Collective processes of an ensemble of spin-1/2 particles

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When the dynamics of a spin ensemble are expressible solely in terms of symmetric processes and collective spin operators, the symmetric collective states of the ensemble are preserved. These many-body states, which are invariant under particle relabeling, can be efficiently simulated since they span a subspace whose dimension is linear in the number of spins. However, many open system dynamics break this symmetry, most notably when ensemble members undergo identical, but local, decoherence. In this paper, we extend the definition of symmetric collective states of an ensemble of spin-1/2 particles in order to efficiently describe these more general collective processes. The corresponding collective states span a subspace that grows quadratically with the number of spins. We also derive explicit formulas for expressing arbitrary, identical, local decoherence in terms of these states. We then investigate the open system dynamics of experimentally relevant nonclassical collective atomic states, including superposition and spin squeezed states, subject to various symmetric but local decoherence models.

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I. INTRODUCTION

The ability to model the open system dynamics of large spin ensembles is crucial to experiments that make use of many atoms, as is often the case in precision metrology [1,2], quantum-information science [3–5], and quantum-optical simulations of condensed matter phenomena [6–8]. Unfortunately, the mathematical description of large atomic spin systems is complicated by the fact that the dimension of the Hilbert space $\mathcal{H}_N$ grows exponentially in the number of atoms N. Realistic simulations of experiments quickly become intractable even for atom numbers smaller than $N\sim 10$. Current experiments, however, often work with atom numbers of more than $N\sim 10^{10}$, meaning that direct simulation of these systems is well beyond feasible. Moreover, simulations over a range $N\sim 1-10$ are far from adequate to discern even the qualitative behavior that would be expected in the $N\gg 1$ limit. Fortunately, it is often the case that experiments involving large spin ensembles respect one or more dynamical symmetries that can be exploited to reduce the effective dimension of the ensemble's Hilbert space. One can then hope to achieve a sufficiently realistic model of experiments without an exponentially large description of the system.

In particular, previous work has focused on the *symmetric collective states* $|\psi_S\rangle$, which are invariant under the permutation of particle labels: $\hat{\Pi}_{ij}|\psi_S\rangle=|\psi_S\rangle$. These states span the subspace $\mathcal{H}_S\subset\mathcal{H}_N$, which grows linearly with the number of particles, $\dim(\mathcal{H}_S)=Nj+1$. However, in order for $\mathcal{H}_S$ to be an invariant subspace, the dynamics of the system must be expressible solely in terms of *symmetric processes*, which are particle permutation invariant, and *collective operators*, which respect the irreducible representation structure of rotations on the spin ensemble. Fortunately, even within this restrictive class, a wide variety of phenomenon may be observed, including spin squeezing [9,10] and zero-temperature phase transitions [6].

In practice, symmetric atomic dynamics are achieved by ensuring that there is identical coupling between all the atoms in the ensemble and the electromagnetic fields (optical, magnetic, microwave, etc.) used to both drive and observe the system [11]. This approximation can be quite good for all of the coherent dynamics, because with sufficient laboratory effort, electromagnetic intensities can be made homogeneous, ensuring that interactions do not distinguish between different atoms in the ensemble. However, incoherent dynamics are often beyond the experimenter's control. Although most types of decoherence are symmetric, they are not generally written using collective operators. Instead they are expressed as identical Lindblad operators for each spin, i.e.,

$$\mathcal{L}[\hat{s}]\hat{\rho}=\sum_{n=1}^{N}\left(\hat{s}^{(n)}\hat{\rho}(\hat{s}^{(n)})^{\dagger}-\frac{1}{2}(\hat{s}^{(n)})^{\dagger}\hat{s}^{(n)}\hat{\rho}-\frac{1}{2}\hat{\rho}(\hat{s}^{(n)})^{\dagger}\hat{s}^{(n)}\right). \tag{1}$$

The fact that decoherence does not preserve $\mathcal{H}_S$ has been well appreciated and the standard practice in experiments that address the collective state of atomic ensembles has been either (i) to model such experiments only in a very short-time limit where decoherence can be approximately ignored; or (ii) to use decoherence models that do respect the particle symmetry, but which are written using only collective operators, even when doing so is not necessarily physically justified. In atomic spin ensembles, for example, a typical source of decoherence comes from spontaneous emission, yet collective radiative processes occur only under specific conditions such as superradiance from highly confined atoms [16] and some cavity-QED or spin-grating settings [15].

In this paper, we generalize the collective states of an ensemble of spin-1/2 particles (qubits) to include states that are preserved under symmetric—but not necessarily collective—transformations. Specifically, we generalize from

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the strict condition of complete permutation invariance to the broader class of states that are indistinguishable across degenerate irreducible representations (irreps) of the rotation group. While the representation theory of the rotation group has been utilized in a wide variety of contexts, such as to protect quantum information from decoherence by encoding it into degenerate irreps with the same total angular momentum [12,13], we utilize relevant aspects of the representation theory to obtain a reduced-dimensional description of quantum maps that act locally but identically on every member of an ensemble of qubits.

Our main result, presented in Eq. (42), enables us to represent arbitrary symmetric Lindblad operators in the collective state basis. We find that the dimension of the Hilbert space $\mathcal{H}_C$ spanned by these generalized collective states scales favorably, $\dim(\mathcal{H}_C)\sim N^2$. This allows for efficient simulation of a broader class of collective spin dynamics and, in particular, allows one to consider the effects of decoherence on previous simulations of symmetric collective spin states. We note that dynamical symmetries for spin-1/2 particles have been studied in the context of decoherence-free quantum-information processing [12,13]. Unlike our work, which uses symmetries to find a reduced description of a quantum system, these works seek to protect quantum information from decoherence by encoding within the degeneracies introduced by dynamical symmetries.

The remainder of the paper is organized as follows. Section II reviews the representation theory of the rotation group, which plays an important role in defining the symmetries related to $\mathcal{H}_S$ and $\mathcal{H}_C$. Section III introduces collective states and Sec. IV defines collective processes over these states. Section V gives an identity for expressing arbitrary symmetric superoperators, e.g., Eq. (1), over the collective states. Section VI leverages this formalism to compare the effect of different decoherence models in nonclassical atomic ensemble states. Section VII concludes.

II. GENERAL STATES OF THE ENSEMBLE

Consider an ensemble of N spin-1/2 particles, with the nth spin characterized by its angular momentum $\hat{\mathbf{j}}^{(n)}=\{\hat{j}_x^{(n)},\hat{j}_y^{(n)},\hat{j}_z^{(n)}\}$. States of the spin ensemble are elements of the composite Hilbert space

$$\mathcal{H}_N=\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}\otimes\cdots\otimes\mathcal{H}^{(N)} \tag{2}$$

with $\dim(\mathcal{H}_N)=2^N$. Pure states of the ensemble, $|\psi\rangle\in\mathcal{H}_N$, are written as

$$|\psi\rangle=\sum_{m_1,m_2,\ldots,m_N} c_{m_1,m_2,\ldots,m_N}|m_1,m_2,\ldots,m_N\rangle \tag{3}$$

with $m_n=\pm\frac{1}{2}$ and where

$$|m_1,m_2,\ldots,m_N\rangle=\left|\frac{1}{2},m_1\right\rangle_1\otimes\left|\frac{1}{2},m_2\right\rangle_2\otimes\cdots\otimes\left|\frac{1}{2},m_N\right\rangle_N \tag{4}$$

satisfies

$$\hat{j}_z^{(n)}|m_1,m_2,\ldots,m_N\rangle=\hbar m_n|m_1,m_2,\ldots,m_N\rangle. \tag{5}$$

When studying the open-system dynamics of the spin ensemble, one must generally consider the density operator

$$\hat{\rho}=\sum_{\substack{m_1,m_2,\ldots,m_N\\ m_1',m_2',\ldots,m_N'}}\rho_{m_1,m_2,\ldots,m_N;m_1',m_2',\ldots,m_N'}\times|m_1,m_2,\ldots,m_N\rangle\langle m_1',m_2',\ldots,m_N'|. \tag{6}$$

States expanded as in Eqs. (5) and (6) are said to be written in the *product basis*.

A. Representations of the rotation group

For a single spin-1/2 particle, a spatial rotation through the Euler angles $R=(\alpha,\beta,\gamma)$ is described by the rotation operator

$$\hat{R}(\alpha,\beta,\gamma)=e^{-i\alpha\hat{j}_z}e^{-i\beta\hat{j}_y}e^{-i\gamma\hat{j}_z}. \tag{7}$$

The basis kets $|\frac{1}{2},m\rangle$ for this particle therefore transform under the rotation R according to

$$\hat{R}\left|\frac{1}{2},m'\right\rangle=\sum_m\mathcal{D}^{1/2}_{m',m}(R)\left|\frac{1}{2},m\right\rangle, \tag{8}$$

where the matrices $\mathcal{D}^{1/2}(R)$ have the elements

$$\mathcal{D}^{1/2}_{m',m}=\left\langle\frac{1}{2},m'\left|\hat{R}(\alpha,\beta,\gamma)\right|\frac{1}{2},m\right\rangle. \tag{9}$$

The rotation matrices $\mathcal{D}^{1/2}(R)$ form a two-dimensional representation of the rotation group.

For the ensemble of N spin-1/2 particles, each component of the ket $|\psi\rangle=|m_1,m_2,\ldots,m_N\rangle$ transforms separately under a rotation so that an arbitrary state transforms as

$$|\psi'\rangle=[\mathcal{D}^{1/2}(R)]^{\otimes N}|\psi\rangle. \tag{10}$$

The rotation matrices $\mathcal{D}(R)=[\mathcal{D}^{1/2}(R)]^{\otimes N}$ provide a reducible representation for the rotation group but can be decomposed into irreducible representations as

$$\mathcal{D}(R)=\bigoplus_{J=J_{\min}}^{J_{\max}}\bigoplus_{i=1}^{d_N^J}\mathcal{D}^{J,i}(R). \tag{11}$$

The quantum number $i(J)=1,2,\ldots,d_N^J$ is used to distinguish between the

$$d_N^J=\frac{N!(2J+1)}{(N/2-J)!(N/2+J+1)!},\quad J_{\min}\leqslant J\leqslant J_{\max}, \tag{12}$$

degenerate irreps with total angular momentum J [14]. That is to say, d_N^J is the number of ways one can combine N spin-1/2 particles to obtain total angular momentum J. The matrix elements of a given irrep $\mathcal{D}^{J,i}(R)$

$$\mathcal{D}^{J,i}_{M,M'}(R)=\langle J,M,i|\mathcal{D}^{1/2}(R)^{\otimes N}|J,M',i\rangle \tag{13}$$

are written in terms of the total angular momentum eigenstates

$$\hat{\mathbf{J}}^2|J,M,i\rangle=J(J+1)|J,M,i\rangle, \tag{14}$$

$$\hat{J}_z|J,M,i\rangle = M|J,M,i\rangle, \tag{15}$$

with $\hat{J}_z=\Sigma_{n=1}^{N}\hat{j}_z^{(n)}$, $J_{\max}=N/2$, and

$$J_{\min} = \begin{cases} \frac{1}{2}, & N \text{ odd}, \\ 0 & N \text{ even}. \end{cases} \tag{16}$$

It is important to note that degenerate irreps have *identical* matrix elements, i.e.,

$$\langle J,M,i|\mathcal{D}^{1/2}(R)^{\otimes N}|J,M',i\rangle = \langle J,M,i'|\mathcal{D}^{1/2}(R)^{\otimes N}|J,M',i'\rangle \tag{17}$$

for all i,i'.

In this representation, pure states are written as

$$|\psi\rangle = \sum_{J=J_{\min}}^{J_{\max}} \sum_{M=-J}^{J} \sum_{i=1}^{d_N^J} c_{J,M,i}|J,M,i\rangle \tag{18}$$

and mixed states as

$$\hat{\rho} = \sum_{J,J'=J_{\min}}^{J_{\max}} \sum_{M,M'=-J,J'}^{J,J'} \sum_{i,i'=1}^{d_N^J,d_N^{J'}} \rho_{J,M,i;J',M',i'}|J,M,i\rangle\langle J',M',i'|. \tag{19}$$

States written in the form of Eq. (18) or (19) are said to be written in the *irrep basis*. We stress that both the product and irrep bases can describe any arbitrary state in $\mathcal{H}_N$.

III. COLLECTIVE STATES

While the representations in Sec. II allow us to express any state of the ensemble of spin-1/2 particles, the irrep basis suggests a scenario in which we could restrict attention to a much smaller subspace of $\mathcal{H}_N$. In particular, the irrep structure of the rotation group, as expressed in Eq. (11), indicates that rotations on the ensemble do not mix irreps and that degenerate irreps transform identically under a rotation.

Following this line of reasoning, we introduce the *collective states* $|\psi_C\rangle$, which span the sub-Hilbert space $\mathcal{H}_C \subset \mathcal{H}_N$. Collective states have the property that degenerate irreps are identical; for pure states, $c_{J,M,i}=c_{J,M,i'}$ for all i and i'. We note that the symmetric collective states mentioned in the Introduction are the collective states with $c_{J,M,i}=0$ unless $J=N/2$ and thus correspond to the largest J value irrep. We also note that

$$\dim \mathcal{H}_C = \sum_{J=J_{\min}}^{J_{\max}} (2J+1) = \begin{cases} \frac{1}{4}(N+3)(N+1) & \text{if } N \text{ is odd}, \\ \frac{1}{4}(N+2)^2 & \text{if } N \text{ is even}. \end{cases} \tag{20}$$

Physically, the collective states reflect an inability to address different degenerate irreps of the same total J. This new symmetry allows us to effectively ignore the quantum number i and write

$$|\psi_C\rangle = \sum_{J=J_{\min}}^{J_{\max}} \sum_{M=-J}^{J} \sum_{i=1}^{d_N^J} c_{J,M,i}|J,M,i\rangle = \sum_{J=J_{\min}}^{J_{\max}} \sum_{M=-J}^{J} \sqrt{d_N^J} c_{J,M}|J,M\rangle, \tag{21}$$

where we have defined effective basis kets

$$|J,M\rangle = \frac{1}{\sqrt{d_N^J}} \sum_{i=1}^{d_N^J} |J,M,i\rangle \tag{22}$$

with effective amplitude $c_{J,M}=c_{J,M,i}$ for all i (since the $c_{J,M,i}$ are equal for collective states).

The factor of $\sqrt{d_N^J}$ serves as normalization, so that we can apply standard spin-J operators to the effective kets without explicitly referencing their constituent degenerate irrep kets $|J,M,i\rangle$. In other words, $|J,M\rangle$ actually represents d_N^J degenerate kets, each with identical probability amplitude coefficients. But since the matrix elements of a spin-J operator are identical for irreps, we need not evaluate them individually.

As an example, consider a rotation operator $\hat{R}$ which necessarily respects the irrep structure of the rotation group. Calculating the expectation value of $\hat{R}$ by expanding the collective state $|\psi_C\rangle$ in the full irrep basis, we have

$$\langle\psi_C|\hat{R}|\psi_C\rangle = \sum_{J,J'} \sum_{M,M'} \sum_{i,i'} c^*_{J,M,i} c_{J',M',i'} \langle J,M,i|\hat{R}|J',M',i'\rangle \tag{23}$$

$$= \sum_{J} \sum_{M,M'} \sum_{i} c^*_{J,M,i} c_{J,M',i} \langle J,M,i|\hat{R}|J,M',i\rangle \tag{24}$$

$$= \sum_{J} \sum_{M,M'} d_N^J c^*_{J,M} c_{J,M'} \langle J,M|\hat{R}|J,M'\rangle, \tag{25}$$

where in going from Eq. (23) to Eq. (24) we set $J=J'$ and $i=i'$ since rotation group elements do not mix irreps. In reaching Eq. (25), we have further used the collective state property that $c_{J,M,i}=c_{J,M,i'}\ \forall\ i,i'$ and the rotation irrep property that $\langle J,M,i|\hat{R}|J,M',i\rangle=\langle J,M,i'|\hat{R}|J,M',i'\rangle\ \forall\ i,i'$ to drop the index i.

Equivalently we can evaluate the expectation using the effective basis kets $|J,M\rangle$ directly:

$$\langle\psi_C|\hat{R}|\psi_C\rangle = \sum_{J,J'} \sum_{M,M'} \sqrt{d_N^J}\sqrt{d_N^{J'}} c^*_{J,M} c_{J',M'} \langle J,M|\hat{R}|J',M'\rangle \tag{26}$$

$$= \sum_{J} \sum_{M,M'} d_N^J c^*_{J,M} c_{J,M'} \langle J,M|\hat{R}|J,M'\rangle. \tag{27}$$

Comparing this to Eq. (25) and recalling that $c_{J,M}=c_{J,M,i}$ for all i, we see that the effective calculation gives the same result.

We can similarly define collective state density operators $\hat{\rho}_C$ that have the properties that (i) there are no coherences between different irrep blocks and (ii) degenerate irrep blocks have identical density matrix elements. The second assumption again means we can effectively drop the index i, since $\rho_{J,M,i;J,M',i}=\rho_{J,M,i';J,M',i'}$ for any i and i'. This allows us to write

$$\hat{\rho}_C = \sum_{J=J_{\min}}^{J_{\max}} \sum_{M,M'=-J}^{J} \rho_{J,M;J,M'} \overline{|J,M\rangle\langle J,M'|} \tag{28}$$

where the effective density matrix elements, written using an outer product with overbar, are related to the irrep matrix elements via

$$\rho_{J,M;J,M'}\overline{|J,M\rangle\langle J,M'|} \coloneqq \frac{1}{d_N^J}\sum_{i=1}^{d_n^J} \rho_{J,M,i;J,M',i}|J,M,i\rangle\langle J,M',i|. \tag{29}$$

Just as for the effective kets, the normalization factor of d_N^J ensures that expectations are correctly calculated using the standard spin-J operators. The density matrix has $\sum_{J=J_{\min}}^{J_{\max}}(2J+1)^2=\frac{1}{6}(N+3)(N+2)(N+1)$ elements.

We stress that the outer product notation with overbar is *different* from naively taking the outer product of the effective kets defined in Eq. (22). Such an approach would involve outer products of kets between different, although degenerate, irreps. Such terms are strictly forbidden by the first property of collective state density operators. Instead, one should consider the effective density operator as a representation of d_N^J identical copies of a spin-J particle. The overbar notation is meant to remind the reader that the outer product beneath should only be interpreted using Eq. (29) to relate back to the irrep basis.

IV. COLLECTIVE PROCESSES

We are now interested in describing quantum processes $\mathcal{L}$ that preserve collective states $\hat{\rho}_C'=\mathcal{L}\hat{\rho}_C$. Writing this explicitly, we must have

$$\sum_{J_1}\sum_{M_1,M_1'} d_N^{J_1}\rho'_{J_1,M_1;J_1,M_1'}\overline{|J_1,M_1\rangle\langle J_1,M_1'|} = \sum_{J_2}\sum_{M_2,M_2'} d_N^{J_2}\rho_{J_2,M_2;J_2,M_2'}\mathcal{L}\overline{|J_2,M_2\rangle\langle J_2,M_2'|}. \tag{30}$$

If we define the action of $\mathcal{L}$ on collective density matrix elements as

$$f^{J,M,M'} = \mathcal{L}\overline{|J,M\rangle\langle J,M'|}, \tag{31}$$

we immediately see that this action must be expressible as

$$f^{J,M,M'} = \sum_{J_1}\sum_{M_1,M_1'} \lambda^{J,M,M'}_{J_1,M_1,M_1'}\overline{|J_1,M_1\rangle\langle J_1,M_1'|} \tag{32}$$

in order for the equality in Eq. (30) to be met. Here $\lambda^{J,M,M'}_{J_1,M_1,M_1'}$ is an arbitrary function of its indices. Any process that preserves collective states by satisfying Eq. (32) is a *collective process*.

Examples of collective processes are those involving *collective angular momentum operators* $\{\hat{J}_x,\hat{J}_y,\ldots\}$ and, more generally, arbitrary *collective operators* $\hat{C}=\sum_{n=1}^{N}\hat{c}^{(n)}$. Since collective operators correspond to precisely the rotations considered when the irrep structure of the rotation group is defined, they can all be written as

$$\hat{C} = \sum_J \sum_{M,M'} c_{J,M,M'}\overline{|J,M\rangle\langle J,M'|}, \tag{33}$$

which cannot couple effective matrix elements with different J.

However, the collective operators define a more restrictive class than an arbitrary collective process, which *can* couple different J blocks, so long as it does not create coherences between them. In fact, if all operators are collective, then the symmetric collective states ($|\psi_S\rangle$) span an invariant subspace of the map. This holds even when considering Lindblad operators that are written in terms of collective operators,

$$\mathcal{L}[\hat{S}]\hat{\rho} = \left[\hat{S}\hat{\rho}\hat{S}_{\dagger} - \frac{1}{2}\hat{S}^{\dagger}\hat{S}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{S}^{\dagger}\hat{S}\right], \tag{34}$$

where $\hat{S}=\sum_n\hat{s}^{(n)}$.

In the following section, we demonstrate that a process of the form

$$f^{J,M,M'} = \sum_{n=1}^{N}\hat{s}^{(n)}\overline{|J,M\rangle\langle J,M'|}(\hat{t}^{(n)})^{\dagger}, \tag{35}$$

which cannot be written solely in terms of collective operators, is nonetheless a collective process. Moreover, if we expand the operators in the spherical Pauli basis via $\hat{s}=\vec{s}\cdot\vec{\sigma}$ and $\hat{t}^{\dagger}=\vec{t}\cdot\vec{\sigma}^{\dagger}$, we find

$$f^{J,M,M'} = \vec{s}\cdot\mathbf{g}(J,M,M',N)\cdot\vec{t} \tag{36}$$

with the tensor $\mathbf{g}(J,M,M',N)$ defined as

$$\mathbf{g}_{qr}(J,M,M',N) = \sum_{n=1}^{N}\hat{\sigma}_q^{(n)}\overline{|J,M\rangle\langle J,M'|}(\hat{\sigma}_r^{(n)})^{\dagger}. \tag{37}$$

The tensor is written as a function of N to coincide with the notation in the following section.

Before deriving a closed form expression for $\mathbf{g}(J,M,M',N)$, we would like to relate it to modeling *symmetric decoherence processes*, which take the form In order to relate $\mathcal{L}[\hat{s}]$ to Eqs. (35) and (37), set $\hat{t}=\hat{s}$ and expand the single spin operator $\hat{s}$ in the spherical Pauli basis

$$\hat{s} = s_I\hat{I} + \sum_q s_q\hat{\sigma}_q = s_I\hat{I} + s_+\hat{\sigma}_+ + s_-\hat{\sigma}_- + s_z\hat{\sigma}_z \tag{38}$$

with the convention $\hbar=1$, $\hat{\sigma}_+=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $\hat{\sigma}_-=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, and $\hat{\sigma}_z=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. The symmetric Lindblad operator of Eq. (37) can be expanded as

$$\mathcal{L}[\hat{s}]\hat{\rho} = \sum_{n=1}^{N}\left(\hat{s}^{(n)}\hat{\rho}(\hat{s}^{(n)})^{\dagger} - \frac{1}{2}(\hat{s}^{(n)})^{\dagger}\hat{s}^{(n)}\hat{\rho} - \frac{1}{2}\hat{\rho}(\hat{s}^{(n)})^{\dagger}\hat{s}^{(n)}\right) = \sum_{n=1}^{N}[\hat{s}^{(n)}\hat{\rho}(\hat{s}^{(n)})^{\dagger}] - \frac{1}{2}\hat{S}_N\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{S}_N \tag{39}$$

with the collective operator $\hat{S}_N$ given by

$$
\begin{aligned}
\hat{S}_N &= \sum_{n=1}^{N} (\hat{s}^{(n)})^{\dagger}\hat{s}^{(n)} \\
&= \left(\frac{1}{2}|s_-|^2 + \frac{1}{2}|s_+|^2 + |s_I|^2 + |s_z|^2\right)N\hat{I} \\
&\quad + (s_-^* s_I - s_-^* s_z + s_I^* s_+ + s_z^* s_+)\hat{J}_+ \\
&\quad + (s_I^* s_- + s_+^* s_I + s_+^* s_Z - s_z^* s_-)\hat{J}_- \\
&\quad + \left(\frac{1}{2}|s_-|^2 - \frac{1}{2}|s_+|^2 + s_I^* s_z + s_z^* s_I\right)\hat{J}_z, \qquad (40)
\end{aligned}
$$

and $\hat{J}_q = \Sigma_{n=1}^{N}\hat{\sigma}_q^{(n)}$ a collective spin operator.

In this form, it is clear that only the first term of the symmetric Lindbladian is not written using collective operators. In fact, if we again expand $\hat{s}^{(n)}$ in the spherical basis, we observe that the only terms which involve noncollective operators are those which do not involve the identity operator,

$$
\begin{aligned}
\sum_{n=1}^{N} [\hat{s}^{(n)}\hat{\rho}(\hat{s}^{(n)})^{\dagger}] &= |s_I|^2 N\hat{\rho} + \sum_{q} (s_q s_I^* \hat{J}_q\hat{\rho} + s_I s_q^* \hat{\rho}\hat{J}_q^{\dagger}) \\
&\quad + \sum_{n=1}^{N}\left(\sum_{q,r} s_q s_r^* \hat{\sigma}_q^{(n)}\hat{\rho}(\hat{\sigma}_q^{(n)})^{\dagger}\right). \qquad (41)
\end{aligned}
$$

The last term here is precisely the tensor evaluation of $\vec{s}\cdot\mathbf{g}(J,M,M',N)\cdot\vec{s}^*$. We now proceed to give an identity for the tensor elements.

V. IDENTITY

Identity 1. Given a collective density matrix element for N spin-1/2 particles, $\overline{|J,M\rangle\langle J,M'|}$, we have

$$
\begin{aligned}
\mathbf{g}_{qr}(J,M,M',N) &= \sum_{n=1}^{N} \hat{\sigma}_q^{(n)}\overline{|J,M\rangle\langle J,M'|}(\hat{\sigma}_r^{(n)})^{\dagger} \\
&= \frac{1}{2J}\left(1 + \frac{\alpha_N^{J+1}}{d_N^J}\frac{2J+1}{J+1}\right) \\
&\quad \times A_q^{J,M}\overline{|J,M_q\rangle\langle J,M_r'|}A_r^{J,M'} \\
&\quad + \frac{\alpha_N^J}{d_N^J 2J}B_q^{J,M}\overline{|J-1,M_q\rangle\langle J-1,M_r'|}B_r^{J,M'} + \frac{\alpha_N^{J+1}}{d_N^J 2(J+1)} \\
&\quad \times D_q^{J,M}\overline{|J+1,M_q\rangle\langle J+1,M_r'|}D_r^{J,M'}, \qquad (42)
\end{aligned}
$$

where $q,r\in\{+,-,z\}$, $M_+=M+1$, $M_-=M-1$, and $M_z=M$,

$$
\alpha_N^J = \sum_{J'=J}^{N/2} d_N^{J'} = \frac{N!}{(N/2-J)!(N/2+J)!}, \qquad (43)
$$

and

$$
A_+^{J,M} = \sqrt{(J-M)(J+M+1)}, \qquad (44a)
$$

$$
A_-^{J,M} = \sqrt{(J+M)(J-M+1)}, \qquad (44b)
$$

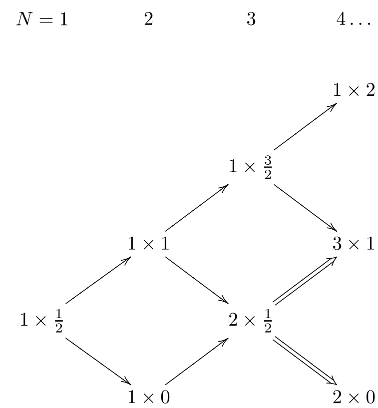

FIG. 1. Degeneracy structure from adding spin-1/2 particles, labeled as $d_N^J\times J$.

$$
A_z^{J,M} = M, \qquad (44c)
$$

and

$$
B_+^{J,M} = \sqrt{(J-M)(J-M-1)}, \qquad (45a)
$$

$$
B_-^{J,M} = -\sqrt{(J+M)(J+M-1)}, \qquad (45b)
$$

$$
B_z^{J,M} = \sqrt{(J+M)(J-M)}, \qquad (45c)
$$

and lastly

$$
D_+^{J,M} = -\sqrt{(J+M+1)(J+M+2)}, \qquad (46a)
$$

$$
D_-^{J,M} = \sqrt{(J-M+1)(J-M+2)}, \qquad (46b)
$$

$$
D_z^{J,M} = \sqrt{(J+M+1)(J-M+1)}. \qquad (46c)
$$

Note that α_N^J and d_N^J are zero if J is negative or $J=N/2$, ensuring that only valid density matrix elements are involved.

In the following sections, we prove Identity 1 inductively. The motivation for the inductive proof comes from the simple recursive structure of adding spin-1/2 particles. As seen in Fig. 1, the d_N^J irreps that correspond to a total spin-J particle composed of N spin-1/2 particles can be split into two groups, depending on how angular momentum was added to reach them. By expressing the N-particle states in terms of bipartite states of a single spin-1/2 particle and a spin-$(N-1)$ particle, we can then evaluate the dynamics independently on either half by assuming that Identity 1 holds. Returning the resulting state to the N-particle basis should then confirm the identity. By inspection, the base case of $N=1$ holds, as the $A_q^{J,M}$ terms reduce to the single spin-1/2 matrix elements. We now proceed to the inductive case.

A. Recursive state structure

In order to apply the inductive hypothesis, we need to express an N-particle state in terms of $(N-1)$-particle states. This recursive structure is best seen by examining Fig. 1, which illustrates the branching structure for adding spin-1/2 particles. For example, the threefold-degenerate $N=4$ spin-1 irreps arise from two different spin additions—addition of a single spin-1/2 particle to the nondegenerate $J=\frac{3}{2}$, $N=3$ irrep and addition to the twofold-degenerate $J=\frac{1}{2}$, $N=3$ irreps. Since we are always adding a spin-1/2 particle, the tree is at most binary. This allows us to recursively decompose the degenerate irreps for a given J in terms of adding a single spin-1/2 particle to the two related $(N-1)$-degenerate irreps.

Recall that for the collective states we defined effective density matrix elements which group degenerate irreps [Eq. (29)]. In order to make the relationship between states of different N clear, in this section we will add the index N to all effective density matrix elements—$\overline{|J,M,N\rangle\langle J,M',N|}$. Similarly, when the collective state is expressed in the irrep basis, we will also use kets with the index N, i.e., $|J,M,N,i\rangle$. Here, the N and i indices indicate that the state is from the ith degenerate total-spin-J irrep that comes from adding N spin-1/2 particles. So that we can leverage the binary branching structure seen in Fig. 1, we also need to relate the N-particle irrep states to the $(N-1)$-particle irrep states. Accordingly, we define $|J,M;\frac{1}{2},J\pm\frac{1}{2},N-1,i_1\rangle$, where the last four entries indicate that the overall N-spin state can be viewed as combining a single spin-1/2 particle with a spin-$(J\pm\frac{1}{2})$ particle. The spin-$(J\pm\frac{1}{2})$ particle is from the i_1th such irrep for $N-1$ spin-1/2 particles. With these definitions, we can now relate the N-particle states to the $(N-1)$-particle states by explicitly tensoring out a single spin-1/2 particle:

$$\overline{|J,M,N\rangle\langle J,M',N|} = \frac{1}{d_N^J}\sum_{i=1}^{d_N^J} |J,M,N,i\rangle\langle J,M',N,i| \tag{47}$$

$$\begin{aligned} &= \frac{1}{d_N^J}\sum_{i_1=1}^{d_{N-1}^{J+1/2}} \left|J,M;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right| \\ &\quad + \frac{1}{d_N^J}\sum_{i_2=1}^{d_{N-1}^{J-1/2}} \left|J,M;\frac{1}{2},J-\frac{1}{2},N-1,i_2\right\rangle\left\langle J,M';\frac{1}{2},J-\frac{1}{2},N-1,i_2\right| \end{aligned} \tag{48}$$

$$\begin{aligned} &= \frac{d_{N-1}^{J+1/2}}{d_N^J}\sum_{m_1,m_1'} {}^{J,M}C^{1/2,m_1}_{J+1/2,M-m_1}\left|\frac{1}{2},m_1\right\rangle\left\langle\frac{1}{2},m_1'\right| \\ &\quad \otimes \overline{\left|J+\frac{1}{2},M-m_1,N-1\right\rangle\left\langle J+\frac{1}{2},M'-m_1',N-1\right|}\,{}^{J,M'}C^{1/2,m_1'}_{J+1/2,M'-m_1'} \\ &\quad + \frac{d_{n-1}^{J-1/2}}{d_N^J}\sum_{m_2,m_2'} {}^{J,M}C^{1/2,m_2}_{J-1/2,M-m_2}\left|\frac{1}{2},m_2\right\rangle\left\langle\frac{1}{2},m_2'\right| \\ &\quad \otimes \overline{\left|J-\frac{1}{2},M-m_2,N-1\right\rangle\left\langle J-\frac{1}{2},M'-m_2',N-1\right|}\,{}^{J,M'}C^{1/2,m_2'}_{J-1/2,M'-m_2'} \end{aligned} \tag{49}$$

with Clebsch-Gordan coefficients ${}^{J,M}C^{j_1,m_1}_{j_2,m_2}=\langle J,M;j_1,j_2|j_1,m_1;j_2,m_2\rangle$ and the m_i,m_i' sums over single-spin projection values $\pm\frac{1}{2}$. In reaching Eq. (50), we made use of the definition of the effective density matrix element for $N-1$ spins given in Eq. (29). With this recursive state definition, we can now start the inductive step of the proof.

B. Application of the inductive hypothesis

In order to prove Identity 1, we must be able to apply the inductive hypothesis to Eq. (42). Ignoring the Clesbsch-Gordan coefficients for the moment, consider an arbitrary term from Eq. (52). The dynamics are distributed as

$$\begin{aligned} &\sum_{n=1}^{N}\sigma_q^{(n)}\left(\left|\frac{1}{2},m_i\right\rangle\left\langle\frac{1}{2},m_i'\right| \otimes \overline{\left|J\pm\frac{1}{2},M-m_i,N-1\right\rangle\left\langle J\pm\frac{1}{2},M-m_i',N-1\right|}\right)\sigma_r^{(n)} \\ &= \mathbf{g}_{qr}\left(\frac{1}{2},m_i,m_i',1\right)\otimes\overline{\left|J\pm\frac{1}{2},M-m_i,N-1\right\rangle\left\langle J\pm\frac{1}{2},M-m_i',N-1\right|} \\ &\quad + \left|\frac{1}{2},m_i\right\rangle\left\langle\frac{1}{2},m_i'\right|\otimes\mathbf{g}_{qr}\left(J\pm\frac{1}{2},M-m_i,M'-m_i',N-1\right). \end{aligned} \tag{50}$$

By extension, all terms in Eq. (50) split the dynamics in this manner, which allows us to apply the inductive hypothesis to evaluate $\mathbf{g}_{qr}(\frac{1}{2},m_i,m_i',1)$ and $\mathbf{g}_{qr}(J\pm\frac{1}{2},M-m_i,M'-m_i',N-1)$. This means evaluating the $\mathbf{g}_{qr}$ terms according to the hypothesis in Eq. (42), after which we rewrite the bipartite states in the N-spin basis.

We have the $\mathbf{g}_{qr}(\frac{1}{2},m_1,m_1',1)$ terms

$$\frac{1}{d_N^J}\sum_{i_1=1}^{d_{N-1}^{J+1/2}}\sum_{J_1=J}^{J+1}\sum_{m_1}\left(A_q^{1/2,m_1}\,{}^{J_1,M_q}G^{1/2,m_1}_{J+1/2,M-m_1}\,{}^{J,M}C^{1/2,m_1}_{J+1/2,M-m_1}\left|J_1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\right)\times\sum_{J_1'=J}^{J+1}\sum_{m_1'}\left(\left\langle J_1',M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|{}^{J,M'}C^{1/2,m_1'}_{J+1/2,M'-m_1'}\,{}^{J_1',M_r'}C^{1/2,m'_{1_r}}_{J+1/2,M'-m_1'}A_r^{1/2,m_1'}\right) \tag{51}$$

and the $\mathbf{g}_{qr}(\frac{1}{2},m_2,m_2',1)$ terms

$$\frac{1}{d_N^J}\sum_{i_1=1}^{d_{N-1}^{J+1/2}}\sum_{J_2=J-1}^{J}\sum_{m_2}\left(A_q^{1/2,m_2}\,{}^{J_2,M_q}G^{1/2,m_2}_{J-1/2,M-m_2}\,{}^{J,M}C^{1/2,m_2}_{J-1/2,M-m_2}\left|J_2,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_2\right\rangle\right)\times\sum_{J_2'=J-1}^{J}\sum_{m_2'}\left(\left\langle J_2',M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_2\right|{}^{J,M'}C^{1/2,m_2'}_{J-1/2,M'-m_2'}\,{}^{J_2',M_r'}C^{1/2,m'_{2_r}}_{J+1/2,M'-m_2'}A_r^{1/2,m_1'}\right). \tag{52}$$

The $\mathbf{g}_{qr}(J+\frac{1}{2},M-m_1,M'-m_1',N-1)$ terms are

$$\frac{1}{d_N^J(2J+1)}\left(1+\frac{\alpha_{N-1}^{J+3/2}}{d_{N-1}^{J+1/2}}\frac{2J+2}{J+3/2}\right)\sum_{i_1=1}^{d_{N-1}^{J+1/2}}\sum_{J_1=J}^{J+1}\sum_{m_1}\left(A_q^{J+1/2,M-m_1}\,{}^{J_1,M_q}C^{1/2,m_1}_{J+1/2,M_q-m_1}\,{}^{J,M}C^{1/2,m_1}_{J+1/2,M-m_1}\left|J_1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\right)\times\sum_{J_1'=J}^{J+1}\sum_{m_1'}\left(\left\langle J_1',M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|{}^{J,M'}C^{1/2,m_1'}_{J+1/2,M'-m_1'}\,{}^{J_1',M_r'}C^{1/2,m_1'}_{J+1/2,M_r'-m_1'}A_r^{J+1/2,M'-m_1'}\right) \tag{53}$$

$$+\frac{\alpha_{N-1}^{J+1/2}}{d_N^Jd_{N-1}^{J-1/2}2(J+1/2)}\sum_{i_1=1}^{d_{N-1}^{J-1/2}}\sum_{J_1=J-1}^{J}\sum_{m_1}\left(B_q^{J+1/2,M-m_1}\,{}^{J_1,M_q}C^{1/2,m_1}_{J-1/2,M_q-m_1}\,{}^{J,M}C^{1/2,m_1}_{J+1/2,M-m_1}\left|J_1,M_q;\frac{1}{2},J-\frac{1}{2},N-1,i_1\right\rangle\right)\times\sum_{J_1'=J-1}^{J}\sum_{m_1'}\left(\left\langle J_1',M_r';\frac{1}{2},J-\frac{1}{2},N-1,i_1\right|{}^{J_1,M'}C^{1/2,m_1'}_{J+1/2,M'-m_1'}\,{}^{J_1',M_r'}C^{1/2,m_1'}_{J-1/2,M_r'-m_1'}B_r^{J+1/2,M'-m_1'}\right) \tag{54}$$

$$+\frac{\alpha_{N-1}^{J+3/2}}{d_N^Jd_{N-1}^{J+3/2}2(J+3/2)}\sum_{i_1=1}^{d_{N-1}^{J+3/2}}\sum_{J_1=J+1}^{J+2}\sum_{m_1}\left(D_q^{J+1/2,M-m_1}\,{}^{J_1,M_q}C^{1/2,m_1}_{J+3/2,M_q-m_1}\,{}^{J,M}C^{1/2,m_1}_{J+1/2,M-m_1}\left|J_1,M_q;\frac{1}{2},J+\frac{3}{2},N-1,i_1\right\rangle\right)\times\sum_{J_1'=J+1}^{J+2}\sum_{m_1'}\left(\left\langle J_1',M_r';\frac{1}{2},J+\frac{3}{2},N-1,i_1\right|{}^{J_1,M'}C^{1/2,m_1'}_{J+1/2,M'-m_1'}\,{}^{J_1',M_r'}C^{1/2,m_1'}_{J+3/2,M_r'-m_1'}D_r^{J+1/2,M'-m_1'}\right) \tag{55}$$

and, lastly, the $\mathbf{g}_{qr}(J-\frac{1}{2},M-m_2,M'-m_2',N-1)$ terms are

$$\frac{1}{d_N^J2(J-1/2)}\left(1+\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J-1/2}}\frac{2J}{J+1/2}\right)\sum_{i_2=1}^{d_{N-1}^{J-1/2}}\sum_{J_2=J-1}^{J}\sum_{m_2}\left(A_q^{J-1/2,M-m_2}\,{}^{J_2,M_q}C^{1/2,m_2}_{J-1/2,M_q-m_2}\,{}^{J,M}C^{1/2,m_2}_{J-1/2,M-m_2}\left|J_2,M_q;\frac{1}{2},J-\frac{1}{2},N-1,i_2\right\rangle\right)\times\sum_{J_2=J-1}^{J}\sum_{m_2'}\left(\left|J_2',M_r';\frac{1}{2},J-\frac{1}{2},N-1,i_2\right\rangle{}^{J,M'}C^{1/2,m_2'}_{J-1/2,M'-m_2'}\,{}^{J_2',M_r'}C^{1/2,m_2'}_{J-1/2,M_r'-m_2'}A_r^{J-1/2,M'-m_2'}\right) \tag{56}$$

$$+\frac{\alpha_{N-1}^{J-1/2}}{d_N^Jd_{N-1}^{J-3/2}2(J-1/2)}\sum_{i_2=1}^{d_{N-1}^{J-3/2}}\sum_{J_2=J-2}^{J-1}\sum_{m_2}\left(B_q^{J-1/2,M-m_2}\,{}^{J_2,M_q}C^{1/2,m_2}_{J-3/2,M_q-m_2}\,{}^{J,M}C^{1/2,m_2}_{J-1/2,M-m_2}\left|J_2,M_q;\frac{1}{2},J-\frac{3}{2},N-1,i_2\right\rangle\right)\times\sum_{J_2=J-2}^{J-1}\sum_{m_2'}\left(\left|J_2',M_r';\frac{1}{2},J-\frac{3}{2},N-1,i_2\right\rangle{}^{J,M'}C^{1/2,m_2'}_{J-1/2,M'-m_2'}\,{}^{J_2',M_r'}C^{1/2,m_2'}_{J-3/2,M_r'-m_2'}B_r^{J-1/2,M'-m_2'}\right) \tag{57}$$

$$\frac{\alpha_{N-1}^{J+1/2}}{d_N^J d_{N-1}^{J+1/2}(2J+1)}\sum_{i_2=1}^{d_{N-1}^{J+1/2}}\sum_{J_2=J}^{J+1}\sum_{m_2}\left(D_q^{J-1/2,M-m_2}\,{}^{J_2,M_q}C^{1/2,m_2}_{J+1/2,M_q-m_2}\,{}^{J,M}C^{1/2,m_2}_{J-1/2,M-m_2}\left|J_2,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_2\right\rangle\right)$$
$$\times\sum_{J_2'=J}^{J+1}\sum_{m_2'}\left(\left|J_2',M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_2\right\rangle{}^{J,M'}C^{1/2,m_2'}_{J-1/2,M'-m_2'}\,{}^{J_2',M_r'}C^{1/2,m_2'}_{J+1/2,M_r'-m_2'}D_r^{J-1/2,M-m_2'}\right). \tag{58}$$

C. Evaluation of the sums

We are now tasked with showing that the terms (52)–(59) sum to $\mathbf{g}_{qr}(J,M,M',N)$ as written in Eq. (42). Before doing so, we observe that the J_i,m_i and J_i',m_i' sums factor in all the equations above. Moreover, if one replaces primed quantities with unprimed ones, the Clebsch-Gordan and A,B,D coefficients of the kets in a given J_i,m_i sum are identical to those of the bras in the related J_i',m_i' sum. Therefore, we focus on simplifying the unprimed sums and then apply those results to the primed sums in order to simplify Eqs. (52)–(59). In the Appendix, we explicitly calculate two representative sums from these equations. The calculations involve manipulating products of Clebsch-Gordan and A,B,D coefficients. Although tedious, the interested and pertinacious reader should have no trouble evaluating them for all relevant sums, finding in particular that the $J\pm 2$ terms vanish. We forego detailing all those manipulations here and simply use the results in both the primed and unprimed terms of the equations above, which then simplify Eq. (52) to

$$\frac{1}{d_N^J(2J+2)^2}\sum_{i_1=1}^{d_{N-1}^{J+1/2}}D_q^{J,M}\left|J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|D_r^{J,M'}$$
$$-A_q^{J,M}\left|J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|D_r^{J,M'}$$
$$-D_q^{J,M}\left|J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|A_r^{J,M'}$$
$$+A_q^{J,M}\left|J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|A_r^{J,M'}, \tag{59}$$

Eq. (53) to

$$\frac{1}{d_N^J 4J^2}\sum_{i_1=1}^{d_{N-1}^{J-1/2}}B_q^{J,M}\left|J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1\right\rangle\left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1\right|B_r^{J,M'}$$
$$+A_q^{J,M}\left|J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1\right\rangle\left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1\right|B_r^{J,M'}$$
$$+B_q^{J,M}\left|J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1\right|A_r^{J,M'}$$
$$+A_q^{J,M}\left|J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1\right|A_r^{J,M'}, \tag{60}$$

Eq. (54) to

$$\frac{1}{d_N^J(2J+1)}\left(1+\frac{\alpha_{N-1}^{J+3/2}}{d_{N-1}^{J+1/2}}\frac{2J+2}{J+3/2}\right)\sum_{i_1=1}^{d_{N-1}^{J+1/2}}\frac{1}{(2J+2)^2}D_q^{J,M}\left|J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|D_r^{J,M'}$$
$$-\frac{2(J+3/2)}{(2J+2)^2}A_q^{J,M}\left|J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|D_r^{J,M'}$$
$$-\frac{2(J+3/2)}{(2J+2)^2}D_q^{J,M}\left|J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|A_r^{J,M'}$$
$$+\frac{(J+3/2)^2}{(J+1)^2}A_q^{J,M}\left|J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|A_r^{J,M'}, \tag{61}$$

Eq. (55) to

$$\frac{\alpha_{N-1}^{J+1/2}}{d_N^J d_{N-1}^{J-1/2} 2J(J+1/2)} \sum_{i_1=1}^{d_{N-1}^{J-1/2}} (J+1)B_q^{J,M} \left| J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| B_r^{J,M'}$$
$$+ A_q^{J,M} \left| J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| B_r^{J,M'}$$
$$+ B_q^{J,M} \left| J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| A_r^{J,M'}$$
$$+ \frac{1}{J+1} A_q^{J,M} \left| J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| A_r^{J,M'}, \tag{62}$$

Eq. (56) to (since $J+2$ terms vanish)

$$\frac{\alpha_N^{J+1}}{d_N^J 2(J+1)} \frac{1}{d_N^{J+1}} \sum_{i_1=1}^{d_{N-1}^{J+3/2}} D_q^{J,M} \left| J+1,M_q;\frac{1}{2},J+\frac{3}{2},N-1,i_1 \right\rangle \left\langle J+1,M_r';\frac{1}{2},J+\frac{3}{2},N-1,i_1 \right| D_r^{J,M'}, \tag{63}$$

Eq. (57) to

$$\frac{1}{d_N^J 4J^2 2(J-1/2)} \left(1 + \frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J-1/2}} \frac{2J}{J+1/2}\right) \sum_{i_1=1}^{d_{N-1}^{J-1/2}} B_q^{J,M} \left| J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| B_r^{J,M'}$$
$$-2\left(J-\frac{1}{2}\right) A_q^{J,M} \left| J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| B_r^{J,M'}$$
$$-2\left(J-\frac{1}{2}\right) B_q^{J,M} \left| J-1,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| A_r^{J,M'}$$
$$+4\left(J-\frac{1}{2}\right)^2 A_q^{J,M} \left| J,M_q;\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J-\frac{1}{2};N-1,i_1 \right| A_r^{J,M'}, \tag{64}$$

Eq. (58) to (since $J-2$ terms vanish)

$$\frac{\alpha_N^J}{d_N^J 2J} \frac{1}{d_N^{J-1}} \sum_{i_1=1}^{d_{N-1}^{J-3/2}} B_q^{J,M} \left| J-1,M_q;\frac{1}{2},J-\frac{3}{2},N-1,i_1 \right\rangle \left\langle J-1,M_r';\frac{1}{2},J-\frac{3}{2},N-1,i_1 \right| B_r^{J,M'}, \tag{65}$$

and Eq. (59) to

$$\frac{\alpha_{N-1}^{J+1/2}}{d_N^J d_{N-1}^{J+1/2}(2J+1)} \sum_{i_1=1}^{d_{N-1}^{j+1/2}} \frac{J}{J+1} D_q^{J,M} \left| J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right\rangle \left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right| D_r^{J,M'}$$
$$+ \frac{1}{J+1} A_q^{J,M} \left| J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right\rangle \left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right| D_r^{J,M'}$$
$$+ \frac{1}{J+1} D_q^{J,M} \left| J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right| A_r^{J,M'}$$
$$+ \frac{1}{J(J+1)} A_q^{J,M} \left| J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right\rangle \left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1 \right| A_r^{J,M'}. \tag{66}$$

D. Recovery of $\mathbf{g}_{qr}(J,M,M',N)$

We now combine the equations from the previous subsection to recover Identity 1 in Eq. (42). Given that density operators in the collective state representation lack coherences between different J irreps, we expect $|J\pm 1\rangle\langle J|$ and $|J\rangle\langle J\pm 1|$ terms to vanish. Since both the $|J\rangle\langle J\pm 1|$ and $|J\pm 1\rangle\langle J|$ terms have the same coefficients, we need only explicitly deal with one of the two. Starting with $|J+1\rangle\langle J|$ coefficients from Eqs. (60), (62), and (67), we find

$$\frac{1}{d_N^J}\left[-\frac{1}{(2J+2)^2}-\frac{(2J+3)}{(2J+1)(2J+2)^2}\left(1+\frac{\alpha_{N-1}^{J+3/2}}{d_{N-1}^{J+1/2}}\frac{2J+2}{J+3/2}\right)+\frac{1}{(J+1)(2J+1)}\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J+1/2}}\right]$$

$$=\frac{1}{d_N^J(2J+2)^2}\left(-1-\frac{(N+1)}{2J+1}+\frac{N+2J+2}{2J+1}\right) \quad (67)$$

$$=0. \quad (68)$$

Similarly, for $|J-1\rangle\langle J|$ coefficients in Eqs. (61), (63), and (65), we have

$$\frac{1}{d_N^J 4J^2}\left[1+\frac{\alpha_{N-1}^{J+1/2}2J}{d_{N-1}^{J-1/2}(J+1/2)}-\left(1+\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J-1/2}}\frac{2J}{J+1/2}\right)\right]=0. \quad (69)$$

Turning to the $J+1$ terms from Eqs. (60), (62), and (67), the coefficients sum to

$$\frac{1}{d_N^J}\left[\frac{1}{(2J+2)^2}+\frac{1}{(2J+1)(2J+2)^2}\left(1+\frac{\alpha_{N-1}^{J+3/2}}{d_{N-1}^{J+1/2}}\frac{2J+2}{J+3/2}\right)+\frac{J}{(J+1)(2J+1)}\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J+1/2}}\right]$$

$$=\frac{1}{d_N^J(2J+2)^2}\left(1+\frac{N+1}{(2J+3)(2J+1)}+\frac{J(N+2J+2)}{2J+1}\right) \quad (70)$$

$$=\frac{1}{d_N^J}\frac{2J+N+4}{8J^2+20J+12}=\frac{1}{d_N^J 2(J+1)}\frac{\alpha_N^{J+1}}{d_N^{J+1}}, \quad (71)$$

which gives overall

$$\frac{\alpha_N^{J+1}}{d_N^J 2(J+1)}\frac{1}{d_N^{J+1}}\sum_{i_1=1}^{d_N^{J+1/2}} D_q^{J,M}\left|J+1,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\left\langle J+1,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|D_r^{J,M'}. \quad (72)$$

The J terms from Eqs. (60), (62), and (67) have coefficients

$$\frac{1}{d_N^J}\left[\frac{1}{(2J+2)^2}+\frac{(2J+3)^2}{(2J+1)(2J+2)^2}\left(1+\frac{\alpha_{N-1}^{J+3/2}}{d_{N-1}^{J+1/2}}\frac{2J+2}{J+3/2}\right)+\frac{1}{J(J+1)(2J+1)}\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J+1/2}}\right]$$

$$=\frac{1}{d_N^J(2J+2)^2}\left(1+\frac{(N+1)(2J+3)}{2J+1}+\frac{N+2J+2}{J(2J+1)}\right) \quad (73)$$

$$=\frac{1}{d_N^J 2J}\left(1+\frac{\alpha_N^{J+1}}{d_N^J}\frac{2J+1}{J+1}\right), \quad (74)$$

which gives overall

$$\frac{1}{2J}\left(1+\frac{\alpha_N^{J+1}}{d_N^J}\frac{2J+1}{J+1}\right)\frac{1}{d_N^J}\sum_{i_2=1}^{d_{N-1}^{J+1/2}} A_q^{J,M}\left|J,M_q;\frac{1}{2},J+\frac{1}{2},N-1,i_1\right\rangle\times\left\langle J,M_r';\frac{1}{2},J+\frac{1}{2},N-1,i_1\right|A_r^{J,M'}. \quad (75)$$

Similarly, the J terms from Eqs. (61), (63), and (65) have coefficients

$$\frac{1}{d_N^J 4J^2}\left[1+\frac{\alpha_{N-1}^{J+1/2}2J}{d_{N-1}^{J-1/2}(J+1/2)(J+1)}+2\left(J-\frac{1}{2}\right)\left(1+\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J-1/2}}\frac{2J}{J+1/2}\right)\right]$$

$$=\frac{1}{d_N^J 4J^2}\left[2J+\frac{N-2J}{2J+1}\left(\frac{1}{J+1}+2J-1\right)\right]=\frac{1}{d_N^J 2J}\left(1+\frac{\alpha_N^{J+1}}{d_N^J}\frac{2J+1}{J+1}\right), \quad (76)$$

which gives

$$\frac{1}{2J}\left(1+\frac{\alpha_N^{J+1}}{d_N^J}\frac{2J+1}{J+1}\right)\frac{1}{d_N^J}\sum_{i_1=1}^{d_{N-1}^{J-1/2}} A_q^{J,M}\left|J,M_q;\frac{1}{2},J-\frac{1}{2},N-1,i_1\right\rangle\left\langle J,M_r';\frac{1}{2},J-\frac{1}{2},N-1,i_1\right|A_r^{J,M'}. \quad (77)$$

And finally, the $J-1$ sums from Eqs. (61), (63), and (65) have coefficients

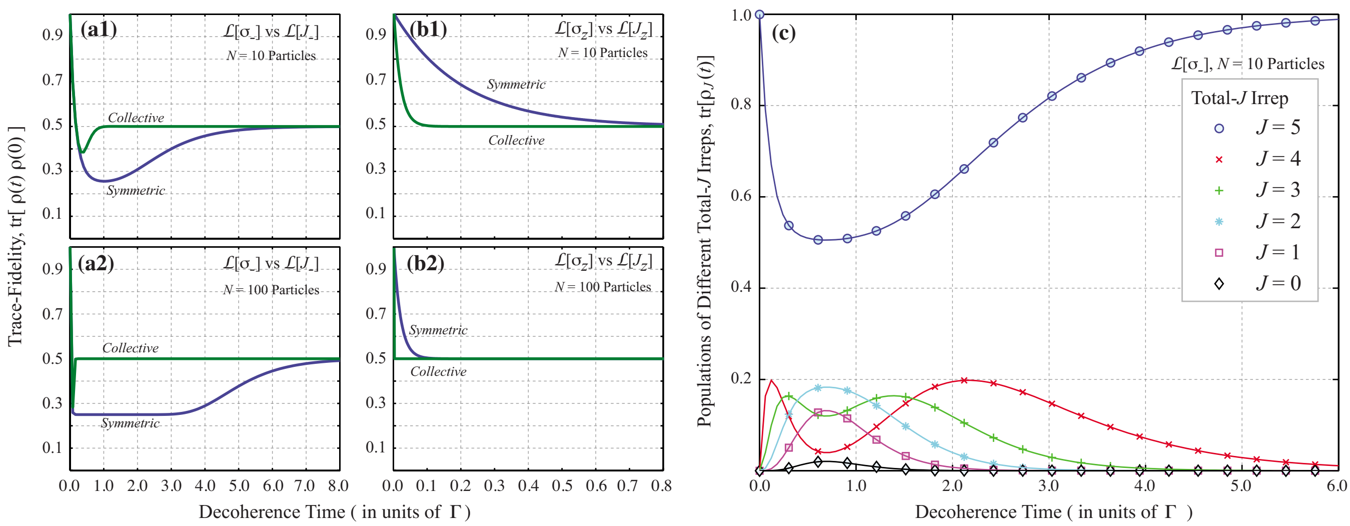

FIG. 2. (Color online) Decoherence of the initial superposition state $|\psi(0)\rangle=(|+N/2\rangle+|-N/2\rangle)/\sqrt{2}$: (a1), (a2) Time-dependent fidelity with the initial state for both symmetric local $\mathcal{L}[\hat{\sigma}_-]$ and collective $\mathcal{L}[\hat{J}_-]$ decoherence for different numbers of particles; (b1), (b2) similar comparison for $\mathcal{L}[\hat{\sigma}_z]$ versus $\mathcal{L}[\hat{J}_z]$; (c) time-dependent populations of different total-J irreps for $\mathcal{L}[\hat{\sigma}_-]$.

$$\frac{1}{d_N^J 4J^2}\left[1+\frac{\alpha_{N-1}^{J+1/2}2J(J+1)}{d_{N-1}^{J-1/2}(J+1/2)}+\frac{1}{2(J-1/2)}\left(1+\frac{\alpha_{N-1}^{J+1/2}}{d_{N-1}^{J-1/2}}\frac{2J}{J+1/2}\right)\right]$$
$$=\frac{1}{d_N^J 4J^2}\left[1+\frac{1}{2J-1}+\frac{N-2J}{2J+1}\left(J+1+\frac{1}{2J-1}\right)\right]=\frac{1}{d_N^J 2J}\frac{\alpha_N^J}{d_N^{J-1}}, \tag{78}$$

which gives

$$\frac{\alpha_N^J}{d_N^J 2J}\frac{1}{d_N^{J-1}}\sum_{i_1=1}^{d_{N-1}^{J-1/2}} B_q^{J,M}\left|J-1,M_q;\frac{1}{2},J-\frac{1}{2},N-1,i_1\right\rangle\times\left\langle J-1,M_r';\frac{1}{2},J-\frac{1}{2},N-1,i_1\right|B_r^{J,M'}. \tag{79}$$

From the definition of $|J,M,N\rangle\langle J,M',N|$ given in Eq. (29), we see that Eqs. (75) and (77) correspond to the $|J,M,N\rangle\langle J,M',N|$ terms in Eq. (42). A similar combination of Eqs. (66) and (80) corresponds to the $J-1$ term and the combination of Eqs. (64) and (73) corresponds to the J term. We have thus shown inductively that Identity 1 holds. ■

VI. EXAMPLES

As discussed in the Introduction, realistic decoherence models for an ensemble of spin particles are often described most aptly by a symmetric sum over local channels. Consider, for example, the open system dynamics governed by the master equation

$$\frac{d\hat{\rho}(t)}{dt}=-i[\hat{H},\hat{\rho}(t)]+\Gamma\mathcal{L}[\hat{s}]\hat{\rho}(t), \tag{80}$$

where $\hat{H}$ (and any measurements performed) are described by collective operators, but the decoherence involves the symmetric Linblad superoperator $\mathcal{L}[\hat{s}]$ of the form in Eq. (37). As this decoherence model does not preserve symmetric states, it has been common practice to consider instead the associated collective process $\mathcal{L}[\hat{S}]$ given in Eq. (34) with $\hat{S}=\Sigma_n\hat{s}^{(n)}$.

To illustrate the difference between symmetric and collective decoherence models, we considered the open system dynamics of two representative problems. First, we compared the dynamics generated by the symmetric-local $\mathcal{L}[\hat{s}]$ versus collective $\mathcal{L}[\hat{S}]$ Linblad master equations applied to an initial superposition (cat) state $|\psi(0)\rangle=(|N/2,+N/2\rangle+|N/2,-N/2\rangle)/\sqrt{2}$. Figures 2(a) and 2(b) depict the fidelity $\mathcal{F}(t)=\langle\psi(0)|\hat{\rho}(t)|\psi(0)\rangle$ evolved under Eq. (80) (with $\hat{H}=0$) for two different types of decoherence channel: Figs. 2(a1) and 2(a2) compare the collective versus symmetric master equations with $\hat{s}=\hat{\sigma}_-$ for $N=10$ and 100 particles, respectively; and Figs. 2(b1) and 2(b2) make a similar comparison for $\hat{s}=\hat{\sigma}_z$. The examples we considered (including some not reported here) suggest that symmetric local decoherence models can generate dynamics that are appreciably different from their collective analogs. This is perhaps not too surprising: for an

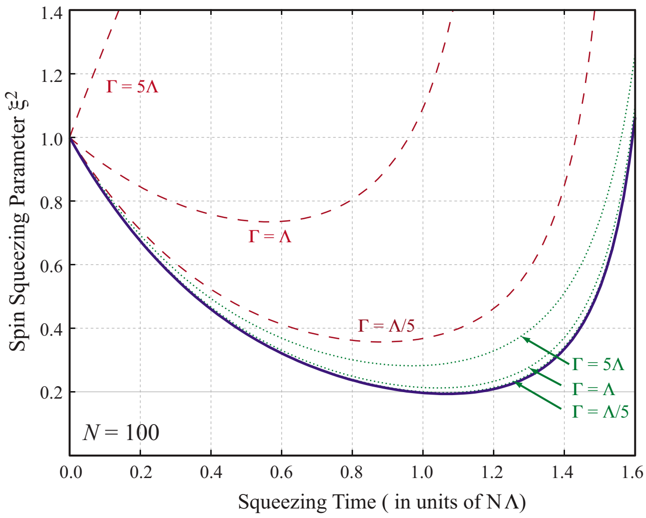

FIG. 3. (Color online) Time evolution of the squeezing parameter ξ^2 for a spin ensemble driven by $\hat{H}=-i\Lambda(\hat{J}_+^2-\hat{J}_-^2)$ subject to $\mathcal{L}[\hat{\sigma}_-]$ (dotted lines) and $\mathcal{L}[\hat{J}_-]$ (dashed lines) with relative decoherence rates $\Gamma=\Lambda/5,\Lambda,5\Lambda$. For comparison, the solid line denotes decoherence-free squeezing.

initially symmetric state, collective decoherence models $\mathcal{L}[\hat{S}]$ confine the dynamics to only maximum-J irreps; symmetric local models $\mathcal{L}[\hat{s}]$ do not necessarily preserve the irrep decomposition of the initial state. Figure 2(c) depicts the norm of each total-J irrep block of the density operator $N_J=\mathrm{tr}[\hat{P}_J\hat{\rho}(t)]$ as a function of time for $\mathcal{L}[\hat{\sigma}_-]$ ($\hat{P}_J=\Sigma_M|J,M\rangle\langle J,M|$). The observation that small-J irreps are only minimally populated suggests that further model reduction by truncating the Hilbert space to only the largest-J blocks could be beneficial.

As a second example, we compared symmetric local versus collective decoherence models applied to dynamically generated spin squeezing under the countertwisting Hamiltonian $\hat{H}=-i\Lambda(\hat{J}_+^2-\hat{J}_-^2)$ [9]. We performed simulations by time-evolving Eq. (80) from the initial spin-coherent state $|N/2,N/2\rangle$ for $N=100$ with $\mathcal{L}[\hat{\sigma}_-]$ and $\mathcal{L}[\hat{J}_-]$. Figure 3 depicts the time-dependent squeezing parameter $\xi^2=N\langle\Delta\hat{J}_y^2\rangle/\langle\hat{J}_z\rangle^2$, each for $\Gamma=\Lambda/5,\Lambda,5\Lambda$. Under the conditions we considered, symmetric local decoherence was evidently less destructive to the squeezing dynamics than collective models. As observed for the cat state dynamics, to a large extent the main effect of symmetric local decoherence is leakage from the maximum-J irrep. But since the driving Hamiltonian $\hat{H}$ involves only collective spin operators, the coherent dynamics decouple for different total J: the population in each irrep block then undergoes its own squeezing, evidently making the dynamics more resistant to symmetric local decoherence than collective processes.

VII. CONCLUSION

We have presented an exact formula for efficiently expressing symmetric processes of an ensemble of spin-1/2 particles. The efficiency is achieved by generalizing the notion of collective spin states to be any such state which does not distinguish degenerate irreps. For a collection of N spin-1/2 particles, the effective Hilbert space dimension grows as N^2, a drastic reduction from the full Hilbert space scaling of 2^N. The collective representation is used in Identity 1, which gives a closed form expression for evaluating noncollective terms from symmetric Lindblad operators. Simulations confirm that symmetric local decoherence models can be drastically different from collective decoherence models. Unfortunately, due to the complicated structure of addition of spin-($J>\frac{1}{2}$) particles [14], our results do not appear to generalize. Nonetheless, we believe that this approach will become a useful tool in analyzing collective spin phenomena and, in particular, accurately considering the role of decoherence in collective spin experiments.

ACKNOWLEDGMENTS

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APPENDIX: EXPLICIT SIMPLIFICATION OF TYPICAL SUMS

In Sec. V C, we simplify the sums in Eqs. (52)–(59) but do not go through the detailed algebra. The work involves manipulating products of Clebsch-Gordan and A,B,D coefficients. In this appendix, we explicitly calculate two representative sums from this set and invite the reader to calculate the remainder in a similar fashion.

First, consider the sums over J_1 and m_1 in Eq. (52), which are representative of sums in Eqs. (52) and (53). For $J_1=J+1$,

$$A_q^{1/2,1/2\;J+1,M_q}C^{1/2,1/2_q}_{J+1/2,M-1/2}\;{}^{J,M}C^{1/2,1/2}_{J+1/2,M-1/2}+A_q^{1/2,-1/2\;J+1,M_q}C^{1/2,-1/2_q}_{J+1/2,M+1/2}\;{}^{J,M}C^{1/2,-1/2}_{J+1/2,M+1/2}$$
$$=\frac{1}{2(J+1)}\times\begin{cases}-\sqrt{(J+M+2)(J+M+1)}, & q=+,\\ \sqrt{(J-M+2)(J-M+1)}, & q=-,\\ \sqrt{(J-M+1)(J+M+1)}, & q=z,\end{cases} \tag{A1}$$

$$=\frac{1}{2J+2}D_q^{J,M}, \tag{A2}$$

and for $J_1=J$

$$\begin{aligned}
&A_q^{1/2,1/2\ J,M_q}C_{J+1/2,M-1/2}^{1/2,1/2_q\ \ J,M}C_{J+1/2,M-1/2}^{1/2,1/2}+A_q^{1/2,-1/2\ J,M_q}C_{J+1/2,M+1/2}^{1/2,-1/2_q\ \ J,M}C_{J+1/2,M+1/2}^{1/2,-1/2}\\
&=-\frac{1}{2(J+1)}\times\begin{cases}\sqrt{(J-M)(J+M+1)}, & q=+,\\ \sqrt{(J+M)(J-M+1)}, & q=-,\\ M, & q=z,\end{cases}
\end{aligned} \tag{A3}$$

$$=-\frac{1}{2J+2}A_q^{J,M}, \tag{A4}$$

where $A_+^{1/2,1/2}=A_-^{1/2,-1/2}=0$, $A_+^{1/2,-1/2}=A_-^{1/2,1/2}=1$, and $A_z^{1/2,\pm 1/2}=\pm 1/2$.

Similarly, consider the sums over J_1 and m_1 in Eq. (55), which is representative of Eqs. (55), (56), and (56)–(59). For $J_1=J-1$, we have

$$\begin{aligned}
&B_q^{J+1/2,M-1/2\ J-1,M_q}C_{J-1/2,M_q-1/2}^{1/2,1/2\ \ J,M}C_{J+1/2,M-1/2}^{1/2,1/2}+B_q^{J+1/2,M+1/2\ J-1,M_q}C_{J-1/2,M_q+1/2}^{1/2,-1/2\ \ J,M}C_{J+1/2,M+1/2}^{1/2,-1/2}\\
&=\sqrt{\frac{(J-M_q)(J-M+1)}{4J(J+1)}}B_q^{J+1/2,M-1/2}\times\left(1+\sqrt{\frac{J+M_q}{J-M_q}}\sqrt{\frac{J+M+1}{J-M+1}}\frac{B_q^{J+1/2,M+1/2}}{B_q^{J+1/2,M-1/2}}\right)\\
&=\sqrt{\frac{(J-M_q)(J-M+1)}{4J(J+1)}}B_q^{J+1/2,M-1/2}\frac{2(J+1)}{J-M+1}=\sqrt{\frac{J+1}{J}}\times\begin{cases}\sqrt{(J-M)(J-M-1)}, & q=+,\\ -\sqrt{(J+M)(J+M-1)}, & q=-,\\ \sqrt{(J+M)(J-M)}, & q=z,\end{cases}=\sqrt{\frac{J+1}{J}}B_q^{J,M}.
\end{aligned} \tag{A5}$$

Similarly, for $J_1=J$, we have

$$\begin{aligned}
&B_q^{J+1/2,M-1/2\ J,M_q}C_{J-1/2,M_q-1/2}^{1/2,1/2\ \ J,M}C_{J+1/2,M-1/2}^{1/2,1/2}+B_qJ+\frac{1}{2},M+\frac{1}{2}^{J,M_q}C_{J-1/2,M_q+1/2}^{1/2,-1/2\ \ J,M}C_{J+1/2,M+1/2}^{1/2,-1/2}\\
&=\sqrt{\frac{(J+M_q)(J-M+1)}{4J(J+1)}}B_q^{J+1/2,M-1/2}\times\left(1-\sqrt{\frac{J-M_q}{J+M_q}}\sqrt{\frac{J+M+1}{J-M+1}}\frac{B_q^{J+1/2,M+1/2}}{B_q^{J+1/2,M-1/2}}\right)\\
&=\sqrt{\frac{(J+M_q)(J-M+1)}{4J(J+1)}}B_q^{J+1/2,M-1/2}\times 2\times\begin{cases}\frac{1}{J-M+1}, & q=+,\\ -\frac{1}{J+M+1}, & q=-,\\ \frac{M}{(J+M)(J-M+1)}, & q=z,\end{cases}\\
&=\sqrt{\frac{1}{J(J+1)}}\times\begin{cases}\sqrt{(J-M)(J+M+1)}, & q=+,\\ \sqrt{(J+M)(J-M+1)}, & q=-,\\ M, & q=z,\end{cases}=\sqrt{\frac{1}{J(J+1)}}A_q^{J,M}.
\end{aligned} \tag{A6}$$

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