Quantum networks on cubelike graphs

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Cubelike graphs are the Cayley graphs of the elementary Abelian group $\mathbb{Z}_2^n$ (e.g., the hypercube is a cubelike graph). We study perfect state transfer between two particles in quantum networks modeled by a large class of cubelike graphs. This generalizes the results of Christandl *et al.* [Phys. Rev. Lett. **92**, 187902 (2004)] and Facer *et al.* [Phys. Rev. A **92**, 187902 (2008)].

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I. INTRODUCTION

In view of applications like the distribution of cryptographic keys [1,2] or communication between registers in quantum devices [3,4], the study of the natural evolution of permanently coupled spin networks has become increasingly important. A special case of interest consists of homogeneous networks of particles coupled by constant and fixed (nearest-neighbor) interactions.

An important feature of these networks is the possibility of faithfully transferring a qubit between specific particles without tuning the couplings or altering the network topology. This phenomenon is usually called *perfect state transfer* (PST). Since quantum networks (and communication networks in general) are naturally associated with undirected or directed graphs, there is a growing amount of literature on the relation between graph-theoretic properties and properties that allow PST (see [5–9]).

In the present paper we will give necessary and sufficient conditions for PST in quantum networks modeled by a relatively large class of *cubelike graphs*. The vertices of a cubelike graph are the binary n-vectors; two vertices u and v are adjacent if and only if their symmetric difference belongs to a chosen set. Equivalently, cubelike graphs on 2^n vertices are the Cayley graphs of the elementary Abelian group of order 2^n [10,11].

Among cubelike graphs, the hypercube is arguably the most famous one, having many applications ranging from switching theory to computer architecture, etc. (see, e.g., [12,13]). There are various and diverse results about quantum dynamics on hypercubes. These are essentially embraced by two areas: continuous-time quantum walks [14] and quantum communication in spin networks [15]. The common ingredient is the use of a Hamiltonian representing the adjacency structure of the graph.

Concerning state transfer, Christandl *et al.* [16] have shown that networks modeled by hypercubes are capable of transporting qubits between pairs of antipodal nodes, perfectly (i.e., with perfect fidelity) and in constant time. Facer *et al.* [17] generalized this observation by considering a family of cubelike graphs whose members have the hypercube as a spanning subgraph (for this reason, these authors coined the term *dressed hypercubes*). Other questions related to quantum dynamics on hypercubes have been addressed in [18–22].

The paper is organized as follows. Section II contains the necessary definitions and the statements of our results. In Theorem 1 we state that there is PST between vertices a and b of a cubelike graph at a time $t=\pi/2$ if the binary strings associated to these vertices satisfy a specific (easily testable) condition. A proof will be given in Sec. III. This is obtained by diagonalizing the Hamiltonians with simple tools from Fourier analysis on $\mathbb{Z}_2^n$.

II. SETUP AND STATEMENTS OF THE RESULTS

Let $\mathbb{Z}_2^n$ be the additive Abelian group $(\mathbb{Z}_2)^{\times n}$. Each element of $\mathbb{Z}_2^n$ is represented as a binary vector of length n. The *zero vector* $\mathbf{0}$ is made up of all 0's. Let $f:\mathbb{Z}_2^n \to \mathbb{Z}_2$ be a Boolean function on n variables and let Ω_f be the set of all binary vectors w in $\mathbb{Z}_2^n$ such that $f(w)=1$; i.e., $\Omega_f=\{w\in\mathbb{Z}_2^n | f(w)=1\}$. Let $d=|\Omega_f|$ be the number of vectors $w\in\mathbb{Z}_2^n$ such that $f(w)=1$. Finally, if w and v are two binary vectors of the same length, then $w\oplus v$ denotes the vector obtained by computing their elementwise addition modulo 2 and $w^T v$ their scalar product.

The *Cayley graph* $X(\Gamma,T)$ of a group Γ with respect to the set $T\subseteq\Gamma$ $(T=T^{-1})$ is the graph with vertex set $V(X)=\{\Gamma\}$ and an edge $\{g,h\}\in E(X)$ if there is $s\in T$ such that $gs=h$. In other words, two nodes in the graph are connected by an edge if the corresponding elements g and h are connected by a member of the group Γ. The set T is also called a *Cayley set*. The Cayley graphs of the form $X(\mathbb{Z}_2^n,\Omega_f)$ are called *cubelike graphs*. Some cubelike graphs are illustrated in Fig. 1.

Notice that this definition embraces every possible set Ω_f. When f is the characteristic function of the standard generating set of $\mathbb{Z}_2^n$, the graph $X(\mathbb{Z}_2^n,\Omega_f)$ is called a *hypercube*—for example, the leftmost graph in Fig. 1. Recall that the *standard generating set* of $\mathbb{Z}_2^n$ is composed of the n vectors

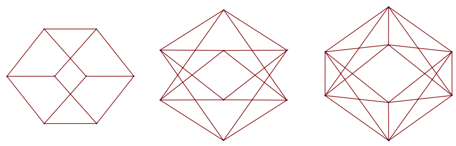

FIG. 1. (Color online) Drawings of three nonisomorphic cubelike graphs on eight vertices.

$(0\ldots010\ldots0)$, where 1 is at position i, with $1\leqslant i\leqslant n$. The *adjacency matrix* of $X(\mathbb{Z}_2^n,\Omega_f)$ is the $2^n\times 2^n$ matrix

$$A_f=\sum_{w\in\Omega_f}\rho_{reg}(w), \tag{1}$$

where $\rho_{reg}(x)$ is the regular permutation representation of $w\in\Omega_f$. In particular, if $w=w_1w_2\cdots w_n$, then

$$\rho_{reg}(w)=\bigotimes_{i=1}^{n}\sigma_x^{w_i},$$

where σ_x is a Pauli matrix. The regular permutation representation of $\mathbb{Z}_2^n$ is the homomorphism from $\mathbb{Z}_2^n$ into the set of permutation matrices of size 2^n.

It is clear that A_f commutes with the adjacency matrix of any other cubelike graph, given that the group $\mathbb{Z}_2^n$ is Abelian.

Now, let us choose a bijection between vertices of $X(\mathbb{Z}_2^n,\Omega_f)$ and the elements of the standard basis $|1\rangle,|2\rangle,\ldots,|N\rangle$ of a Hilbert space $\mathcal{H}\cong\mathbb{C}^N$, where $N=2^n$. This is the usual space of n qubits. If we look at the single excitation case in the *XY* model, the evolution of a network of spin-1/2 quantum mechanical particles on the vertices of $X(\mathbb{Z}_2^n,\Omega_f)$ can be seen as induced by the adjacency matrix A_f, which then plays the role of a Hamiltonian (for details see [16] or [17]). In light of this observation, given two vectors $a,b\in\mathbb{Z}_2^n$, the (unnormalized) *transition amplitude* between a and b induced by A_f is the expression

$$\begin{aligned}T(a,b)&=\langle b|e^{-iA_ft}|a\rangle\\&=\sum_{w\in\mathbb{Z}_2^n}(-1)^{a^Tw}e^{-i\lambda_wt}(-1)^{b^Tw}\\&=\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}e^{-i\lambda_wt},\end{aligned} \tag{2}$$

where $t\in\mathbb{R}^+$. This is written by diagonalizing A_f (see Sec. III). The *fidelity* of state transfer between a and b is then $F(a,b)=|T(a,b)|$. By definition, the evolution under A_f is *periodic* if there is $t\in\mathbb{R}^+$ such that $F(a,a)=1$ for every $a\in\mathbb{Z}_2^n$. Graph-theoretic properties responsible for periodic evolution (sometimes also called *perfect revival*) have been considered in the literature [6,9,23–25].

With the next proposition, we show that every network modeled by a cubelike graph has a periodic evolution. The period is π and it does not depend on the number of vertices of the graph. Equivalently, it does not depend on the dimension of $\mathcal{H}$.

Theorem 1. Let $X(\mathbb{Z}_2^n,\Omega_f)$ be a cubelike graph and let $a,b\in\mathbb{Z}_2^n$.

(i) For $t=\pi$, we have $F(a,b)=1$ if and only if $a=b$.

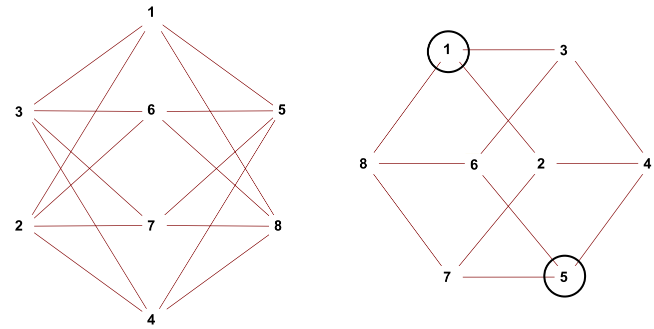

FIG. 2. (Color online) Left: The graph $X(\mathbb{Z}_2^n,C)$, where $C=\{(100),(010),(001),(111)\}$. This graph is isomorphic to the complete bipartite graph $K_{4,4}$. Right: the graph $X(\mathbb{Z}_2^n,C_1)$, where $C_1=\{(010),(001),(111)\}$. This graph turns out to be isomorphic to the hypercube of dimension 3. Since $100=010\oplus001\oplus111$, there is PST between 000 and 100 at time $\pi/2$, given that $000\oplus100=100$. Also, there is PST for the pairs {010, 110}, {001, 101}, and {011, 111}.

(ii) For $t=\pi/2$, we have $F(a,b)=1$ if $a\oplus b=u$ and $u=\oplus_{w\in\Omega_f}w\neq\mathbf{0}$.

As a simple consequence of this statement, we have various ways to *route* information between any two nodes of a network whose vertices correspond to the elements of $\mathbb{Z}_2^n$. Let f be a Boolean function such that $\Omega_f=\{w_1,\ldots,w_r\}$ is a generating set of $\mathbb{Z}_2^n$. Let $\oplus_{w_i\in\Omega_f}w_i=w\neq\mathbf{0}$. Let us define $C=\{w,w_1,\ldots,w_r\}$ and $C_i=C\backslash w_i$. Since the sum of the elements of C_i is nonzero, the Cayley graph $X(\mathbb{Z}_2^n,C_i)$ has PST between a and b such that $a\oplus b=w_i$ at time $\pi/2$. The simplest case arises when we chose $r=n$ and take the vectors $w_1,\ldots,w_r$ to be the standard generating set of $\mathbb{Z}_2^n$. Then $X(\mathbb{Z}_2^n,C)$ is the *folded d-cube* and, by using a suitable sequence of the graphs $X(\mathbb{Z}_2^n,C_i)$, we can arrange PST from the zero vector to any desired element of $\mathbb{Z}_2^n$.

For example, consider the case $d=3$. We write $w_1=(100)$, $w_2=(010)$, and $w_3=(001)$. Then $w=(111)$. The graph $X(\mathbb{Z}_2^3,C)$ is illustrated in Fig. 1, left. Since $w_1=w_2\oplus w_3\oplus w$, there is PST between 000 and $w_1=100$ at time $\pi/2$, given that $000\oplus100=100$ (see Fig. 2, right). Also, there is PST for the pairs {010, 110}, {001, 101}, and {011, 111}. Notice that $X(\mathbb{Z}_2^n,C_1)$ is isomorphic to the three-dimensional hypercube.

For generic dimension, the hypercube is $X(\mathbb{Z}_2^n,E)$, where $E=\{(10\ldots0),(010\ldots0),\ldots,(0\ldots01)\}$ and $\vec{J}_n=\oplus_{w\in E}w$, the all-1st vector of length 2^n. Since the Hamming distance between two different elements $a,b\in\mathbb{Z}_2^n$ is exactly their distance in $X(\mathbb{Z}_2^n,E)$, we have PST between any two antipodal vertices of the hypercube, as was already observed in [16,17]. Recall that the distance between two vertices in a graph is the length of the geodesic (equivalently, the shortest path) connecting the vertices. Two vertices are said to be *antipodal* if their distance is the *diameter* of the graph—i.e., the longest among all the geodesics. Antipodal vertices in Cayley graphs are connected via a sequence of all elements of the Cayley set. It remains as an open problem to verify that whenever there is PST between two vertices of a cubelike graph, then the vertices are antipodal.

III. PROOF OF THE THEOREM

The *abstract Fourier transform* of a Boolean function f is the rational valued function $\tilde{f}:\mathbb{Z}_2^n \rightarrow \mathbb{Q}$ which defines the coefficients of f with respect to the orthonormal basis of the functions

$$Q_w(x) = (-1)^{w^T x}, \quad (3)$$

that is,

$$\tilde{f}(w) = 2^{-n} \sum_{x \in \mathbb{Z}_2^n} (-1)^{w^T x} f(x). \quad (4)$$

Then

$$f(x) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{w^T x} \tilde{f}(w) \quad (5)$$

is the Fourier expansion of f. Note that the zeroth-order Fourier coefficient is equal to the probability that the function takes the value 1—i.e., $\tilde{f}(\mathbf{0}) = \frac{d}{2^n}$—while the other Fourier coefficients measure the correlation between the function and the parity of subsets of its arguments.

Using a vector representation for the functions f and $\tilde{f}$, and considering the natural ordering of the binary vectors $w \in \mathbb{Z}_2^n$, one can derive a convenient matrix formulation for the transform pair: $f = H_n \tilde{f}$ and $\tilde{f} = \frac{1}{2^n} H_n f$, where H_n is the Hadamard transform matrix. Given a function $f:\mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$, the set Ω_f defines the Cayley graph $X(\mathbb{Z}_2^n, \Omega_f)$, whose spectrum coincides, up to a factor 2^n, with the Fourier spectrum of the function $\frac{1}{2^n} H_n A_f H_n = D_f$, where A_f is the adjacency matrix of $X(\mathbb{Z}_2^n, \Omega_f)$ and D_f is a $2^n \times 2^n$ diagonal matrix. In particular, for $w \in \mathbb{Z}_2^n$, we have $\lambda_w = 2^n \tilde{f}(w)$. Theorem 1 needs the following two technical lemmas.

Lemma 2. Let f be a Boolean function such that $\oplus_{w \in \Omega_f} w = \mathbf{0}$. Then, for all $v \in \mathbb{Z}_2^n$,

$$\lambda_v = d - 4k_v, \quad (6)$$

where $d = |\Omega_f|$ and $k_v \in \mathbb{N}$, with $0 \leq k_v \leq \lfloor d/2 \rfloor$.

Proof. Let $a \in \Omega_f$, $a \neq \mathbf{0}$, and let $\Omega_f' = \Omega \setminus \{a\}$. Since $\oplus_{w \in \Omega_f} w = \mathbf{0}$, we have that $a = \oplus_{w \in \Omega_f'} w$. Now observe that, for any $v \in \mathbb{Z}_2^n$,

$$\begin{aligned} \lambda_v &= \sum_{w \in \mathbb{Z}_2^n} (-1)^{w^T v} f(w) \\ &= \sum_{w \in \Omega_f} (-1)^{w^T v} \\ &= \sum_{w \in \Omega_f'} (-1)^{w^T v} + (-1)^{a^T v} \\ &= \sum_{w \in \Omega_f'} (-1)^{w^T v} + (-1)^{(\oplus_{w \in \Omega_f'} w)^T v} \\ &= \sum_{w \in \Omega_f'} (-1)^{w^T v} + \prod_{w \in \Omega_f'} (-1)^{w^T v}. \end{aligned} \quad (7)$$

Let

$$n_v = |\{w \in \Omega' | (-1)^{w^T v} = -1\}|$$

and

$$p_v = |\{w \in \Omega' | (-1)^{w^T v} = 1\}|.$$

Note that $p_v = |\Omega_f'| - n_v = (d-1) - n_v$. We have

$$\lambda_v = p_v - n_v + (-1)^{n_v} = d - 1 - 2n_v + (-1)^{n_v}. \quad (8)$$

Thus, if n_v is an even number, $n_v = 2k_v$, with $k_v \in \mathbb{N}$, we get

$$\lambda_v = d - 1 - 4k_v + 1 = d - 4k_v. \quad (9)$$

Otherwise, if $n_v = 2k_v - 1$ is odd, we get

$$\lambda_v = d - 1 - 2(2k_v - 1) - 1 = d - 4k_v. \quad (10)$$

Since, for all v, $-d \leq \lambda_v \leq d$, we finally get $0 \leq k_v \leq \lfloor d/2 \rfloor$. ■

Lemma 3. Let f be a Boolean function such that $\oplus_{w \in \Omega_f} w = u \neq \mathbf{0}$.

(i) If $u \notin \Omega_f$, then, for all $v \in \mathbb{Z}_2^n$,

$$\lambda_v = \begin{cases} d - 4k_v, & \text{if } u^T v \text{ is even}, \\ d - 4k_v + 2, & \text{if } u^T v \text{ is odd}, \end{cases}$$

where $d = |\Omega_f|$ and $k_v \in \mathbb{N}$, $0 \leq k_v \leq \lfloor \frac{d+1}{2} \rfloor$.

(ii) If $u \in \Omega_f$, then, for all $v \in \mathbb{Z}_2^n$,

$$\lambda_v = \begin{cases} d - 4k_v, & \text{if } u^T v \text{ is even}, \\ d - 4k_v - 2, & \text{if } u^T v \text{ is odd}, \end{cases}$$

where $d = |\Omega_f|$ and $k_v \in \mathbb{N}$, $0 \leq k_v \leq \lfloor \frac{d-1}{2} \rfloor$.

Proof. (i) Consider the function g such that $\Omega_g = \Omega_f \cup \{u\}$. Let μ_v denote the eigenvalues of the Cayley graph associated to g. As $\oplus_{w \in \Omega_g} w = \mathbf{0}$, from Lemma 2, we get

$$\mu_v = |\Omega_g| - 4k_v = d + 1 - 4k_v, \quad (11)$$

with $0 \leq k_v \leq \lfloor \frac{d+1}{2} \rfloor$. Now, observe that

$$\begin{aligned} \mu_v &= \sum_{w \in \mathbb{Z}_2^n} (-1)^{w^T v} g(w) \\ &= \sum_{w \in \Omega_g} (-1)^{w^T v} \\ &= \sum_{w \in \Omega_f} (-1)^{w^T v} + (-1)^{u^T v} \\ &= \lambda_v + (-1)^{u^T v}. \end{aligned} \quad (12)$$

Thus,

$$\lambda_v = \mu_v - (-1)^{u^T v} = d + 1 - 4k_v - (-1)^{u^T v}, \quad (13)$$

and the thesis immediately follows.

(ii) Consider the function g such that $\Omega_g = \Omega_f \setminus \{u\}$. Let μ_v denote the eigenvalues of the Cayley graph associated to g. As $u = \oplus_{w \in \Omega_f} w = \oplus_{w \in \Omega_g} \oplus u$, we have $\oplus_{w \in \Omega_g} w = \mathbf{0}$. By applying Lemma 2, we obtain

$$\mu_v = |\Omega_g| - 4k_v = d - 1 - 4k_v, \quad (14)$$

with $0 \leq k_v \leq \lfloor \frac{d-1}{2} \rfloor$. Now, as in (i) observe that

$$\begin{aligned}\lambda_v &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{w^Tv}f(w)\\ &= \sum_{w\in\Omega_f}(-1)^{w^Tv}\\ &= \sum_{w\in\Omega_g}(-1)^{w^Tv}+(-1)^{u^Tv}\\ &= \mu_v+(-1)^{u^Tv}.\end{aligned} \tag{15}$$

So we get

$$\lambda_v = d-1-4k_v+(-1)^{u^Tv}, \tag{16}$$

concluding the proof of the lemma. ■

Proof of Theorem 1. (i) All eigenvalues λ_w are integers with the same parity. In particular, they are all odd if $d=|\Omega_f|$ is odd, and all even, otherwise. Thus, by Eq. (2), for $t=\pi$ we have

$$\begin{aligned}T(a,b) &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}e^{-i\lambda_w\pi}\\ &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}\cos(\lambda_w\pi)\\ &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}(-1)^{\lambda_w}\\ &= (-1)^d\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}.\end{aligned} \tag{17}$$

Thus the statement follows since

$$\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}=\begin{cases}2^n, & a\oplus b=\mathbf{0}, \text{ i.e., } a=b,\\ 0, & \text{otherwise}.\end{cases}$$

(ii) First, let us suppose that f is such that $\oplus_{w\in\Omega_f} w=\mathbf{0}$. Then, applying Lemma 2, we can see that

$$\begin{aligned}T(a,b) &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}e^{-i\lambda_w\pi/2}\\ &= \sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}e^{-i(d-4k_w)\pi/2}\\ &= e^{-id\pi/2}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}e^{i2k_w\pi}\\ &= e^{-id\pi/2}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}.\end{aligned} \tag{18}$$

The thesis is verified, since $\Sigma_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}=2^n$ if and only if $a\oplus b=\mathbf{0}=\oplus_{w\in\Omega_f} w$, but here $a\neq b$. The remaining case is when $\oplus_{w\in\Omega_f} w=u\neq\mathbf{0}$. Applying Lemma 3, we can write

$$\begin{aligned}T(a,b) &= \sum_{\substack{w\in\mathbb{Z}_2^n\\ u^Tw \text{ even}}}(-1)^{(a\oplus b)^Tw}e^{-i(d-4k_w)\pi/2} + \sum_{\substack{w\in\mathbb{Z}_2^n\\ u^Tw \text{ odd}}}(-1)^{(a\oplus b)^Tw}e^{-i(d-4k_w\pm 2)\pi/2}\\ &= e^{-id\pi/2}\left(\sum_{\substack{w\in\mathbb{Z}_2^n\\ u^Tw \text{ even}}}(-1)^{(a\oplus b)^Tw}e^{i2k_w\pi}-\sum_{\substack{w\in\mathbb{Z}_2^n\\ u^Tw \text{ odd}}}(-1)^{(a\oplus b)^Tw}e^{i2k_w\pi}\right)\\ &= e^{-id\pi/2}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b)^Tw}(-1)^{u^Tw} = e^{-id\pi/2}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b\oplus u)^Tw}.\end{aligned} \tag{19}$$

The statement holds since $\Sigma_{w\in\mathbb{Z}_2^n}(-1)^{(a\oplus b\oplus u)^Tw}=2^n$ if and only if $a\oplus b\oplus u=\mathbf{0}$; i.e., $a\oplus b=u=\oplus_{w\in\Omega_f} w$. ■

IV. CONCLUSION

We have given a necessary and sufficient condition for PST in quantum networks modeled by a large class of cubelike graphs.

A special case is left open: when $\oplus_{w\in\Omega_f} w=\mathbf{0}$. Numerical evidence suggests that cubelike graphs with this property do not allow PST. In addition, it is natural to ask whether quantum dynamics on cubelike graphs can help in getting useful information about the corresponding Boolean functions [26–30].

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