathbf{p}$  and  $-\mathbf{p}$  is obtained by setting  $E=p^2/m$ . Comparing with the expression for the  $T$ -matrix element in Eq. (3), we obtain the phase shift in Eq. (5).

In Ref. [10], the contact  $C$  in the Tan relations was identified as the integral over space of the expectation value of

the local operator  $\lambda_0^2(\Lambda) \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1^{(\Lambda)}$  (see footnote 1). Matrix elements of the operator, which is the product of four quantum fields, are ultraviolet divergent, as indicated by the superscript  $(\Lambda)$ , but the dependence of the matrix elements on  $\Lambda$  is precisely canceled by the prefactor  $\lambda_0^2(\Lambda)$ . Since this operator has ultraviolet finite matrix elements, we suppress the dependence on  $\Lambda$  and denote it by  $\lambda_0^2 \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1$ . The contact can be expressed as<sup>2</sup>

$$C = \int d^3R \langle X | \lambda_0^2 \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1(\mathbf{R}) | X \rangle. \quad (20)$$

This expression makes explicit the dependence of the contact on the state  $|X\rangle$ . It also reveals that the contact is an extensive quantity.

For some of the Tan relations, the derivations in Ref. [10] followed in a straightforward way from the renormalization of the quantum field theory once one realizes that  $\lambda_0^2 \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1$  is a finite operator. For example, the *energy relation* in Eq. (9) can be derived by separating the interaction term  $\int d^3R \langle X | \mathcal{I} | X \rangle$  into two terms by using the identity

$$\lambda_0 = -\frac{\Lambda}{2\pi^2} \lambda_0^2 + \frac{1}{4\pi a} \lambda_0^2. \quad (21)$$

The *adiabatic relation* in Eq. (11) can be derived from the Feynman-Hellman theorem,

$$dE/da = \int d^3R \langle X | \partial \mathcal{H} / \partial a | X \rangle, \quad (22)$$

together with the identity

$$\frac{d}{da} \lambda_0 = \frac{1}{4\pi a^2} \lambda_0^2. \quad (23)$$

The *pressure relation* in Eq. (12) can be derived by applying dimensional analysis to the free energy density (or thermodynamic potential density)  $\mathcal{F}$ , which requires

$$\left[ T \frac{\partial}{\partial T} - \frac{1}{2} a \frac{\partial}{\partial a} \right] \mathcal{F} = \frac{5}{2} \mathcal{F}. \quad (24)$$

The *virial theorem* in Eq. (13) can be derived by applying dimensional analysis to the energy  $E = \int d^3R \langle X | \mathcal{H} | X \rangle$ , which requires

$$\left[ \omega \frac{\partial}{\partial \omega} - \frac{1}{2} a \frac{\partial}{\partial a} \right] E = E. \quad (25)$$

The expression for the *inelastic loss rate*  $\Gamma$  in Eq. (14) can be derived from the adiabatic relation in Eq. (11) by identifying  $-\frac{1}{2}\Gamma$  as the imaginary part of the energy  $E$  that results from adding a small negative imaginary part to the scattering length  $a$  to take into account the effects of the inelastic scattering channels.

The derivations of the tail of the momentum distribution in Eq. (8) and the expression for the contact density  $C$  in terms of the local pair density in Eq. (10) require the OPE.

<sup>1</sup>In Ref. [10], the authors used the operator  $\psi_1^{\dagger} \psi_2^{\dagger} \psi_1 \psi_2$  in which the last two quantum fields appear in the opposite order. The ordering we use here is preferred because it makes the operator positive.

<sup>2</sup>We denote the product of local operators  $\mathcal{O}_A(\mathbf{R})$  and  $\mathcal{O}_B(\mathbf{R})$  at the same point in space by  $\mathcal{O}_A \mathcal{O}_B(\mathbf{R})$ .



The number density operator for atoms with spin  $\sigma$  is  $\psi_\sigma^\dagger \psi_\sigma(\mathbf{R})$ . The momentum distribution for atoms with spin  $\sigma$  in the state  $|X\rangle$  can be expressed as

$$\rho_\sigma(\mathbf{k}) = \langle X | \tilde{\psi}_\sigma^\dagger(\mathbf{k}) \tilde{\psi}_\sigma(\mathbf{k}) | X \rangle, \quad (26)$$

where  $\tilde{\psi}_\sigma(\mathbf{k})$  is the Fourier transform of the quantum field  $\psi_\sigma(\mathbf{r})$ . The expression for the momentum distribution in terms of the quantum field is

$$\rho_\sigma(\mathbf{k}) = \int d^3R \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} \left\langle X \left| \psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) \right| X \right\rangle. \quad (27)$$

The OPE for the bilocal operator inside the expectation value has the form

$$\psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) = \sum_n C_{\sigma,n}(\mathbf{r}) \mathcal{O}_n(\mathbf{R}), \quad (28)$$

where the sum is over all possible local operators. The local operators can be expressed as products of any number of quantum fields  $\psi_\sigma(\mathbf{R})$  and an equal number of quantum fields  $\psi_\sigma^\dagger(\mathbf{R})$ , with any number of gradients applied to those fields. If the OPE in Eq. (28) is inserted into Eq. (27), the momentum distribution reduces to

$$\rho_\sigma(\mathbf{k}) = \sum_n \left( \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} C_{\sigma,n}(\mathbf{r}) \right) \int d^3R \langle X | \mathcal{O}_n(\mathbf{R}) | X \rangle. \quad (29)$$

Since the OPE is an asymptotic expansion for small  $\mathbf{r}$ , Eq. (29) is an asymptotic expansion for large  $\mathbf{k}$ . If a Wilson coefficient  $C_{\sigma,n}(\mathbf{r})$  can be expanded as a power series in the vector  $\mathbf{r}$ , the corresponding integral in Eq. (29) can be expressed in terms of the Dirac delta function in  $\mathbf{k}$  and derivatives of the Dirac delta function. A power-law tail in  $\rho_\sigma(\mathbf{k})$  at large  $\mathbf{k}$  can arise only from a term whose Wilson coefficient is not an analytic function of the vector  $\mathbf{r}$  at  $\mathbf{r}=\mathbf{0}$ .

As shown in Ref. [10], the OPE in Eq. (28) can be expressed as

$$\begin{aligned} \psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) &= \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}) + \frac{1}{2}\mathbf{r} \cdot [\psi_\sigma^\dagger \nabla \psi_\sigma(\mathbf{R}) \\ &\quad - \nabla \psi_\sigma^\dagger \psi_\sigma(\mathbf{R})] \\ &\quad - \frac{r}{8\pi} \lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}) + \dots, \end{aligned} \quad (30)$$

where we have written explicitly all terms whose Wilson coefficients go to zero more slowly than  $r^2$  as  $r \rightarrow 0$ . The first two terms on the right-hand side of the OPE in Eq. (30) can be obtained by multiplying the Taylor expansions of the two operators. The third term arises from quantum fluctuations involving pairs of atoms with small separations. Its Wilson coefficient is proportional to  $r=|\mathbf{r}|$ , which is not an analytic function of the vector  $\mathbf{r}$  at  $\mathbf{r}=\mathbf{0}$ . Its Fourier transform at nonzero values of  $\mathbf{k}$  is given by

$$\int d^3r r e^{i\mathbf{k}\cdot\mathbf{r}} = -\frac{8\pi}{k^4}. \quad (31)$$

This can be derived by differentiating the Fourier transform of  $1/r$ . The corresponding term in Eq. (29) gives a power-law tail in the momentum distribution as follows:

$$\rho_\sigma(\mathbf{k}) \rightarrow \int d^3R \langle X | \lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}) | X \rangle / k^4. \quad (32)$$

Comparing with Eq. (8), we obtain the expression for the contact  $C$  in Eq. (20).

The expression for  $C$  in terms of the local pair density in Eq. (10) follows from the OPE for  $\psi_1^\dagger \psi_1(\mathbf{R}+\mathbf{r})$  and  $\psi_2^\dagger \psi_2(\mathbf{R}+\mathbf{r}')$ . In Ref. [10], it was shown that the OPE includes the operator  $\lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$  with a Wilson coefficient proportional to  $|\mathbf{r}'-\mathbf{r}|^{-2}$  as follows:

$$\begin{aligned} \psi_1^\dagger \psi_1(\mathbf{R}+\mathbf{r}) \psi_2^\dagger \psi_2(\mathbf{R}+\mathbf{r}') &= \frac{1}{16\pi^2 |\mathbf{r}'-\mathbf{r}|^2} \lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}) \\ &\quad + \dots. \end{aligned} \quad (33)$$

Only the most singular term as  $\mathbf{r}' \rightarrow \mathbf{r}$  is shown explicitly on the right-hand side. Integrating the left-hand side over  $\mathbf{r}'$  and  $\mathbf{r}$  inside a small ball of radius  $s$ , we obtain an operator that counts the number of pairs of atoms with spins 1 and 2 inside that ball. If the right-hand side of Eq. (33) is integrated over the same region and multiplied by  $4\pi/s$ , the only term that survives in the limit  $s \rightarrow 0$  is the one that is shown explicitly. Taking the expectation value of both sides, we obtain the expression for the contact density  $\mathcal{C}$  in Eq. (10).

#### IV. RESONANCE MODEL

A practical method for controlling the scattering length of atoms is by exploiting Feshbach resonances. A Feshbach resonance occurs at a value  $B_0$  of the magnetic field for which a molecular state in a closed channel with a higher two-atom threshold is in resonance with a pair of atoms at threshold in the channel of interest. Near the Feshbach resonance, the dependence of the scattering length  $a$  on the magnetic field  $B$  can be approximated by

$$a(B) = a_{\text{bg}} \left( 1 - \frac{\Delta}{B - B_0} \right), \quad (34)$$

where  $a_{\text{bg}}$  is the background scattering length far from the resonance and  $B_0 + \Delta$  is the position of a zero of the scattering length.

A natural description for atoms near a Feshbach resonance is provided by the resonance model, in which the molecule responsible for the resonance is treated as a point particle. The resonance model has three parameters  $\nu$ ,  $g$ , and  $\lambda$  that can be defined by the  $S$ -wave phase shift given in Eq. (6). The scattering length and the effective range are

$$a = \frac{1}{4\pi} \left( \lambda - \frac{g^2}{m\nu} \right), \quad (35a)$$

$$r_s = -8\pi \left( \lambda - \frac{g^2}{m\nu} \right)^{-2} \frac{g^2}{m^2\nu^2}. \quad (35b)$$

The effective range is negative definite. The scattering length in Eq. (35a) can be made arbitrarily large by tuning  $\nu$  to near 0. In this case, the limiting value of the effective range is  $r_s \rightarrow -8\pi/g^2$ . The standard expression for the scattering length near a Feshbach resonance in Eq. (34) can be reproduced by taking  $\nu$  to be linear in the magnetic field  $B$  while  $g^2$  and  $\lambda$  are constants:

$$\nu = -\mu(B - B_0), \quad (36a)$$

$$g^2 = -4\pi m a_{\text{bg}} \mu \Delta, \quad (36b)$$

$$\lambda = 4\pi a_{\text{bg}}. \quad (36c)$$

The parameter  $\mu$  in Eqs. (36) is the difference between the magnetic moment of the molecule and twice the magnetic moment of an isolated atom. Away from the Feshbach resonance, the parameter  $\nu$  can be interpreted as the detuning energy of the molecular state relative to the two-atom threshold. Near the Feshbach resonance, the strong coupling between the molecular state and two-atom scattering states makes the interpretation of  $\nu$  more complicated.

The resonance model was first formulated as a local quantum field theory by Kaplan [21] and by Kokkelmans *et al.* [22]. In the sector that consists of states containing two atoms or one diatomic molecule, the model can be solved analytically. The solution reveals that the model is renormalizable, at least in this sector. Kaplan used dimensional regularization to remove ultraviolet divergences, thus avoiding the need for explicit renormalization of the parameters [21]. Kokkelmans *et al.* [22] derived the renormalizations of the parameters that are required to make the observables independent of the ultraviolet momentum cutoff. They used numerical solutions of mean-field integral equations for this model to demonstrate the feasibility for achieving superfluidity in ultracold gas of fermionic atoms.

In the quantum field theory formulation of the resonance model for fermionic atoms with two spin states, there are three quantum fields: fermionic fields  $\psi_1$  and  $\psi_2$  that annihilate atoms and a bosonic field  $\phi$  that annihilates a diatomic molecule. The Hamiltonian density for the resonance model is the sum of a kinetic term  $\mathcal{T}$ , an interaction term  $\mathcal{I}$ , and an external potential term  $\mathcal{V}$ :  $\mathcal{H} = \mathcal{T} + \mathcal{I} + \mathcal{V}$ . The individual terms are

$$\mathcal{T} = \frac{1}{2m} \nabla \psi_1^\dagger \cdot \nabla \psi_1^{(\Lambda)} + \frac{1}{2m} \nabla \psi_2^\dagger \cdot \nabla \psi_2^{(\Lambda)} + \frac{1}{4m} \nabla \phi^\dagger \cdot \nabla \phi, \quad (37a)$$

$$\mathcal{I} = \nu_0 \phi^\dagger \phi + \frac{g_0}{m} (\phi^\dagger \psi_2 \psi_1^{(\Lambda)} + \psi_1^\dagger \psi_2^\dagger \phi^{(\Lambda)}) + \frac{\lambda_0}{m} \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1^{(\Lambda)}, \quad (37b)$$

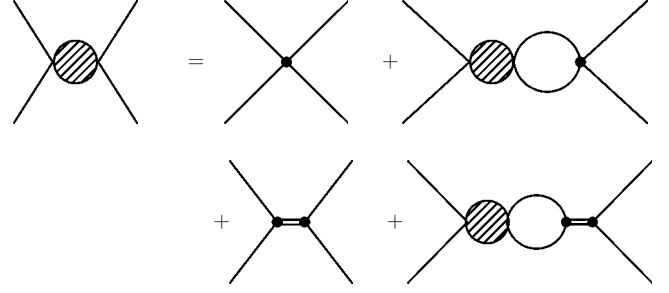


FIG. 1. Integral equation for the scattering of a pair of atoms in the resonance model. Single lines denote atom propagators and double lines denote molecule propagators. The blob represents the amplitude  $iA(E)$ . The solid dots represent the interaction vertices  $-i\lambda_0/m$  and  $-ig_0/m$ .

$$\mathcal{V} = V(\mathbf{R})(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + 2\phi^\dagger \phi). \quad (37c)$$

To avoid ultraviolet divergences, an ultraviolet cutoff  $\Lambda$  must be imposed on the momenta of virtual particles. The superscripts  $(\Lambda)$  on some of the operators in Eqs. (37a) and (37b) indicate that they have matrix elements that are ultraviolet divergent. For the model to have a nontrivial finite limit as  $\Lambda \rightarrow \infty$ , the parameters  $\nu_0$ ,  $g_0$ , and  $\lambda_0$  in the interaction term in Eq. (37b) must depend on  $\Lambda$ . We will refer to them as the *bare parameters*. The parameters  $\nu$ ,  $g$ , and  $\lambda$  in the phase shift in Eq. (6) do not depend on  $\Lambda$ . We will refer to them as the *renormalized parameters*. The values for the bare parameters that are required to reproduce the phase shift in Eq. (6) are

$$\lambda_0(\Lambda) = Z(\Lambda)\lambda, \quad (38a)$$

$$g_0(\Lambda) = Z(\Lambda)g, \quad (38b)$$

$$\nu_0(\Lambda) = \nu - [1 - Z(\Lambda)] \frac{g^2}{m\lambda}, \quad (38c)$$

where the renormalization constant  $Z$  is

$$Z(\Lambda) = \left( 1 - \frac{\lambda\Lambda}{2\pi^2} \right)^{-1}. \quad (39)$$

There are two simple combinations of the parameters that are renormalization invariants.

$$\frac{g}{\lambda} = \frac{g_0}{\lambda_0}, \quad (40a)$$

$$\nu - \frac{g^2}{m\lambda} = \nu_0 - \frac{g_0^2}{m\lambda_0}. \quad (40b)$$

The amplitude for the scattering of a pair of atoms can be calculated by solving the Lippmann-Schwinger integral equation, which is represented diagrammatically in Fig. 1. After evaluating the loop integral using Eq. (A14a), the integral equation reduces to an algebraic equation as follows:

$$\mathcal{A}(E) = -\frac{1}{m} \left( \lambda_0 + \frac{g_0^2/m}{E - \nu_0 + i\epsilon} \right) \times \left[ 1 + m\mathcal{A}(E) \left( \frac{\Lambda}{2\pi^2} - \frac{1}{4\pi} \sqrt{-mE - i\epsilon} \right) \right], \quad (41)$$

where  $\Lambda$  is an ultraviolet momentum cutoff. The solution  $\mathcal{A}(E)$  depends on the total energy  $E$  of the pair of atoms in the center-of-mass frame and not separately on their momenta. After substituting Eqs. (38) for the bare parameters, the amplitude becomes independent of the ultraviolet cutoff:

$$\mathcal{A}(E) = -\frac{1}{m} \left[ \left( \lambda + \frac{g^2/m}{E - \nu + i\epsilon} \right)^{-1} - \frac{1}{4\pi} \sqrt{-mE - i\epsilon} \right]^{-1}. \quad (42)$$

The  $T$ -matrix element for scattering of a pair of atoms with momenta  $+\mathbf{p}$  and  $-\mathbf{p}$  is obtained by setting  $E = p^2/m$ . Comparing with the expression for the  $T$ -matrix element in Eq. (3), we obtain the phase shift in Eq. (6).

Since the resonance model can be formulated as a quantum field theory, the methods used to derive the Tan relations in Ref. [10] can be applied equally well to this model. Because the resonance model is a local quantum field theory, we can use the OPE to identify the contact density operator. Because it is renormalizable, we can use standard renormalization methods to derive the analogs of some of the other Tan relations. In the following seven sections, we deduce the generalized Tan relations for the resonance model.

## V. TAIL OF THE MOMENTUM DISTRIBUTION

The Tan relation in Eq. (8) states that the momentum distributions  $\rho_\sigma(\mathbf{k})$  have power-law tails proportional to  $1/k^4$ , with the coefficient for both spin states given by the contact  $C$ . We will use the OPE to show that the momentum distributions in the resonance model also have power-law tails proportional to  $1/k^4$  and we will express the contact  $C$  as the integral over space of the expectation value of a local operator.

Our starting point is the expression for the momentum distribution  $\rho_\sigma(\mathbf{k})$  in Eq. (27), which involves the expectation value of  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$ . Inserting the OPE in Eq. (28) for this bilocal operator, we obtain the asymptotic expansion of  $\rho_\sigma(\mathbf{k})$  for large  $\mathbf{k}$  in Eq. (29). A power-law tail in  $\rho_\sigma(\mathbf{k})$  can only come from terms in the OPE whose Wilson coefficients  $C_{\sigma,n}(\mathbf{r})$  are not analytic functions of  $\mathbf{r}$  at  $\mathbf{r}=0$ . Such terms can arise from short-distance quantum fluctuations involving pairs of atoms.

The Wilson coefficients on the right-hand side of the general OPE in Eq. (1) can be determined by a procedure called *matching*. First, matrix elements of the bilocal operator  $\mathcal{O}_A(\mathbf{R} - \frac{1}{2}\mathbf{r})\mathcal{O}_B(\mathbf{R} + \frac{1}{2}\mathbf{r})$  between appropriate states are calculated and expanded in powers of  $r=|\mathbf{r}|$ . Second, matrix elements of the local operators  $\mathcal{O}_C(\mathbf{R})$  between the same states are calculated. Finally, the coefficients  $C_{A,B}^C(\mathbf{r})$  are determined by demanding that the expansions in powers of  $r$  on both sides of the equation match. The coefficient of  $\mathcal{O}_C(\mathbf{R})$  can be determined by matching matrix elements between any

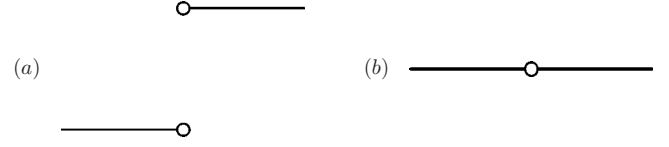


FIG. 2. Diagrammatic representations of (a) the bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$  and (b) local operators that annihilate one atom and create one atom, such as  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$ .

states  $|X\rangle$  and  $|X'\rangle$  for which  $\langle X'|\mathcal{O}_C(\mathbf{R})|X\rangle$  is nonzero. We can exploit the fact that the OPE is an operator equation by choosing the simplest possible states for the matching.

The matrix elements of the operators in the OPE can be evaluated using diagrammatic methods by applying the Feynman rules given in the Appendix. The bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$  can be represented diagrammatically by a pair of open dots with an atom line ending at the dot associated with  $\psi_\sigma^\dagger$  and an atom line beginning at the dot associated with  $\psi_\sigma$ , as in Fig. 2(a). The atom lines should be labeled by the spin  $\sigma$ , but we suppress that label. Local operators that annihilate one atom and create one atom, such as  $\psi_\sigma^\dagger\psi_\sigma$  and  $\psi_\sigma^\dagger\nabla^j\psi_\sigma - \nabla^j\psi_\sigma^\dagger\psi_\sigma$ , can be represented diagrammatically by a single open dot, with an atom line ending at the dot and another atom line beginning at the dot, as in Fig. 2(b). Local operators that annihilate a pair of atoms with spins 1 and 2 or a molecule and create a pair of atoms or a molecule, such as  $\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$ ,  $\psi_1^\dagger\psi_2^\dagger\phi$ ,  $\phi^\dagger\psi_2\psi_1$ , and  $\phi^\dagger\phi$ , can be represented diagrammatically by a single open dot, with two atom lines or a molecule line ending at the dot and two atom lines or a molecule line beginning at the dot, as in Fig. 3.

The Wilson coefficients for operators that annihilate one atom and create one atom can be obtained by matching matrix elements between the one-atom states consisting of a single atom with specified momentum and spin. For the bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$ , the matrix element between one-atom states is given by the diagram in Fig. 2(a). The expression for the matrix element is simply the Feynman rule for the operator, which is an exponential function of  $\mathbf{r}$ . For local operators that annihilate one atom and create one atom, such as  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$ , the matrix element is given by the diagram in Fig. 2(b). Their Wilson coefficients can be determined by matching the matrix elements between the one-atom states. The Wilson coefficients are identical to those that would be obtained by multiplying the Taylor expansions of the operators  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})$  and  $\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$ .

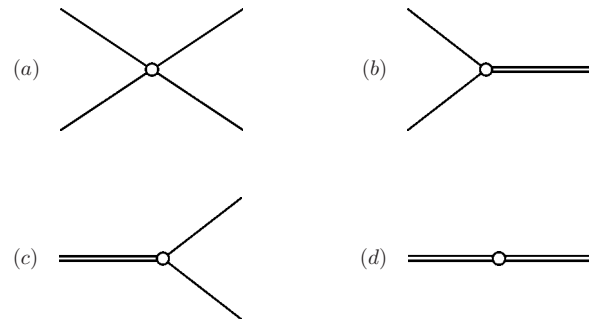


FIG. 3. Diagrammatic representations of the local operators (a)  $\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$ , (b)  $\psi_1^\dagger\psi_2^\dagger\phi$ , (c)  $\phi^\dagger\psi_2\psi_1$ , and (d)  $\phi^\dagger\phi$ .

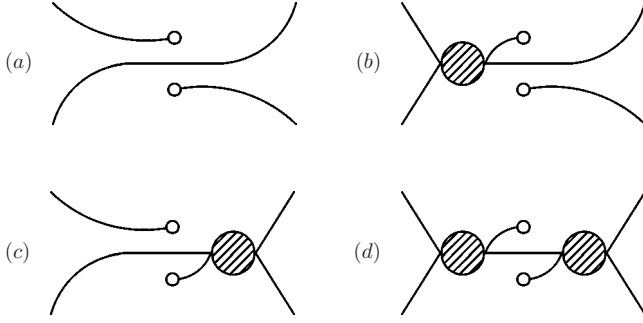


FIG. 4. The diagrams for matrix elements of the bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$  between two-atom scattering states. The open dots represent the operators.

The Wilson coefficients of local operators that annihilate one atom and create one atom, such as  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$ , could also be determined by matching matrix elements between two-atom states, but the calculations are more complicated. The diagrams for the matrix element of the bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$  are shown in Fig. 4. The blobs represent the infinite series of diagrams that are summed up by the integral equation in Fig. 1. The contributions from each of the three diagrams in Figs. 4(a)–4(c) are analytic functions of  $\mathbf{r}$ . The diagram in Fig. 4(d) is evaluated in Sec. 2 a of the Appendix. It is an exponential function of  $r=|\mathbf{r}|$ , so it is not analytic at  $\mathbf{r}=0$ . The diagrams for the matrix element of a local operator that annihilates one atom and creates one atom, such as  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$ , are shown in Fig. 5. Given the Wilson coefficients determined by matching matrix elements between one-atom states, the three diagrams in Figs. 5(a)–5(c) match the expansions in powers of  $\mathbf{r}$  of the corresponding diagrams in Figs. 4(a)–4(c). The diagram in Fig. 5(d) matches the even powers of  $r$  in the expansion of the diagram in Fig. 4(d) in powers of  $r$ . The diagram in Fig. 4(d) is evaluated in Sec. 2 b of the Appendix. However, for operators whose Wilson coefficient is an odd function of  $\mathbf{r}$ , such as  $\psi_\sigma^\dagger\nabla\psi_\sigma - \nabla\psi_\sigma^\dagger\psi_\sigma$ , the diagram in Fig. 4(d) vanishes. The odd powers of  $r$  in the expansion of the diagram in Fig. 4(d), which are not analytic at  $\mathbf{r}=0$ , must be matched by more complicated operators whose matrix elements between one-atom states are zero. The next most complicated operators with nontrivial matrix elements between two-atom scattering

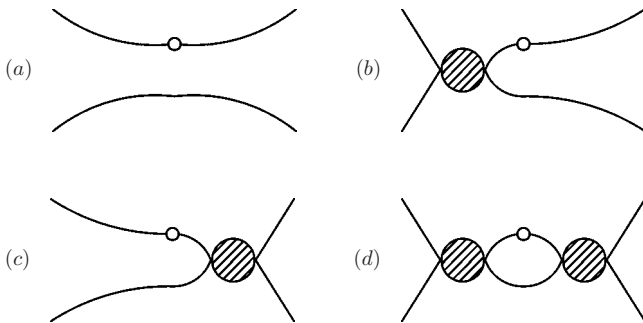


FIG. 5. The diagrams for matrix elements of local operators that annihilate an atom and create an atom, such as  $\psi_\sigma^\dagger\psi_\sigma$  or  $\psi_\sigma^\dagger\nabla\psi_\sigma - \nabla\psi_\sigma^\dagger\psi_\sigma$ , between two-atom scattering states. The open dot represents the operator.

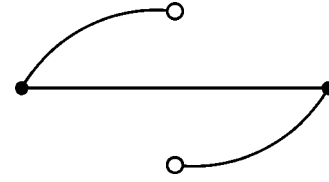


FIG. 6. The lines in the diagram of Fig. 4(d) whose energies and momenta must be integrated over. The solid dots are vertices to which are attached either a pair of atom lines or a molecule line.

states have one factor of  $\psi_2\psi_1$  or  $\phi$  and one factor of  $\psi_1^\dagger\psi_2^\dagger$  or  $\phi^\dagger$ .

To identify the operators on the right-hand side of the OPE whose matrix elements between one-atom states are zero but whose matrix elements between two-atom scattering states are nonzero, we consider the lines in the diagram in Fig. 4(d) whose energies and momenta are integrated over. Those lines are shown in Fig. 6, where the momenta are labeled. The solid dots are vertices whose Feynman rule is  $-i\lambda_0/m$  if a pair of atom lines is attached and  $-ig_0/m$  if a molecule line is attached. Nonanalytic dependence on  $\mathbf{r}$  can arise only from the integration region with large momentum  $\mathbf{q}$ . There can be a significant contribution from this region only if the coordinates associated with the end points of the lines carrying momentum  $\mathbf{q}$  are close together. Contributions from this region therefore correspond to shrinking those lines to a single point. If we include the lines attached to the solid dots in Fig. 6, the shrinking of the large-momentum lines to a point gives one of the four vertices in Fig. 3. These four vertices correspond to the operators  $\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$ ,  $\phi^\dagger\psi_2\psi_1$ ,  $\psi_1^\dagger\psi_2^\dagger\phi$ , and  $\phi^\dagger\phi$ , respectively. They appear with relative weights  $\lambda_0^2$ ,  $\lambda_0g_0$ ,  $\lambda_0g_0$ , and  $g_0^2$ , respectively. It is convenient to introduce a composite operator  $\Phi(\mathbf{R})$  defined by

$$\Phi(\mathbf{R}) = \lambda_0\psi_2\psi_1^{(\Lambda)}(\mathbf{R}) + g_0\phi(\mathbf{R}). \quad (43)$$

The simplest combination of operators that corresponds to the large-momentum region of the diagram in Fig. 4(d) is therefore  $\Phi^\dagger\Phi(\mathbf{R})$ .

The Wilson coefficient of the  $\Phi^\dagger\Phi(\mathbf{R})$  term can be determined by matching matrix elements between two-atom scattering states. The matrix element for  $\Phi^\dagger\Phi(\mathbf{R})$  can be represented by the sum of the four diagrams in Fig. 7, together with 12 other diagrams in which there is no scattering of the two incoming lines or no scattering of the two outgoing lines or both. The sum of all these diagrams is calculated in Sec. 2 c of the Appendix. The dependence on the initial and final states is exactly what is required to match the term linear in  $r$  in the expansion of the diagram in Fig. 4(d). By matching, we determine the Wilson coefficient of the operator  $\Phi^\dagger\Phi$  to be  $-r/(8\pi)$ . Thus the analog of the OPE in Eq. (30) in the resonance model is

$$\begin{aligned} \psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right)\psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) &= \psi_\sigma^\dagger\psi_\sigma(\mathbf{R}) + \frac{1}{2}\mathbf{r} \cdot [\psi_\sigma^\dagger\nabla\psi_\sigma(\mathbf{R}) \\ &\quad - \nabla\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})] - \frac{r}{8\pi}\Phi^\dagger\Phi(\mathbf{R}) \\ &\quad + \cdots \end{aligned} \quad (44)$$



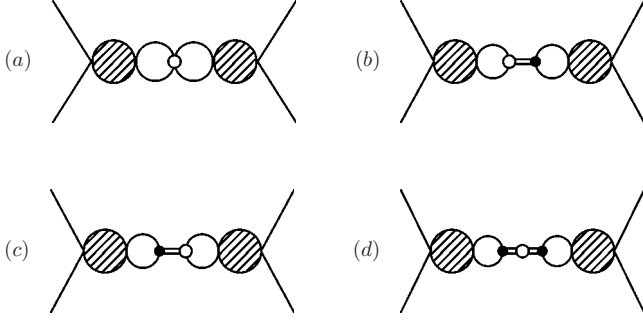


FIG. 7. Diagrams for matrix elements of the local operator  $\Phi^\dagger\Phi$  between two-atom scattering states. The open dot represents the operator. For each of the diagrams shown, there are three other diagrams with no scattering of the atoms in the initial state or in the final state or both.

We can now identify the coefficient  $C$  of the  $1/k^4$  tail in the momentum distribution in Eq. (8). The leading term in the asymptotic expansion of the momentum distribution in Eq. (29) comes from the term with operator  $\Phi^\dagger\Phi$ . The Fourier integral of its Wilson coefficient  $-r/(8\pi)$  is given by Eq. (31). Comparing with Eq. (8), we find that the contact in the resonance model is

$$C = \int d^3R \langle X | \Phi^\dagger\Phi(\mathbf{R}) | X \rangle, \quad (45)$$

where  $\Phi(\mathbf{R})$  is the composite operator defined by Eq. (43). In subsequent sections, we will suppress the state  $|X\rangle$  and denote the expectation value  $\langle X | \mathcal{O} | X \rangle$  of an operator in that state simply by  $\langle \mathcal{O} \rangle$ .

## VI. ENERGY RELATION

Tan's energy relation in Eq. (9) is a decomposition of the energy into terms that are separately insensitive to the range of the interactions between the atoms. In a quantum field theoretic formulation of the problem, the analogous relation is a decomposition of the energy into terms that are insensitive to the ultraviolet cutoff. To accomplish this separation for the resonance model, we use the facts that  $\phi^\dagger\phi$  and  $\Phi^\dagger\Phi$  are operators with ultraviolet-finite matrix elements. For  $\Phi^\dagger\Phi$ , this follows from it being the contact density operator. For  $\phi^\dagger\phi$ , this follows from an adiabatic relation derived later in Sec. VIII. Using the expressions for the bare coupling constants in Eqs. (38), the interaction term  $\mathcal{I}$  in the Hamiltonian density in Eq. (37b) can be expressed in the form

$$\mathcal{I} = \left( \nu - \frac{g^2}{m\lambda} \right) \phi^\dagger\phi + \left( \frac{1}{m\lambda} - \frac{\Lambda}{2\pi^2 m} \right) \Phi^\dagger\Phi. \quad (46)$$

We can now decompose the sum of the kinetic energy density and the interaction energy density into five terms, each of which has ultraviolet-finite matrix elements.

$$\begin{aligned} \mathcal{T} + \mathcal{I} = & \frac{1}{2m} \sum_{\sigma} \left( \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)} - \frac{\Lambda}{2\pi^2} \Phi^\dagger\Phi \right) + \frac{1}{4m} \nabla \phi^{\dagger} \cdot \nabla \phi \\ & + \left( \nu - \frac{g^2}{m\lambda} \right) \phi^{\dagger}\phi + \frac{1}{m\lambda} \Phi^{\dagger}\Phi. \end{aligned} \quad (47)$$

To see that the term that is summed over  $\sigma$  in Eq. (47) has ultraviolet finite matrix elements, we consider the integral over space of its expectation value as follows:

$$\langle T_{\sigma}^{(\text{sub})} \rangle = \frac{1}{2m} \int d^3R \left( \langle \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)} \rangle - \frac{\Lambda}{2\pi^2} \langle \Phi^\dagger\Phi \rangle \right). \quad (48)$$

This differs from the expectation value of the kinetic energy of atoms with spin  $\sigma$  by a subtraction that is linear in the ultraviolet cutoff  $\Lambda$ . By expressing the quantum fields  $\psi$  and  $\psi^\dagger$  in terms of their Fourier transforms and using the expression for the momentum distribution  $\rho_{\sigma}(\mathbf{k})$  in Eq. (26), we can express the subtracted kinetic energy in Eq. (48) in the form

$$\langle T_{\sigma}^{(\text{sub})} \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} \left( \rho_{\sigma}(\mathbf{k}) - \frac{C}{k^4} \right), \quad (49)$$

where  $C$  is the contact given in Eq. (45). We have expressed the factor of  $\Lambda$  in Eq. (48) as an integral over momentum space. Since the tail of the momentum distributions at large  $\mathbf{k}$  has the form in Eq. (8), the subtraction in Eq. (49) makes the integral over  $\mathbf{k}$  convergent in the limit  $\Lambda \rightarrow \infty$ .

The energy relation for the resonance model is obtained by taking the expectation value of  $\mathcal{H} = \mathcal{T} + \mathcal{I} + \mathcal{V}$  and integrating over space. It can be expressed in the form

$$\begin{aligned} E = & \langle V \rangle + \langle T_1^{(\text{sub})} \rangle + \langle T_2^{(\text{sub})} \rangle + \langle T_{\text{mol}} \rangle \\ & + \int d^3R \left[ \left( \nu - \frac{g^2}{m\lambda} \right) \langle \phi^{\dagger}\phi \rangle + \frac{1}{m\lambda} \langle \Phi^{\dagger}\Phi \rangle \right], \end{aligned} \quad (50)$$

where  $\langle V \rangle$  is the energy associated with the external potential,

$$\langle V \rangle = \int d^3R V(\mathbf{R}) \langle \psi_1^{\dagger}\psi_1 + \psi_2^{\dagger}\psi_2 + 2\phi^{\dagger}\phi \rangle, \quad (51)$$

and  $\langle T_{\text{mol}} \rangle$  is the kinetic energy of the molecules.

$$\langle T_{\text{mol}} \rangle = \frac{1}{4m} \int d^3R \langle \nabla \phi^{\dagger} \cdot \nabla \phi \rangle. \quad (52)$$

## VII. LOCAL PAIR DENSITY

The Tan relation in Eq. (10) expresses the contact density  $C$  in terms of the local pair density  $N_{\text{pair}}(\mathbf{R}, s)$ , which is the number of pairs of atoms in a ball of volume  $\frac{4}{3}\pi s^3$  centered at  $\mathbf{R}$ . To obtain the analogous relation in the resonance model, we consider the OPE of the number density operators  $\psi_1^{\dagger}\psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r})$  and  $\psi_2^{\dagger}\psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r})$ . The matrix elements of the product of these operators between two-atom scattering states can be represented diagrammatically by the sum of the four diagrams in Fig. 8. The three diagrams in Figs.

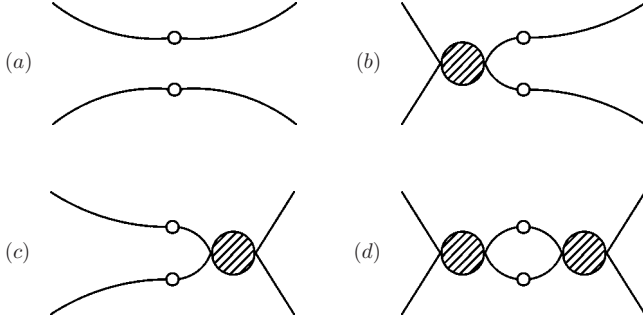


FIG. 8. The diagrams for the matrix element of the bilocal operator  $\psi_1^\dagger \psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r}) \psi_2^\dagger \psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r})$  between two-atom scattering states. The open dots represent the operators.

8(b)–8(d) involve integrals over the four-momenta of atoms. The integrated momenta run through the lines that connect the operator vertices to the blobs. Wilson coefficients that are not analytic at  $\mathbf{r}=0$  must come from the large-momentum regions of those integrals. The corresponding operators can be deduced by shrinking the lines carrying the large momentum to a point. Shrinking those lines in the diagram in Fig. 8(d) produces the operator vertices in Fig. 3 in the combination that corresponds to the operator  $\Phi^\dagger \Phi(\mathbf{R})$ . The Wilson coefficient of this operator is calculated in Sec. 2 d of the Appendix. It is proportional to  $1/r^2$ , so it is singular at  $\mathbf{r}=0$ . For the diagrams in Figs. 8(b) and 8(c), shrinking the lines carrying large momentum produces operator vertices that correspond to the operators  $\psi_1^\dagger \psi_2^\dagger \Phi(\mathbf{R})$  and  $\Phi^\dagger \psi_2 \psi_1(\mathbf{R})$ , respectively. The Wilson coefficient of these operators are proportional to  $1/r$ . Thus the most singular term in the OPE is

$$\psi_1^\dagger \psi_1\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_2^\dagger \psi_2\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) \rightarrow \frac{1}{16\pi^2 r^2} \Phi^\dagger \Phi(\mathbf{R}). \quad (53)$$

We can use the OPE in Eq. (53) to obtain a simple interpretation of the contact density  $\mathcal{C}$ . It can be expressed in the form

$$\psi_1^\dagger \psi_1(\mathbf{R} + \mathbf{r}) \psi_2^\dagger \psi_2(\mathbf{R} + \mathbf{r}') \rightarrow \frac{1}{16\pi^2 |\mathbf{r}' - \mathbf{r}|^2} \Phi^\dagger \Phi(\mathbf{R}), \quad (54)$$

where on the right-hand side we have kept only the most singular term as  $\mathbf{r}' \rightarrow \mathbf{r}$ . If we integrate the left side of Eq. (54) over both  $\mathbf{r}$  and  $\mathbf{r}'$  inside the ball of volume  $\frac{4}{3}\pi s^3$  centered at  $\mathbf{R}$ , we obtain an operator that in the absence of interactions would give the product of the number of atoms with spin 1 and the number of atoms with spin 2 in that ball. Thus the expectation value  $N_{\text{pair}}(\mathbf{R}, s)$  of this operator can be interpreted as the number of pairs of atoms with spins 1 and 2 inside that ball. Integrating the right-hand side of Eq. (54) over  $\mathbf{r}$  and  $\mathbf{r}'$  inside that same ball and taking the expectation value, we obtain the limiting behavior of  $N_{\text{pair}}(\mathbf{R}, s)$  as  $s \rightarrow 0$ .

$$N_{\text{pair}}(\mathbf{R}, s) \rightarrow \frac{s^4}{4} \langle \Phi^\dagger \Phi(\mathbf{R}) \rangle. \quad (55)$$

Since  $\langle \Phi^\dagger \Phi(\mathbf{R}) \rangle$  is the contact density  $\mathcal{C}$ , the Tan relation in Eq. (10) is satisfied without modification in the resonance model.

### VIII. ADIABATIC RELATIONS

Tan's adiabatic relation in Eq. (11) gives the rate of change in the energy  $E$  of a state due to a change in the interaction parameter  $a$ . In the resonance model, there are three interaction parameters:  $\nu$ ,  $g$ , and  $\lambda$ . According to the expressions for these parameters in Eq. (36),  $\nu$  can be changed experimentally by varying the magnetic field while  $g$  and  $\lambda$  are essentially constants. Nevertheless, we will consider the effects of changing all three parameters.

The Feynman-Hellman theorem for variations in the parameter  $\nu$  is

$$\partial E / \partial \nu = \int d^3 R \langle \partial \mathcal{H} / \partial \nu \rangle. \quad (56)$$

There are analogous equations for the variations in the parameters  $g$  and  $\lambda$ . Since the resonance model is renormalizable, the Hamiltonian density operator  $\mathcal{H} = \mathcal{T} + \mathcal{V}$  given by Eqs. (37) has ultraviolet-finite matrix elements. More explicitly, if Eqs. (38) are used to eliminate the bare parameters  $\nu_0$ ,  $g_0$ , and  $\lambda_0$  in favor of the renormalized parameters  $\nu$ ,  $g$ , and  $\lambda$ , the matrix elements of  $\mathcal{H}$  have finite limits as  $\Lambda \rightarrow \infty$ , where  $\Lambda$  is the ultraviolet cutoff. Since  $\mathcal{H}$  has ultraviolet-finite matrix elements for any values of  $\nu$ ,  $g$ , and  $\lambda$ , its derivatives  $(\partial / \partial \nu) \mathcal{H}$ ,  $(\partial / \partial g) \mathcal{H}$ , and  $(\partial / \partial \lambda) \mathcal{H}$  must also have ultraviolet-finite matrix elements. The partial derivatives with the ultraviolet cutoff  $\Lambda$  held fixed can be evaluated by using Eqs. (38) for the bare parameters. The expressions for the partial derivatives are simplest if they are expressed in terms of the field  $\phi$  and the composite operator  $\Phi$  defined in Eq. (43):

$$\nu \frac{\partial}{\partial \nu} \mathcal{H} = \nu \phi^\dagger \phi, \quad (57a)$$

$$g \frac{\partial}{\partial g} \mathcal{H} = -\frac{2g^2}{m\lambda} \phi^\dagger \phi + \frac{g}{m\lambda} (\phi^\dagger \Phi + \Phi^\dagger \phi), \quad (57b)$$

$$\lambda \frac{\partial}{\partial \lambda} \mathcal{H} = \frac{g^2}{m\lambda} \phi^\dagger \phi - \frac{g}{m\lambda} (\phi^\dagger \Phi + \Phi^\dagger \phi) + \frac{1}{m\lambda} \Phi^\dagger \Phi. \quad (57c)$$

The right-hand sides have been expressed as linear combinations of the operators  $\phi^\dagger \phi$ ,  $\phi^\dagger \Phi + \Phi^\dagger \phi$ , and  $\Phi^\dagger \Phi$ , with coefficients that are functions of renormalized parameters. Since the three partial derivatives of  $\mathcal{H}$  in Eqs. (57) must have ultraviolet-finite matrix elements, these three operators must also have ultraviolet-finite matrix elements.

The adiabatic relations for the resonance model can be obtained by taking the expectation values of the partial derivatives in Eqs. (57) and then integrating over all space:

$$\nu \frac{\partial}{\partial \nu} E = \nu \int d^3R \langle \phi^\dagger \phi \rangle, \quad (58a)$$

$$g \frac{\partial}{\partial g} E = \int d^3R \left( -\frac{2g^2}{m\lambda} \langle \phi^\dagger \phi \rangle + \frac{g}{m\lambda} \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle \right), \quad (58b)$$

$$\lambda \frac{\partial}{\partial \lambda} E = \int d^3R \left( \frac{g^2}{m\lambda} \langle \phi^\dagger \phi \rangle - \frac{g}{m\lambda} \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle + \frac{1}{m\lambda} \langle \Phi^\dagger \Phi \rangle \right). \quad (58c)$$

### IX. PRESSURE RELATION

For a homogeneous system, the external potential is a constant:  $V(\mathbf{r}) = -\mu$ , where  $\mu$  is the chemical potential. The thermodynamic properties of the system are determined by the free energy density (or thermodynamic potential density)  $\mathcal{F}$ . It can be expressed in terms of the trace of the partition function as follows:

$$\exp(-\beta \mathcal{F}V) = \text{Tr} \exp\left(-\beta \int d^3R \mathcal{H}\right), \quad (59)$$

where  $\beta = 1/T$  and  $V$  is the volume. The free energy density  $\mathcal{F}$  is a function of  $T$ ,  $\mu$ , and the interaction parameters  $\nu$ ,  $g$ , and  $\lambda$ . The parameter  $\nu$  has dimensions of energy. The mass  $m$ , together with  $\hbar = 1$ , can be used as a conversion constant to obtain energy scales  $1/(m\lambda^2)$  and  $g^4/m$  associated with the two other interaction parameters. The free energy density  $\mathcal{F}$  has dimensions of energy per volume, which can be expressed as  $(\text{energy})^{5/2}$  up to factors of  $m$  and  $\hbar$ . Thus dimensional analysis requires that the free energy density satisfy

$$\left[ T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} + \nu \frac{\partial}{\partial \nu} + \frac{1}{4} g \frac{\partial}{\partial g} - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} \right] \mathcal{F} = \frac{5}{2} \mathcal{F}. \quad (60)$$

The logarithmic derivatives on the left-hand side of Eq. (60) can be determined by applying them to the logarithms of both sides of Eq. (59). In the case of the interaction parameters  $\nu$ ,  $g$ , and  $\lambda$ , the logarithmic derivatives of  $\mathcal{F}$  are simply the expectation values of the logarithmic derivatives of the Hamiltonian density given in Eqs. (57). In the case of the temperature and the chemical potential, the logarithmic derivatives of  $\mathcal{F}$  are

$$T \frac{\partial}{\partial T} \mathcal{F} = \mathcal{F} - \mathcal{E} + \mu n, \quad (61a)$$

$$\mu \frac{\partial}{\partial \mu} \mathcal{F} = -\mu n, \quad (61b)$$

where  $n$  is the expectation value of the total number density.

$$n = \langle \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + 2\phi^\dagger \phi \rangle. \quad (62)$$

Because the Hamiltonian density  $\mathcal{H}$  includes the term  $\mathcal{V}$  in Eq. (37c), it is actually the grand Hamiltonian density for the grand canonical ensemble. Its expectation value is  $\langle \mathcal{H} \rangle = \mathcal{E}$

$-\mu n$ , where  $\mathcal{E}$  is the energy density. Because the system is homogeneous, the pressure is simply  $\mathcal{P} = -\mathcal{F}$ . Combining together all the terms in Eq. (60), we obtain

$$\begin{aligned} \mathcal{E} = & \frac{3}{2} \mathcal{P} + \left( \nu - \frac{g^2}{m\lambda} \right) \langle \phi^\dagger \phi \rangle \\ & + \frac{3g}{4m\lambda} \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle - \frac{1}{2m\lambda} \langle \Phi^\dagger \Phi \rangle. \end{aligned} \quad (63)$$

This is the *pressure relation* for the resonance model.

### X. VIRIAL THEOREM

If the trapping potential  $V(\mathbf{R})$  has simple behavior under scaling the coordinates  $\mathbf{R} = (X, Y, Z)$ , the virial theorem gives simple relations between various contributions to the energy. The most important case for cold atoms is a harmonic trapping potential. For simplicity, we consider only the case of an isotropic harmonic potential as follows:

$$V(\mathbf{R}) = \frac{1}{2} m \omega^2 \mathbf{R}^2. \quad (64)$$

The generalization to the case of an anisotropic harmonic potential is straightforward.

The virial theorem can be derived simply from dimensional analysis. Let us assume that the state of the system can be specified by a set of integer quantum numbers. The energy  $E = \int d^3r \langle \mathcal{H} \rangle$  for any such state must be a function of the frequency  $\omega$  and the interaction parameters  $\nu$ ,  $g$ , and  $\lambda$ . Since  $E$  has dimensions of energy, dimensional analysis requires that it satisfy

$$\left[ \omega \frac{\partial}{\partial \omega} + \nu \frac{\partial}{\partial \nu} + \frac{1}{4} g \frac{\partial}{\partial g} - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} \right] E = E. \quad (65)$$

If we consider the system at a nonzero temperature  $T$ , the differential operator in Eq. (65) would also include a term  $T(\partial/\partial T)$ .

The action of the differential operator in Eq. (65) on the Hamiltonian density can be obtained by inserting the partial derivatives in Eqs. (57) as follows:

$$\begin{aligned} & \left[ \omega \frac{\partial}{\partial \omega} + \nu \frac{\partial}{\partial \nu} + \frac{1}{4} g \frac{\partial}{\partial g} - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} \right] \mathcal{H} \\ &= 2V(\mathbf{R}) (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + 2\phi^\dagger \phi) + \left( \nu - \frac{g^2}{m\lambda} \right) \phi^\dagger \phi \\ &+ \frac{3g}{4m\lambda} (\phi^\dagger \Phi + \Phi^\dagger \phi) - \frac{1}{2m\lambda} \Phi^\dagger \Phi. \end{aligned} \quad (66)$$

The virial theorem for the resonance model is obtained by taking the expectation value, integrating over space, and setting this equal to  $E$  as follows:

$$E = 2\langle V \rangle + \int d^3R \left[ \left( \nu - \frac{g^2}{m\lambda} \right) \langle \phi^\dagger \phi \rangle + \frac{3g}{4m\lambda} \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle - \frac{1}{2m\lambda} \langle \Phi^\dagger \Phi \rangle \right], \quad (67)$$

where  $\langle V \rangle$  is the energy from the external potential given by Eq. (51).

## XI. INELASTIC TWO-BODY LOSS RATE

If the atoms have other spin states with lower energy than the spin states  $\sigma=1, 2$  of interest, a pair of atoms with spins 1 and 2 may be able to scatter inelastically into the lower spin states. The molecular state responsible for the Feshbach resonance may also be able to decay into pairs of atoms in lower spin states. If the lower spin states interact weakly with the spin states 1 and 2, the primary effects of transitions to the lower spin states is the disappearance of atoms from the spin states 1 and 2 and the disappearance of the Feshbach molecule. The inclusive effects of the lower spin states on states near the threshold for the spin states 1 and 2 can be taken into account through anti-Hermitian terms in the Hamiltonian density. These terms must be anti-Hermitian to account for the loss of probability from transitions to the lower spin states, which are not taken into account as explicit degrees of freedom.

If the energy gap between the lower spin states and the spin states 1 and 2 is much larger than the energies of interest, the anti-Hermitian terms in the Hamiltonian density can be chosen to be local operators. The rate of disappearance of molecules through their decays into pairs of atoms in lower spin states is proportional to the number density of the molecules. The resulting effects can be taken into account through the operator  $\phi^\dagger \phi$  with a negative imaginary coefficient. The rate of disappearance of pairs of atoms through inelastic scattering is proportional to the local density of pairs of atoms. The resulting effects can be taken into account through the operator  $\psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$  with a negative imaginary coefficient. If the decay of the molecule and the inelastic scattering of a pair of atoms can produce the same final states, the resulting interference effects can be taken into account through the operators  $\phi^\dagger \psi_2 \psi_1$  and  $\psi_1^\dagger \psi_2^\dagger \phi$ . The large energy gap  $E_{\text{gap}}$  is essential for allowing the effects of transitions into the lower spin states to be taken into account through local operators. The atoms created by the decay of a molecule or by the inelastic scattering of atoms with spins 1 and 2 will emerge with large momentum given approximately by  $(mE_{\text{gap}})^{1/2}$ . By the uncertainty principle, a localized wave packet corresponding to such an atom can be traced back to a region of origin whose size is approximately  $\hbar/(mE_{\text{gap}})^{1/2}$ . But the regions of origin of the two atoms must also be separated by a distance at most of order  $\hbar/(mE_{\text{gap}})^{1/2}$ . The reason for this is that the interactions that change the spin state of the atom from 1 or 2 to a lower spin state must also deliver a momentum kick of order  $(mE_{\text{gap}})^{1/2}$ . This large momentum can be transferred between the two final state atoms only if they are close together. Even if it is transferred through an intermediate atom with spin 1 or 2, that atom must

be far off its energy shell and therefore can propagate only over a short distance. As long as  $\hbar/(mE_{\text{gap}})^{1/2}$  is small compared to the length scales that are described explicitly by the field theory, the effects of the transitions to lower spin states can be taken into account systematically through local operators.

The operators  $\psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$ ,  $\phi^\dagger \psi_2 \psi_1 + \psi_1^\dagger \psi_2^\dagger \phi$ , and  $\phi^\dagger \phi$  required to take into account the leading effects of the lower spin states are ones that appear already in the interaction term in the Hamiltonian density given in Eq. (37b). To give anti-Hermitian terms in the Hamiltonian density, these operators must be multiplied by imaginary coefficients. Thus the leading effects of the lower spin states can be taken into account by adding negative imaginary parts to the bare parameters  $\nu_0$ ,  $g_0$ , and  $\lambda_0$ . We now consider the effect of the non-Hermitian terms in the Hamiltonian. The expectation value of those terms can be expressed as  $-i\Gamma/2$ , where

$$\Gamma \approx -2 \int d^3R \left[ \text{Im } \nu_0 \langle \phi^\dagger \phi \rangle + \frac{\text{Im } g_0}{m} \langle \phi^\dagger \psi_2 \psi_1^{(\Lambda)} + \psi_1^\dagger \psi_2^\dagger \phi^{(\Lambda)} \rangle + \frac{\text{Im } \lambda_0}{m} \langle \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1^{(\Lambda)} \rangle \right]. \quad (68)$$

The leading effect of these terms on a state with definite energy  $E$  is to change its time dependence from  $\exp(-iEt/\hbar)$  to  $\exp[-i(E - i\Gamma/2)t/\hbar]$ . The probability in such a state decreases with time at the rate  $\Gamma/\hbar$ . Thus  $\Gamma$  is the rate at which atoms with spins 1 and 2 and Feshbach molecules are depleted by transitions to pairs of atoms in lower spin states.

The loss rate in Eq. (68) can be expressed in terms of expectation values of operators with ultraviolet-finite matrix elements by eliminating  $\psi_2 \psi_1$  in favor of the operator  $\Phi$  defined in Eq. (43). Since the coefficients of these expectation values must also be ultraviolet finite, they are expressible in terms of the renormalized parameters  $\nu$ ,  $g$ , and  $\lambda$ . Using Eqs. (38) to eliminate the bare parameters and expanding to first order in the imaginary parts of the renormalized parameters, we obtain

$$\Gamma \approx -2 \int d^3R \left[ \left( \text{Im } \nu - \frac{2g}{m\lambda} \text{Im } g + \frac{g^2}{m\lambda^2} \text{Im } \lambda \right) \langle \phi^\dagger \phi \rangle + \left( \frac{1}{m\lambda} \text{Im } g - \frac{g}{m\lambda^2} \text{Im } \lambda \right) \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle + \frac{1}{m\lambda^2} \text{Im } \lambda \langle \Phi^\dagger \Phi \rangle \right]. \quad (69)$$

This is the generalization in the resonance model of the Tan relation for the *inelastic two-body loss rate*.

## XII. CONSTRAINTS FOR LARGE SCATTERING LENGTH

The Tan relations in Eqs. (8)–(14) express various properties of an arbitrary state consisting of atoms with two spin states and a large scattering length in terms of the contact  $C$  or the contact density  $\mathcal{C}$ . These relations hold exactly in the zero-range model. The contact density operator in this model was identified in Ref. [10] to be  $\lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$ . In the resonance model, the generalizations of the Tan relations derived



in Secs. V–XI involve the expectation values of three distinct local operators:  $\Phi^\dagger\Phi$ ,  $\phi^\dagger\Phi + \Phi^\dagger\phi$ , and  $\phi^\dagger\phi$ , where  $\Phi$  is the composite operator defined in Eq. (43). The contact density operator is identified in Eq. (45) to be  $\Phi^\dagger\Phi$ . The same operator appears in the energy relation in Eq. (50) and in the local pair density in Eq. (55). The three adiabatic relations in Eqs. (58), the virial theorem in Eq. (67), the pressure relation in Eq. (63), and the two-body inelastic loss rate in Eq. (69) involve various linear combinations of the expectation values of the three local operators.

The scattering length is the only interaction parameter in the zero-range model. The three parameters of the resonance model provide three distinct length scales:  $|m\nu|^{-1/2}$ ,  $g^{-2}$ , and  $|\lambda|$ . There are two limits in which the scattering length given by Eq. (35a) becomes arbitrarily large compared to the other two length scales. In particular, it becomes large compared to the effective range given by Eq. (35b). One of these limits is  $\lambda \rightarrow \pm\infty$  with  $\nu$  and  $g$  fixed, in which case  $a \rightarrow \lambda/(4\pi)$ . The other limit is  $\nu \rightarrow 0$  with  $g$  and  $\lambda$  fixed, in which case  $a \rightarrow -g^2/(4\pi m\nu)$ . In either of these two limits, the universality of systems with large scattering length requires the resonance model to reduce to the zero-range model. The generalized Tan relations must therefore reduce to the original Tan relations. We will find that this imposes constraints on matrix elements of  $\phi^\dagger\phi$  and  $\phi^\dagger\Phi + \Phi^\dagger\phi$ .

#### A. $\lambda \rightarrow \pm\infty$ with $\nu$ and $g$ fixed

We first consider the case of a large scattering length  $a$  obtained by increasing  $|\lambda|$  with  $\nu$  and  $g$  fixed. The adiabatic relation in Eq. (58c) can be expressed as an adiabatic relation for variations in  $a$  as follows:

$$\frac{d}{da}E = \frac{4\pi}{m\lambda^2} \int d^3R (g^2\langle\phi^\dagger\phi\rangle - g\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle + \langle\Phi^\dagger\Phi\rangle). \quad (70)$$

When  $\lambda$  is sufficiently large, this must reduce to Tan's adiabatic relation in Eq. (11), which can be expressed as

$$\frac{d}{da}E = \frac{4\pi}{m\lambda^2} \int d^3R \langle\Phi^\dagger\Phi\rangle + O(1/\lambda^3). \quad (71)$$

This implies a constraint on the matrix elements in Eq. (70) as  $\lambda \rightarrow \pm\infty$ .

$$g^2\langle\phi^\dagger\phi\rangle - g\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle = O(1/\lambda). \quad (72)$$

We can obtain an independent constraint on the matrix elements by using the virial theorem, which for the resonance model is given in Eq. (67). When  $\lambda$  is sufficiently large, this must reduce to Tan's virial theorem in Eq. (13), which can be expressed as

$$E = 2\langle V \rangle - \frac{1}{2m\lambda} \int d^3R \langle\Phi^\dagger\Phi\rangle + O(1/\lambda^2). \quad (73)$$

This implies a constraint on the matrix elements in Eq. (67) as  $\lambda \rightarrow \pm\infty$ :

$$\left(\nu - \frac{g^2}{m\lambda}\right)\langle\phi^\dagger\phi\rangle + \frac{3g}{4m\lambda}\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle = O(1/\lambda^2). \quad (74)$$

If  $\nu \neq 0$ , the constraints in Eqs. (72) and (74) imply

$$\langle\phi^\dagger\phi\rangle = O(1/\lambda^2), \quad (75a)$$

$$\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle = O(1/\lambda). \quad (75b)$$

These constraints are consistent with the decoupling of the molecular field  $\phi$  in the limit  $\lambda \rightarrow \pm\infty$ . Matrix elements of operators involving the field  $\phi$  are suppressed by a factor of  $1/\lambda$  for every factor of  $\phi$  in the operator.

#### B. $\nu \rightarrow 0$ with $g$ and $\lambda$ fixed

We next consider the case of a large scattering length  $a$  obtained by decreasing  $|\nu|$  with  $g$  and  $\lambda$  fixed. According to Eq. (36a), this can be achieved by tuning the magnetic field to the position  $B_0$  of the Feshbach resonance. The adiabatic relation in Eq. (58a) can be expressed as an adiabatic relation for variations in  $a$  as follows:

$$\frac{d}{da}E = \frac{4\pi m\nu^2}{g^2} \int d^3R \langle\phi^\dagger\phi\rangle. \quad (76)$$

When  $\nu$  is sufficiently small, this must reduce to Tan's adiabatic relation in Eq. (11), which can be expressed as

$$\frac{d}{da}E = \frac{4\pi m\nu^2}{g^4} \int d^3R \langle\Phi^\dagger\Phi\rangle + O(\nu^3). \quad (77)$$

This implies that in the limit  $\nu \rightarrow 0$ , the expectation value of  $\phi^\dagger\phi$  can be expressed in terms of the contact density as follows:

$$g^2\langle\phi^\dagger\phi\rangle = \langle\Phi^\dagger\Phi\rangle + O(\nu). \quad (78)$$

We can obtain an independent constraint on the matrix elements by using the virial theorem, which for the resonance model is given in Eq. (67). When  $\nu$  is sufficiently small, this must reduce to Tan's virial theorem in Eq. (13), which can be expressed as

$$E = 2\langle V \rangle + \frac{\nu}{2g^2} \int d^3R \langle\Phi^\dagger\Phi\rangle + O(\nu^2). \quad (79)$$

This implies a constraint on the matrix elements in Eq. (67) as  $\nu \rightarrow 0$ :

$$\begin{aligned} &\left(\nu - \frac{g^2}{m\lambda}\right)\langle\phi^\dagger\phi\rangle + \frac{3g}{4m\lambda}\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle - \frac{1}{2m\lambda}\langle\Phi^\dagger\Phi\rangle \\ &= \frac{\nu}{2g^2}\langle\Phi^\dagger\Phi\rangle + O(\nu^2). \end{aligned} \quad (80)$$

Combined with the constraint in Eq. (78), this implies that the expectation value of  $\phi^\dagger\Phi + \Phi^\dagger\phi$  can also be expressed in terms of the contact density as follows:

$$g\langle\phi^\dagger\Phi + \Phi^\dagger\phi\rangle = 2\langle\Phi^\dagger\Phi\rangle + O(\nu). \quad (81)$$

It also implies a constraint on the terms of order  $\nu$  on the right-hand sides of Eqs. (78) and (81):

$$\begin{aligned}
& (g^2 \langle \phi^\dagger \phi \rangle - \langle \Phi^\dagger \Phi \rangle) - \frac{3}{4} (g \langle \phi^\dagger \Phi + \Phi^\dagger \phi \rangle - 2 \langle \Phi^\dagger \Phi \rangle) \\
&= \frac{\lambda m \nu}{2g^2} \langle \Phi^\dagger \Phi \rangle + O(\nu^2).
\end{aligned} \tag{82}$$

### XIII. EFFECTIVE RANGE MODEL

If we set  $\lambda=0$  in the  $S$ -wave phase shift for the resonance model, which is given by Eq. (6), it reduces to

$$k \cot \delta_0(k) = \frac{4\pi m \nu}{g^2} - \frac{4\pi}{g^2} k^2. \tag{83}$$

This is the phase shift of the effective range model in Eq. (7), with the scattering length and the effective range given by

$$a = -\frac{g^2}{4\pi m \nu}, \tag{84a}$$

$$r_s = -\frac{8\pi}{g^2}. \tag{84b}$$

Note that the effective range is negative.

In the simplest quantum field theory formulation of the effective range model [20], there are three quantum fields: fermionic fields  $\psi_1$  and  $\psi_2$  that annihilate atoms and a bosonic field  $\phi$  that annihilates a diatomic molecule. The Hamiltonian density  $\mathcal{H}=\mathcal{T}+\mathcal{I}+\mathcal{V}$  for the effective range model is the sum of the kinetic term in Eq. (37a), the external potential term in Eq. (37c), and the interaction term

$$\mathcal{I} = \nu_0 \phi^\dagger \phi + \frac{g}{m} (\phi^\dagger \psi_2 \psi_1^{(\Lambda)} + \psi_1^\dagger \psi_2^\dagger \phi^{(\Lambda)}). \tag{85}$$

To avoid ultraviolet divergences, an ultraviolet cutoff  $\Lambda$  must be imposed on the momenta of virtual particles. The only renormalization required in this model is an additive renormalization of the parameter  $\nu_0$  as follows:

$$\nu_0(\Lambda) = \nu + \frac{g^2 \Lambda}{2\pi^2 m}. \tag{86}$$

The effective range model can also be formulated as a local quantum field theory with fermion fields  $\psi_1$  and  $\psi_2$  only and with a momentum-dependent interaction [20]. In this formulation of the model, a rather intricate renormalization of the parameters is required to obtain the phase shift in Eq. (83) in the limit  $\Lambda \rightarrow \infty$ .

The generalized Tan relations for the effective range model can be derived by using the same methods that we used for the resonance model. However, they can be derived more easily by starting from the generalized Tan relations for the resonance model and taking the limit  $\lambda \rightarrow 0$ . This limit is not trivial, because some of the generalized Tan relations for the resonance model involve factors of  $1/\lambda$ . The limit  $\lambda \rightarrow 0$  can be taken by expanding the composite operator  $\Phi$  defined in Eq. (43) through first order in  $\lambda$ . Using the expressions for the bare coupling constants in Eqs. (38a) and (38b) to expand  $\lambda_0$  and  $g_0$  in powers of  $\lambda$ , we obtain

$$\Phi(\mathbf{R}) = g\phi(\mathbf{R}) + \lambda \left[ \psi_2 \psi_1^{(\Lambda)}(\mathbf{R}) + \frac{g\Lambda}{2\pi^2} \phi(\mathbf{R}) \right] + O(\lambda^2). \tag{87}$$

We denote the composite operator defined by the coefficient of  $\lambda$  by  $\Phi'(\mathbf{R})$ :

$$\Phi'(\mathbf{R}) = \psi_2 \psi_1^{(\Lambda)}(\mathbf{R}) + \frac{g\Lambda}{2\pi^2} \phi(\mathbf{R}). \tag{88}$$

The contact  $C$  in the resonance model is given in Eq. (45). From the expansion in Eq. (87), we can see that the limit  $\lambda \rightarrow 0$  can be taken simply by replacing  $\Phi$  by  $g\phi$ . Thus the contact in the effective range model is

$$C = \int d^3R g^2 \langle X | \phi^\dagger \phi(\mathbf{R}) | X \rangle. \tag{89}$$

We proceed to enumerate the generalized Tan relations for the effective range model as follows:

(1) *Tail of the momentum distribution.* This is given by Eq. (8), where  $C$  is the contact in Eq. (89).

(2) *Energy relation.* The decomposition of the energy into terms that are ultraviolet finite can be obtained by taking the limit  $\lambda \rightarrow 0$  in Eq. (50) as follows:

$$\begin{aligned}
E &= \langle V \rangle + \langle T_1^{(\text{sub})} \rangle + \langle T_2^{(\text{sub})} \rangle + \langle T_{\text{mol}} \rangle \\
&+ \int d^3R \left( \nu \langle \phi^\dagger \phi \rangle + \frac{g}{m} \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle \right),
\end{aligned} \tag{90}$$

where  $\Phi'$  is the composite operator defined by Eq. (88). The subtracted kinetic energy for a single spin state is obtained from Eq. (49) by replacing  $\Phi$  by  $g\phi$  as follows:

$$\langle T_\sigma^{(\text{sub})} \rangle = \frac{1}{2m} \int d^3R \left( \langle \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)} \rangle - \frac{g^2 \Lambda}{2\pi^2} \langle \phi^\dagger \phi \rangle \right). \tag{91}$$

(3) *Local pair density.* The contact density  $C=g^2 \langle \phi^\dagger \phi \rangle$  is related to the local pair density  $N_{\text{pair}}(\mathbf{R}, s)$  by Eq. (10).

(4) *Adiabatic relation.* The adiabatic relation for variations in  $\nu$  is the same as in the resonance model, where it is given by Eq. (58a). The adiabatic relation for variations in  $g$  can be obtained from Eq. (58b) by taking the limit  $\lambda \rightarrow 0$ . The adiabatic relations are therefore

$$\nu \frac{\partial}{\partial \nu} E = \nu \int d^3R \langle \phi^\dagger \phi \rangle, \tag{92a}$$

$$g \frac{\partial}{\partial g} E = \frac{g}{m} \int d^3R \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle. \tag{92b}$$

(5) *Pressure relation.* The relation between the energy density  $\mathcal{E}$  and the pressure  $\mathcal{P}$  in a homogeneous system can be obtained from Eq. (63) by taking the limit  $\lambda \rightarrow 0$ :

$$\mathcal{E} = \frac{3}{2} \mathcal{P} + \nu \langle \phi^\dagger \phi \rangle + \frac{g}{4m} \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle. \tag{93}$$

(6) *Virial theorem.* The virial theorem for a system with a

harmonic trapping potential can be obtained from Eq. (67) by taking the limit  $\lambda \rightarrow 0$ :

$$E = 2\langle V \rangle + \int d^3R \left( \nu \langle \phi^\dagger \phi \rangle + \frac{g}{4m} \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle \right). \quad (94)$$

The virial theorem for the effective range model has been derived previously by Werner [26]. His result can be expressed in the form

$$E = 2\langle V \rangle - \frac{a}{2} \frac{\partial}{\partial a} E - \frac{r_s}{2} \frac{\partial}{\partial r_s} E. \quad (95)$$

This is just the generalization of the dimensional analysis condition for the energy in Eq. (25) for a model with two interaction parameters  $a$  and  $r_s$  with dimensions of length. If we use Eqs. (84) to change variables from  $a$  and  $r_s$  to  $\nu$  and  $g$  and if we use the adiabatic relations in Eqs. (92) to express the derivatives in terms of matrix elements, we obtain the virial theorem in Eq. (94).

(7) *Inelastic two-body loss rate.* The rate at which low-energy atoms and molecules are depleted by transitions to pairs of atoms in lower spin states can be obtained from Eq. (69) by setting  $\text{Im } \lambda = 0$  and then taking the limit  $\lambda \rightarrow 0$ . It can be obtained more simply by using the adiabatic relations in Eqs. (92) to determine the change in the energy due to small imaginary changes in the parameters  $\nu$  and  $g$  as follows:

$$\Gamma \approx -2 \int d^3R \left( \text{Im } \nu \langle \phi^\dagger \phi \rangle + \frac{\text{Im } g}{m} \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle \right). \quad (96)$$

The scattering length in Eq. (84a) can be made arbitrarily large by taking the limit  $\nu \rightarrow 0$  with  $g$  fixed. In this limit, the universality of systems with large scattering length requires the effective range model to reduce to the zero-range model. This imposes a constraint on the expectation values of the two operators that appear in the generalized Tan relations. The constraint can be derived by demanding that the virial theorem in Eq. (94) reduce to the Tan relation in Eq. (13) as  $\nu \rightarrow 0$ . It can also be derived from the constraint for the resonance model in Eq. (80) by taking the limit  $\lambda \rightarrow 0$ . They both give the same constraint as follows:

$$g \langle \phi^\dagger \Phi' + \Phi'^\dagger \phi \rangle = -2m\nu \langle \phi^\dagger \phi \rangle + O(\nu^2). \quad (97)$$

#### XIV. SUMMARY

The Tan relations enumerated in Sec. II are universal relations between various properties of an arbitrary system consisting of fermions with two spin states and a large scattering length. These relations should be satisfied provided the energy, temperature, and number density of the system are low enough that the interactions between the atoms are accurately described by the  $S$ -wave phase shift in Eq. (5), which depends only on the scattering length  $a$ . They are satisfied exactly in the zero-range model, in which the phase shift is given by Eq. (5) up to arbitrarily high energies.

In Ref. [10], quantum field theory methods were used to derive the Tan relations. The zero-range model can be for-

mulated as a local quantum field theory. Using this formulation, the Tan relations were derived by using standard renormalization methods together with the OPE. The contact density operator in this model was identified in Ref. [10] to be  $\lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$ . One advantage of using quantum field theory methods is that it is straightforward to derive the generalizations of the Tan relations for any system that can be formulated as a renormalizable local quantum field theory. In this paper, we used quantum field theory methods to derive the generalized Tan relations for the resonance model. We also derived them for the effective range model by using the fact that it can be obtained as a limit of the resonance model.

The resonance model is defined by the  $S$ -wave phase shift in Eq. (6), which depends on three interaction parameters  $\nu$ ,  $g$ , and  $\lambda$ . In Secs. V–XI, we derived the generalized Tan relations for the resonance model. We identified the contact density operator to be  $\Phi^\dagger \Phi$ , where  $\Phi$  is the composite operator defined in Eq. (43). The generalized Tan relations involve expectation values of linear combinations of three local operators:  $\Phi^\dagger \Phi$ ,  $\phi^\dagger \Phi + \Phi^\dagger \phi$ , and  $\phi^\dagger \phi$ . The scattering length in the resonance model can be made arbitrarily large by tuning  $\nu \rightarrow 0$ . The condition that the generalized Tan relations reduce in this limit to the original Tan relations implies the constraints on the expectation values of  $\Phi^\dagger \Phi$ ,  $\phi^\dagger \Phi + \Phi^\dagger \phi$ , and  $\phi^\dagger \phi$  that are given in Eqs. (78), (81), and (82).

The effective range model is defined by the  $S$ -wave phase shift in Eq. (7), which depends on the two parameters  $a$  and  $r_s < 0$ . It can be obtained from the resonance model by setting  $\lambda = 0$ , in which case the  $S$ -wave phase shift is given by Eq. (83) in terms of the parameters  $\nu$  and  $g$ . In Sec. XIII, we enumerated the generalized Tan relations for the effective range model. We identified the contact density operator to be  $g^2 \phi^\dagger \phi$ . The generalized Tan relations involve expectation values of linear combinations of two local operators:  $\phi^\dagger \phi$  and  $\phi^\dagger \Phi' + \Phi'^\dagger \phi$ , where  $\Phi'$  is the composite operator defined in Eq. (88). The scattering length in the effective Range Model can be made arbitrarily large by tuning  $\nu \rightarrow 0$ , which implies  $|a| \gg |r_s|$ . The condition that the generalized Tan relations reduce in this limit to the original Tan relations implies the constraint on the expectation values of  $\phi^\dagger \phi$  and  $\phi^\dagger \Phi' + \Phi'^\dagger \phi$  that is given in Eq. (97).

The Tan relations can be tested experimentally by using cold trapped atoms near a Feshbach resonance, as discussed at the end of Sec. II. Measurements of the momentum distributions can be used to determine the contact  $C$  and the sum  $\langle T \rangle + \langle I \rangle$  of the kinetic and interaction energies. Measurements of the density profiles can be used to determine the potential energy  $\langle V \rangle$ . The virial theorem in Eq. (13) is a nontrivial relation between  $\langle T \rangle + \langle I \rangle$ ,  $\langle V \rangle$ , and  $C$ . The adiabatic relation in Eq. (11) gives a constraint on the variations in  $\langle T \rangle + \langle I \rangle$ ,  $\langle V \rangle$ , and  $C$  from changing the scattering length. Both of these relations can be tested experimentally. Since the ability to change the scattering length is essential both for determining  $C$  and  $\langle T \rangle + \langle I \rangle$  and for testing the adiabatic relation, the Feshbach resonance plays a crucial role. The resonance model provides a more detailed microscopic model for atoms near a Feshbach resonance than the zero-range model. Our results on the generalized Tan relations in the resonance model should be useful for quantifying the theo-

retical errors in experimental tests of the Tan relations.

### ACKNOWLEDGMENTS

This research was supported in part by the Department of Energy under Grants No. DE-FG02-05ER15715 and No. DE-FC02-07ER41457 and by the National Science Foundation under Grant No. PHY-0653312.

### APPENDIX: DIAGRAMMATIC CALCULATIONS

#### 1. Feynman rules

The results for the matrix elements given in Secs. V and VII can be calculated relatively easily by applying a set of rules—known generally in quantum field theory as Feynman rules. The calculation of matrix elements of operators reduces to drawing all relevant Feynman diagrams, using the Feynman rules to write down mathematical expressions for the matrix elements, and evaluating any integrals that appear in those expressions.

##### a. Propagators and vertices

We first give the Feynman rules for the resonance model that are required to calculate Green's functions and  $T$ -matrix elements. A Feynman diagram consists of atom lines (represented by single lines) and molecule lines (represented by double lines) connected by vertices of two types. The mathematical expression for a diagram is obtained by applying the following rules:

(1) *Four-momenta*. Assign a four momentum  $(p_0, \mathbf{p})$ , where  $p_0$  is the energy and  $\mathbf{p}$  is the momentum, to every external line. The initial and final four momenta are constrained by overall conservation of energy and momentum. The four momentum of the internal lines are constrained only by conservation of energy and momentum at each vertex.

(2) *Propagator factors*. For each internal line with four momentum  $(q_0, \mathbf{q})$ , include the factor  $i/[q_0 - q^2/(2m) + i\epsilon]$  for an atom line and  $i/[q_0 - \nu - q^2/(4m) + i\epsilon]$  for a molecule line.

(3) *Vertex factors*. For each two-atom-to-two-atom vertex, include the factor  $-i\lambda_0/m$ . For each two-atom-to-molecule vertex, include the factor  $-ig_0/m$ .

(4) *Loop momenta*. If there are four momenta  $(k_0, \mathbf{k})$  that are not determined by the four momenta of the external lines, integrate over them using the measure  $\int d^3k dk_0/(2\pi)^4$ . The integrals over  $k_0$  can be evaluated using contour integration.

The connected Green's function for  $N$  atoms to evolve into  $N$  atoms can be expressed as the sum of all connected diagrams with  $N$  incoming atom lines and  $N$  outgoing atom lines. An example is the amplitude  $\mathcal{A}(E)$  for  $N=2$  in Eq. (42), which can be expressed as the infinite sum of diagrams generated by iterating the integral equation in Fig. 1. If this amplitude occurs as a subdiagram of a Feynman diagram, the Feynman rule for the subdiagram is  $i\mathcal{A}[E - P^2/(4m)]$ , where  $E$  and  $\mathbf{P}$  are the total energy and momentum flowing through the subdiagram.

The  $T$ -matrix element for  $N$  atoms to scatter into  $N$  atoms is obtained by setting the energy for each external atom line

equal to the value required by the nonrelativistic energy-momentum relation:  $p_0 = p^2/(2m)$ . The  $T$ -matrix elements involving diatomic molecules are more complicated and will not be discussed here.

##### b. Operator vertices

To calculate matrix elements of composite operators, we also need Feynman rules for the operators. Energy and momentum can flow into and out of the vertices for the operators. For simplicity, we consider only operators at a fixed time  $t=0$ . The operators listed below in Eqs. (A1) annihilate an atom and create an atom, so the operator vertex has an incoming atom line and an outgoing atom line. If the incoming and outgoing momenta are  $\mathbf{k}$  and  $\mathbf{k}'$ , the Feynman rules for the operator vertices are

$$\psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right)\psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right): \exp\left[i\mathbf{k} \cdot \left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right)\right] \exp\left[-i\mathbf{k}' \cdot \left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right)\right], \quad (\text{A1a})$$

$$\psi_\sigma^\dagger\psi_\sigma(\mathbf{R}): \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}], \quad (\text{A1b})$$

$$\psi_\sigma^\dagger \nabla^j \psi_\sigma(\mathbf{R}) - \nabla^j \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}): i(\mathbf{k} + \mathbf{k}')^j \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}]. \quad (\text{A1c})$$

The operators listed below in Eqs. (A2) annihilate a pair of atoms or a molecule and create a pair of atoms or a molecule. If the total momenta entering and leaving the vertex are  $\mathbf{K}$  and  $\mathbf{K}'$ , respectively, the Feynman rules for the operator vertices are

$$\psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}): \exp[i(\mathbf{K} - \mathbf{K}') \cdot \mathbf{R}], \quad (\text{A2a})$$

$$\psi_1^\dagger \psi_2^\dagger \phi(\mathbf{R}): \exp[i(\mathbf{K} - \mathbf{K}') \cdot \mathbf{R}], \quad (\text{A2b})$$

$$\phi^\dagger \psi_2 \psi_1(\mathbf{R}): \exp[i(\mathbf{K} - \mathbf{K}') \cdot \mathbf{R}], \quad (\text{A2c})$$

$$\phi^\dagger \phi(\mathbf{R}): \exp[i(\mathbf{K} - \mathbf{K}') \cdot \mathbf{R}]. \quad (\text{A2d})$$

We will consider matrix elements of these operators only between states for which the total momenta  $\mathbf{K}$  and  $\mathbf{K}'$  are both zero, in which case the Feynman rules in Eqs. (A2) all reduce to 1.

The matrix element of a composite operator between initial and final states consisting of atoms is the sum of all diagrams in which each external atom line is connected either to an operator vertex or to other external atom lines.

#### 2. Matrix elements between two-atom scattering states

In this section, we calculate the matrix elements of composite operators between two-atom scattering states. We denote the state consisting of a single atom with momentum  $\mathbf{p}$



and spin  $\sigma$  by  $|\mathbf{p}, \sigma\rangle$ . A two-atom scattering state is labeled by the momenta and the spins of the two particles:  $|\mathbf{p}_1, \sigma_1; \mathbf{p}_2, \sigma_2\rangle$ . For simplicity, we will calculate matrix elements only between two-atom scattering states consisting of two atoms with different spins, total momentum 0, and total energy  $E$ . These states, which we denote by  $|\mathbf{p}, 1; -\mathbf{p}, 2\rangle$ , are labeled by a vector  $\mathbf{p}$  whose magnitude is  $|\mathbf{p}| = (mE)^{1/2}$ .

#### a. Matrix element of $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$

The matrix element of the bilocal operator  $\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})$  is given by the sum of the four diagrams in Fig. 4. The only one of these diagrams that involves an integral over the four momentum of an atom is Fig. 4(d). Using the Feynman rules, the expression for this diagram is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{4(d)} = i\mathcal{A}^2(E) \int \frac{d^3q dq_0}{(2\pi)^4} \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{[q_0 - q^2/(2m) + i\epsilon]^2 [E - q_0 - q^2/(2m) + i\epsilon]}, \quad (\text{A3})$$

where  $\mathcal{A}(E)$  is the amplitude in Eq. (42) and  $E = p^2/m$ . The integral over  $q_0$  can be evaluated using contours. The resulting momentum integral is given in Eq. (A13b). The final result is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \psi_\sigma^\dagger\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \psi_\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{4(d)} = \frac{im^2}{8\pi p} \mathcal{A}^2(E) \exp(ipr). \quad (\text{A4})$$

Unlike the first three diagrams in Fig. 4, this diagram is not an analytic function of  $\mathbf{r}$  at  $\mathbf{r}=0$ .

#### b. Matrix element of $\psi_\sigma^\dagger\psi_\sigma$

The matrix element of the local operator  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$  is given by the sum of the four diagrams in Fig. 5. These diagrams are equal to the  $\mathbf{r} \rightarrow 0$  limits of the corresponding diagrams for the bilocal operator in Fig. 4. For the first three diagrams, this can be seen easily from the Feynman rules. This is not as obvious for the fourth diagram, because it involves an integral over the four momentum of an atom. Using the Feynman rules, the expression for the diagram in Fig. 4(d) is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \psi_\sigma^\dagger\psi_\sigma(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{5(d)} = i\mathcal{A}^2(E) \int \frac{d^3q dq_0}{(2\pi)^4} \frac{1}{[q_0 - q^2/(2m) + i\epsilon]^2 [E - q_0 - q^2/(2m) + i\epsilon]}. \quad (\text{A5})$$

The integral over  $q_0$  can be evaluated using contours. The resulting momentum integral is given in Eq. (A14b). The final result is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \psi_\sigma^\dagger\psi_\sigma(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{5(d)} = \frac{im^2}{8\pi p} \mathcal{A}^2(E). \quad (\text{A6})$$

This result matches the  $r^0$  term in the expansion of Eq. (A4) in powers of  $r$ . This is consistent with the Wilson coefficient of  $\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$  in the OPE in Eq. (44) being simply 1.

#### c. Matrix element of $\Phi^\dagger\Phi$

The matrix element for  $\Phi^\dagger\Phi(\mathbf{R})$  can be represented by the sum of the four diagrams in Fig. 7, together with 12 other diagrams in which there is no scattering of the two incoming atoms or the two outgoing atoms or both. Using the Feynman rules, the expression for the diagram in Fig. 7(a) is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{7(a)} = \lambda_0^2 \left[ -i\mathcal{A}(E) \int \frac{d^3q dq_0}{(2\pi)^4} \frac{1}{[q_0 - q^2/(2m) + i\epsilon][E - q_0 - q^2/(2m) + i\epsilon]} \right]^2. \quad (\text{A7})$$

The integral over  $q_0$  can be evaluated using contours. The resulting momentum integral is given in Eq. (A14a). The result is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \lambda_0^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{7(a)} = \lambda_0^2 \left[ m\mathcal{A}(E) \left( \frac{\Lambda}{2\pi^2} + \frac{ip}{4\pi} \right) \right]^2. \quad (\text{A8})$$

The expressions for the three diagrams obtained from Fig. 7(a) by omitting the scattering of the atoms in the initial state or final state or both are obtained from Eq. (A8) by replacing one or both of the factors in square brackets by 1. The expressions for the three diagrams in Figs. 7(b)–7(d) are obtained by replacing one or both of the factors of  $\lambda_0$  by  $(g_0^2/m)/(E - \nu_0)$ . The sum of all 16 diagrams is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \Phi^\dagger \Phi(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle = \left( \lambda_0 + \frac{g_0^2/m}{E - \nu_0} \right)^2 \left[ 1 + m\mathcal{A}(E) \left( \frac{\Lambda}{2\pi^2} + \frac{ip}{4\pi} \right) \right]^2. \quad (\text{A9})$$

By using the integral equation in Eq. (41), this can be simplified to

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \Phi^\dagger \Phi(\mathbf{R}) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle = m^2 \mathcal{A}^2(E), \quad (\text{A10})$$

where  $\mathcal{A}(E)$  is the amplitude in Eq. (42) and  $E = p^2/m$ . This result matches the  $r^1$  term in the expansion of Eq. (A4) in powers of  $r$  if the Wilson coefficient of  $\Phi^\dagger \Phi(\mathbf{R})$  in the OPE in Eq. (30) is  $-r/(8\pi)$ .

#### d. Matrix element of $\psi_1^\dagger \psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r}) \psi_2^\dagger \psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r})$

The matrix element of the bilocal operator  $\psi_1^\dagger \psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r}) \psi_2^\dagger \psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r})$  is given by the sum of the four diagrams in Fig. 8. The diagram in Fig. 8(a) is simply 1. The two diagrams in Figs. 8(b) and 8(c) involve an integral over the four momentum of an atom. The diagram in Fig. 8(d) involves two such integrals. Using the Feynman rules, the expression for this diagram is

$$\langle \mathbf{p}', 1; -\mathbf{p}', 2 | \psi_1^\dagger \psi_1 \left( \mathbf{R} - \frac{1}{2}\mathbf{r} \right) \psi_2^\dagger \psi_2 \left( \mathbf{R} + \frac{1}{2}\mathbf{r} \right) | \mathbf{p}, 1; -\mathbf{p}, 2 \rangle_{8(d)} = \left[ -i\mathcal{A}(E) \int \frac{d^3 q dq_0}{(2\pi)^4} \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{[q_0 - q^2/(2m) + i\epsilon][E - q_0 - q^2/(2m) + i\epsilon]} \right]^2. \quad (\text{A11})$$

The integral over  $q_0$  can be evaluated using contours. The resulting momentum integral is given in Eq. (A13a). The final result is

$$\left\langle \mathbf{p}', 1; -\mathbf{p}', 2 \left| \psi_1^\dagger \psi_1 \left( \mathbf{R} - \frac{1}{2}\mathbf{r} \right) \psi_2^\dagger \psi_2 \left( \mathbf{R} + \frac{1}{2}\mathbf{r} \right) \right| \mathbf{p}, 1; -\mathbf{p}, 2 \right\rangle_{8(d)} = \left[ \frac{m}{4\pi r} \mathcal{A}(E) \exp(ipr) \right]^2. \quad (\text{A12})$$

The term proportional to the  $r^{-2}$  term in the expansion of Eq. (A12) in powers of  $r$  can be matched by the matrix element of  $\Phi^\dagger \Phi(\mathbf{R})$  in Eq. (A10) if the Wilson coefficient of this operator is  $1/(16\pi^2 r^2)$ , in accord with Eq. (53).

#### e. Momentum integrals

We list here the momentum integrals that arise in the calculation of matrix elements in the preceding sections. The calculations of the matrix elements of bilocal operators in Secs. 2 a and 2 d require the following Fourier transforms:

$$\int \frac{d^3 q}{(2\pi)^3} \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{q^2 - p^2 - i\epsilon} = \frac{1}{4\pi r} \exp(ipr), \quad (\text{A13a})$$

$$\int \frac{d^3 q}{(2\pi)^3} \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{(q^2 - p^2 - i\epsilon)^2} = \frac{i}{8\pi p} \exp(ipr). \quad (\text{A13b})$$

The calculations of the matrix elements of local operators in Secs. 2 b and 2 c require the following integrals:

$$\int \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2 - p^2 - i\epsilon} = \frac{\Lambda}{2\pi^2} + \frac{ip}{4\pi}, \quad (\text{A14a})$$

$$\int \frac{d^3 q}{(2\pi)^3} \frac{1}{(q^2 - p^2 - i\epsilon)^2} = \frac{i}{8\pi p}. \quad (\text{A14b})$$

The integral in Eq. (A14a) is ultraviolet divergent. It has been evaluated using an ultraviolet cutoff  $|\mathbf{q}| < \Lambda$ .

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