Generalized transformation optics from triple spacetime metamaterials

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In this paper, various extensions of the design strategy for transformation media are proposed. We show that it is possible to assign different transformed spaces to the field strength tensor (electric field and magnetic induction) and to the excitation tensor (displacement field and magnetic field), respectively. In this way, several limitations of standard transformation media can be overcome. In particular, it is possible to provide a geometric interpretation of nonreciprocal as well as indefinite materials. We show that these transformations can be complemented by a continuous version of electric-magnetic duality and comment on the relation to the complementary approach of field-transforming metamaterials.

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I. INTRODUCTION

In the field of metamaterials, artificial electromagnetic materials, the use of spacetime transformations as a design tool for new materials has been proved very successful recently [1–3]. The basic idea of this concept is that a metamaterial mimics a transformed, but empty space. The light rays follow the trajectories according to Fermat's principle in this transformed (electromagnetic) space instead of laboratory space. This allows one to design in an efficient way materials with various characteristics such as invisibility cloaks [1,2,4], perfect lenses [3], magnification devices [5], an optical analogue of the Aharonov-Bohm effect, or even artificial black holes [3]. Still the media relations accessible in this way are rather limited, in particular nonreciprocal or indefinite media (materials exhibiting strong anisotropy) are not covered. But these types of material also have been linked to some of the mentioned concepts, in particular perfect lenses [6,7] and hyperlenses [8]. This raises the question whether there exists an extension of the concept of transformation media such as to cover those materials as well and to provide a geometric interpretation thereof.

In this paper we propose an extension of this type. As in Refs. [1–3] our concept is based on diffeomorphisms locally represented as coordinate transformations. Therefore many of our results allow a geometric interpretation similar to the one of Refs. [3,9] as opposed to another recently suggested route to overcome the restrictions of diffeomorphism transforming media [10,11]. The starting point of our considerations is Maxwell's equations in possibly curved but vacuous space:[1]

$$\nabla_i B^i = 0, \quad \nabla_0 B^i + \epsilon^{ijk}\partial_j E_k = 0, \tag{1}$$

$$\nabla_i \mathcal{D}^i = \rho, \quad \epsilon^{ijk}\partial_j \mathcal{H}_k - \nabla_0 \mathcal{D}^i = j^i. \tag{2}$$

Here, ∇_i is the covariant derivative in three dimensions

$$\nabla_i A^i = (\partial_i + \Gamma^i_{ij})A^j = \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}A^i), \tag{3}$$

with the space metric γ_{ij} and its determinant γ.

For many manipulations it will be advantageous to use relativistically covariant quantities. Therefore, Eqs. (1) and (2) are rewritten in terms of the field strength tensor $F_{\mu\nu}$, the excitation tensor $H^{\mu\nu}$, and a four-current J^μ (see the Appendix):

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \quad D_\nu H^{\mu\nu} = -J^\mu, \quad D_\mu J^\mu = 0. \tag{4}$$

The four-dimensional covariant derivative D_μ is defined analogously to (3), whereby the space metric is replaced by the spacetime metric $g_{\mu\nu}$ and its volume element $\sqrt{-g}$.

We wish to analyze these equations of motion from the point of view of transformation media. All transformation materials have in common that they follow as a transformation from a (not necessarily source-free) vacuum solution of the equations of motion, which maps this solution onto a solution of the equations of motion of the transformation material.[2] The crucial ingredient in the definition of transformation media then is the class of transformations to be considered. As space of all transformations we restrict ourselves to all *linear* transformations in four-dimensional spacetime. Consequently, all media exhibit linear constitutive relations, which may be written within the covariant formulation as [12]

$$H^{\mu\nu} = \frac{1}{2}\chi^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{5}$$

In vacuum one obtains[3]

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[1]Here and in the following we use Einstein's summation convention, in which a summation over all repeated indices is assumed: $A^iB_i = \Sigma_i A^iB_i$. For latin indices this sum runs over the values 1,2,3 (spatial indices), while for greek indices it runs from 0 to 4 with $x^0 = t$ being time.

[2]Strictly speaking this applies to transformations which are regular everywhere, only. Several singular transformations have been proposed in the literature in the context of metamaterials, e.g., the invisibility cloak [1,2,4]. In this case a careful study of the global solution is indispensable, as has been done for the case of the cloak in Ref. [23].

[3]Throughout the paper, natural units with $\epsilon_0 = \mu_0 = c = 1$ are used. Notice that the corresponding relation in Ref. [3] differs from the one used here. According to our conventions $F^{\mu\nu} = H^{\mu\nu}$ in vacuum, while there $\sqrt{-g}F^{\mu\nu} = H^{\mu\nu}$.

$$\chi^{\mu\nu\rho\sigma} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}), \tag{6}$$

such that the standard results $\vec{E}=\vec{\mathcal{D}}$ and $\vec{B}=\vec{\mathcal{H}}$ emerges.

These transformations and the ensuing media properties (5) have the advantage of being relativistically invariant and thus very easy to handle. However, they do not include any frequency dependence and remain strictly real, which perhaps is the most severe restriction that follows from the coordinate transformation approach. As long as the linear transformations are seen as transformations of spacetime (rather than of the fields) this restriction is not surprising, though. Indeed, from energy conservation it follows that it is not possible to model a process of absorption by the medium as a local transformation of spacetime (notice that the spacetime itself is not dynamical and thus cannot contribute to the energy).

II. DIFFEOMORPHISM TRANSFORMING METAMATERIALS

Obviously, the concept of transformation materials as sketched above is related to symmetry transformations, as those are by definition linear transformations that map a solution of the equations of motion onto another one. Therefore it is worth working out this relation in some more detail.

A symmetry is a transformation which leaves the source-free[4] action of the theory, here

$$\mathcal{S} = \int d^4x\sqrt{-g}F_{\mu\nu}H^{\mu\nu}, \tag{7}$$

invariant, whereby surface terms are dropped. It straightforwardly follows that a symmetry transformation applied on a solution of the equations of motion still solves the latter. In the above action a general, not necessarily flat, spacetime is considered. The symmetries of this action are well known: these are the U(1) gauge symmetry of electromagnetism and the symmetries of spacetime (diffeomorphisms). The gauge symmetry cannot help in designing materials as the media relations are formulated exclusively in terms of gauge-invariant quantities. However, diffeomorphisms change the media relations, as is pointed out, e.g., in Ref. [13] and as it has been applied to metamaterials in Ref. [3]. Thus one way to define transformation media is the following.

Definition 1. A transformation material follows from a symmetry transformation applied to a vacuum solution of Maxwell's equations. This vacuum solution need not be source-free.

The space of all possible transformation materials of this kind has been derived in Ref. [3]; here we briefly want to summarize the result of that paper. The starting point is the observation that a curved space in Maxwell's equations looks like a medium. Indeed, in empty but possibly curved space

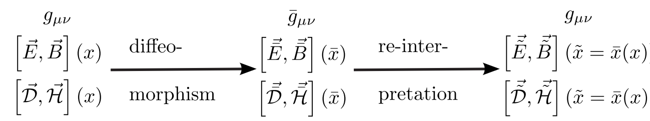

FIG. 1. Diffeomorphism transforming metamaterials according to Ref. [3].

the constitutive relation among the electromagnetic fields is found by exploiting

$$F_{0i} = (g_{00}g_{ij} - g_{0j}g_{i0})H^{0j} + g_{0k}g_{il}H^{kl}, \tag{8}$$

$$H^{ij} = 2g^{i0}g^{jk}F_{0k} + g^{ik}g^{jl}F_{kl}, \tag{9}$$

which in terms of the space vectors reads

$$\mathcal{D}^i = \frac{g^{ij}}{\sqrt{-g_{00}}}E_j - \frac{g_{0j}}{g_{00}}\epsilon^{jil}\mathcal{H}_l, \tag{10}$$

$$B^i = \frac{g^{ij}}{\sqrt{-g_{00}}}\mathcal{H}_j + \frac{g_{0j}}{g_{00}}\epsilon^{jil}E_l. \tag{11}$$

Thus empty space can appear like a medium with permeability and permittivity $\epsilon^{ij}=\mu^{ij}=g^{ij}/\sqrt{-g_{00}}$ and with bianisotropic couplings $\xi^{ij}=-\kappa^{ij}=\epsilon^{lij}g_{0l}/g_{00}$.

Now, as the basic idea of Ref. [3], if empty space can appear like a medium, a medium should also be able to appear as empty space. One starts with electrodynamics *in vacuo*; we call these fields $F_{\mu\nu}$ and $H^{\mu\nu}$ with flat metric $g_{\mu\nu}$. Now we apply a diffeomorphism, locally represented as a coordinate transformation $x^\mu \rightarrow \bar{x}^\mu(x)$. As the equations of motion by definition are invariant under diffeomorphisms, all relations remain the same with the fields F, H and the metric $g_{\mu\nu}$ replaced by the new barred quantities. As a last step one reinterprets in the dynamical equations (1) and (2) the coordinates $\bar{x}^\mu$ as the original ones x^μ, while keeping $\bar{g}_{\mu\nu}$ in the constitutive relation. To make this possible some fields must be rescaled in order to transform barred covariant derivatives (containing $\bar{g}$) into unbarred ones (containing g.) The situation of diffeomorphism transforming metamaterials is illustrated in Fig. 1, which also summarizes our notation. As a more technical remark it should be noted that this manipulation is possible as we consider just Maxwell's theory on a curved background rather than Einstein-Maxwell theory (general relativity coupled to electrodynamics.) In the former case the metric is an external parameter and thus this manipulation is possible as long as none of the involved quantities depends explicitly on the metric.

To keep the whole discussion fully covariant the fields are transformed at the level of the field strength and excitation tensor (rather than at the level of space vectors as was done in Ref. [3]). To transform the covariant derivatives $\bar{D}_\mu$ into the original D_μ we have to apply the rescalings

$$\tilde{H}^{\mu\nu} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\bar{H}^{\mu\nu}, \quad \tilde{J}^\mu = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\bar{J}^\mu. \tag{12}$$

In addition the transformation $x^\mu \rightarrow \bar{x}^\mu(x)$ may not preserve the orientation of the manifold, which technically means that

[4]Sources are external parameters and thus should be set to zero for a symmetry transformation. Even together with sources the symmetry can be restored, if an appropriate transformation rule of the sources is defined.

the Levi-Cività tensor changes sign [3]. This is corrected by introducing the sign ambiguity

$$\tilde{F}_{\mu\nu} = \pm \bar{F}_{\mu\nu}, \tag{13}$$

with the plus sign for orientation-preserving and the minus for nonpreserving transformations. These new fields again exist in the original space with metric $g_{\mu\nu}$, but now the space is filled with a medium with

$$\tilde{\chi}^{\mu\nu\rho\sigma} = \pm \frac{1}{2}\frac{\sqrt{-\bar{g}}}{\sqrt{-g}}(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}), \tag{14}$$

or, in terms of space vectors,

$$\tilde{\mathcal{D}}^i = s\frac{\bar{g}^{ij}}{\sqrt{-\bar{g}_{00}}}\frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\tilde{E}_j - \frac{\bar{g}_{0j}}{\bar{g}_{00}}\epsilon^{jil}\tilde{\mathcal{H}}_l, \tag{15}$$

$$\tilde{B}^i = s\frac{\bar{g}^{ij}}{\sqrt{-\bar{g}_{00}}}\frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\tilde{\mathcal{H}}_j + \frac{\bar{g}_{0j}}{\bar{g}_{00}}\epsilon^{jil}\tilde{E}_l, \tag{16}$$

with $s=\pm 1$ being the sign in (13) and (14). As can be seen, the media properties are restricted to reciprocal materials ($\epsilon=\epsilon^T$, $\mu=\mu^T$, $\kappa=\chi^T$), which, in addition, obey $\epsilon=\mu$. This result has been obtained in Ref. [3] in a slightly different way and encompasses the transformations in Refs. [1,14]. We do not want to go into further details of this approach but refer to the review [9], where its geometric optics interpretation is discussed in detail. Indeed, light travels in transformation media of this type along null geodesics of the electromagnetic space $\bar{x}^\mu$, which allows (with some restrictions to be discussed in Sec. V) a simple and intuitive interpretation of the transformation.

III. TRIPLE SPACETIME METAMATERIALS

Despite the variety of applications of diffeomorphism transforming metamaterials some results suggest a search for extensions. Indeed, there exist, e.g., designs of super- and hyperlenses that make use of indefinite materials (strong anisotropy) [6–8]. Although both concepts should be perfectly understandable in terms of transformation media, the specific material relations used in these works do not fall under the class of diffeomorphism transforming metamaterials.

To understand a possible route to generalize the concept of diffeomorphism transforming media we have to consider again their basis, namely symmetry transformation. The concept of symmetries is used to identify different solutions of the equations of motion that effectively describe the same physics. By means of reinterpretation in the last step of Fig. 1, such symmetry transformations can be used as a simple tool to derive within a restricted class of constitutive relations new media properties in a geometrically intuitive and completely algebraic way.

Nonetheless, within the concept of metamaterials it is not important that the transformed solution in principle describes the same physics as the original one. Still, one may want to keep the possibility of mapping source-free solutions onto other source-free solutions in a straightforward way, as only in this way do we have an effective control over passive media and do not risk introducing exotic sources such as magnetic monopoles. Furthermore, a geometric interpretation of the transformations is kept, which is advantageous in many applications. To weaken the conditions on transformation materials while keeping the advantages of symmetry transformations we thus propose the following definition.

Definition 2. Consider the set of all transformations T which map a source-free solution of the equations of motion (1) and (2) onto another source-free solution. A transformation material is a material obtained by applying a transformation T onto a (not necessarily source-free) vacuum solution.

There are two types of extensions contained in this definition compared to the previous section.

(1) There exist transformations that leave the equations of motion invariant, but change the action by a constant and thus are not symmetry transformations. A transformation of this type is the so-called electric-magnetic duality. Its effect will briefly be discussed in Sec. III A.

(2) We do allow for transformations which leave all Maxwell's equations (1) and (2) invariant, but change the media relations (5). This indeed generalizes the concept in an important way.

To see the origin of the second extension it is important to realize that the equations of motion of electrodynamics separate into two different sets [Eqs. (1) and (2), respectively] with mutually exclusive field content. This characteristic is not just an effect of our notation, but as has been shown, e.g., in Refs. [15,16], the equations of motion of electrodynamics can be derived from first principles without using explicitly the constitutive relation $H=H(F)$. As the two sets of equations are separately invariant under diffeomorphisms it should be possible to assign *different* transformed spaces to $H=(\vec{\mathcal{D}},\vec{\mathcal{H}})$ and $F=(\vec{E},\vec{B})$. In other words, it must be possible to distort the spaces (or the coordinates) of the field strength tensor and the excitation tensor separately, whereby the resulting transformation material per constructionem satisfies all conditions of the Definition 2. The ensuing constitutive relation as well as the solutions of the equations of motion still follow (almost) as simple as in the case of Ref. [3].

To prove the potential of this method we have to extend the notation compared to the previous section: as before laboratory space has metric $g_{\mu\nu}$, its fields *in vacuo* are $H=(\vec{\mathcal{D}},\vec{\mathcal{H}})$ and $F=(\vec{E},\vec{B})$; the fields of the transformation material (existing in the space with metric $g_{\mu\nu}$) are again labeled with a tilde. The transformed space of the field strength tensor has metric $\bar{g}_{\mu\nu}$ and fields $\bar{F}=(\vec{\bar{E}},\vec{\bar{B}})$, the one of the excitation tensor $\bar{\bar{g}}_{\mu\nu}$ and $\bar{\bar{H}}=(\vec{\bar{\bar{\mathcal{D}}}},\vec{\bar{\bar{\mathcal{H}}}})$. This new transformation is illustrated in Fig. 2. Applying the two transformations

$$\bar{x}^\mu = \bar{x}^\mu(x), \quad \bar{\bar{x}}^\mu = \bar{\bar{x}}^\mu(x) \tag{17}$$

to the constitutive relation (5) with χ being the vacuum relation (6) yields

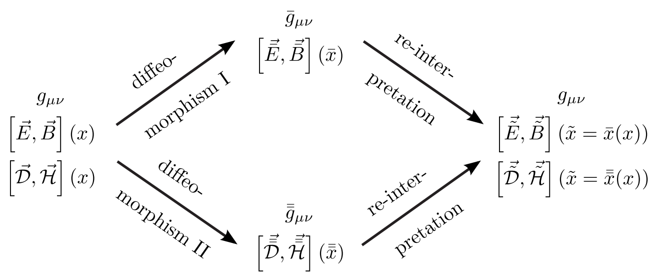

FIG. 2. Illustration and notation of the generalized "triple spacetime metamaterials." Notice that the diffeomorphism I acts only on the fields $\vec E$ and $\vec B$, while diffeomorphism II acts on $\vec{\mathcal D}$ and $\vec{\mathcal H}$.

$$\bar{\bar H}^{\mu\nu} = \frac{1}{2}\frac{\partial\bar{\bar x}^\mu}{\partial x^\lambda}\frac{\partial\bar{\bar x}^\nu}{\partial x^\tau}(g^{\lambda\alpha}g^{\tau\beta}-g^{\lambda\beta}g^{\tau\alpha})\frac{\partial\bar x^\rho}{\partial x^\alpha}\frac{\partial\bar x^\sigma}{\partial x^\beta}\bar F_{\rho\sigma}. \tag{18}$$

Introducing the notation

$$g^{\bar{\bar\mu}\bar\nu} = \frac{\partial\bar{\bar x}^\mu}{\partial x^\rho}\frac{\partial\bar x^\nu}{\partial x^\sigma}g^{\rho\sigma} = \bar{\bar g}^{\mu\rho}\frac{\partial\bar x^\nu}{\partial\bar{\bar x}^\rho} = \frac{\partial\bar{\bar x}^\mu}{\partial\bar x^\rho}\bar g^{\rho\nu}, \tag{19}$$

the relation may be written as

$$\bar{\bar H}^{\mu\nu} = \frac{1}{2}(g^{\bar{\bar\mu}\bar\rho}g^{\bar{\bar\nu}\bar\sigma}-g^{\bar{\bar\mu}\bar\sigma}g^{\bar{\bar\nu}\bar\rho})\bar F_{\rho\sigma}. \tag{20}$$

It should be noted that $g^{\bar{\bar\mu}\bar\nu}$ in Eq. (19) is no longer a metric, in particular it need not be symmetric in its indices and it need not have signature (3, 1).

To derive the new constitutive relations in the original (laboratory) space we proceed analogously to the previous section. All fields have to be rescaled in order to obey the equations of motion in the original space with metric $g_{\mu\nu}$, which implies

$$\tilde F_{\mu\nu} = \pm\bar F_{\mu\nu}, \quad \tilde H^{\mu\nu} = \frac{\sqrt{-\bar{\bar g}}}{\sqrt{-g}}\bar{\bar H}^{\mu\nu}, \quad \tilde J^\mu = \frac{\sqrt{-\bar{\bar g}}}{\sqrt{-g}}\bar{\bar J}^\mu. \tag{21}$$

Thus the constitutive relation becomes

$$\tilde H^{\mu\nu} = \tilde\chi^{\mu\nu\rho\sigma}\tilde F_{\rho\sigma} = \pm\frac{1}{2}\frac{\sqrt{-\bar{\bar g}}}{\sqrt{-g}}(g^{\bar{\bar\mu}\bar\rho}g^{\bar{\bar\nu}\bar\sigma}-g^{\bar{\bar\mu}\bar\sigma}g^{\bar{\bar\nu}\bar\rho})\tilde F_{\rho\sigma}, \tag{22}$$

where the sign refers to the possible change of orientation in the transformation $x^\mu\to\bar x^\mu$. For the equivalent relation in terms of space vectors the notation

$$\epsilon_{\mu\nu\rho\sigma} = \bar s\frac{\sqrt{-g}}{\sqrt{-\bar g}}\bar\epsilon_{\mu\nu\rho\sigma} = \bar{\bar s}\frac{\sqrt{-g}}{\sqrt{-\bar{\bar g}}}\bar{\bar\epsilon}_{\mu\nu\rho\sigma} \tag{23}$$

is used, where $\bar s$ and $\bar{\bar s}$ are the respective signs due to the change of orientation in the transformations to laboratory space. Now it easily follows from (A17)–(A20) that

$$\mathfrak{A}^{ij} = -\bar s\frac{\sqrt{-\bar{\bar g}}}{\sqrt{\gamma}}(g^{\bar{\bar 0}\bar 0}g^{\bar{\bar i}\bar j}-g^{\bar{\bar 0}\bar j}g^{\bar{\bar i}\bar 0}), \tag{24}$$

$$\mathfrak{B}_{ij} = -\bar{\bar s}\frac{\sqrt{\gamma}}{\sqrt{-\bar g}}(g_{\bar{\bar 0}\bar 0}g_{\bar{\bar i}\bar j}-g_{\bar{\bar 0}\bar j}g_{\bar{\bar i}\bar 0}), \tag{25}$$

$$\mathfrak{C}^i{}_j = -\frac{\bar s}{2}\frac{\sqrt{-\bar{\bar g}}}{\sqrt{\gamma}}\epsilon_{ikl}(g^{\bar{\bar k}\bar 0}g^{\bar{\bar l}\bar j}-g^{\bar{\bar k}\bar j}g^{\bar{\bar l}\bar 0}), \tag{26}$$

$$\mathfrak{D}^i{}_j = \frac{\bar s}{2}\frac{\sqrt{-\bar{\bar g}}}{\sqrt{\gamma}}\epsilon_{jkl}(g^{\bar{\bar 0}\bar k}g^{\bar{\bar i}\bar l}-g^{\bar{\bar 0}\bar l}g^{\bar{\bar i}\bar k}), \tag{27}$$

which are the defining tensors of the Boys-Post relation. After some algebra the Tellegen relation

$$\tilde{\mathcal D}^i = -\bar s\frac{\sqrt{-\bar{\bar g}}}{\sqrt{\gamma g_{\bar{\bar 0}\bar 0}}}g^{\bar{\bar i}\bar j}\tilde E_j - \bar s\bar{\bar s}\frac{\sqrt{-\bar g}\sqrt{-\bar{\bar g}}}{\gamma g_{\bar{\bar 0}\bar 0}}g^{\bar{\bar i}\bar k}g^{\bar{\bar 0}\bar l}\epsilon_{klm}g^{\bar{\bar m}\bar j}\tilde{\mathcal H}_j, \tag{28}$$

$$\tilde B^i = -\bar{\bar s}\frac{\sqrt{-\bar g}}{\sqrt{\gamma g_{\bar{\bar 0}\bar 0}}}g^{\bar{\bar i}\bar j}\tilde{\mathcal H}_j + \bar s\bar{\bar s}\frac{\sqrt{-\bar g}\sqrt{-\bar{\bar g}}}{\gamma g_{\bar{\bar 0}\bar 0}}g^{\bar{\bar i}\bar k}\epsilon_{klm}g^{\bar{\bar l}\bar 0}g^{\bar{\bar m}\bar j}\tilde E_j \tag{29}$$

is found, which in the limit of $\bar{\bar g}_{\mu\nu}=\bar g_{\mu\nu}$ is equivalent to Eqs. (15) and (16). An important comment is in order: due to the different transformations applied to $H^{\mu\nu}$ and $F_{\mu\nu}$, respectively, the constitutive relation (22), or (28) and (29), relates fields from *different* spacetime points in the original space, e.g., $\tilde E_i(\tilde x=\bar x(x))$ refers the field $E_i(x)$ at a different point x^μ in the original space than $\tilde{\mathcal D}^i(\tilde x=\bar{\bar x}(x))$ does.

Let us comment on the more technical parts of this result. In Sec. II we saw that transformation materials derived from symmetry transformations are restricted to reciprocal materials with $\epsilon=\mu$. These restrictions can be overcome partially with the above result.

(a) As $g^{\bar{\bar i}\bar j}=(g^{\bar{\bar j}\bar i})^T$ it follows that the permittivity and permeability are related as

$$\bar{\bar s}\sqrt{-\bar{\bar g}}\mu^{ij} = \bar s\sqrt{-\bar g}\epsilon^{ji}. \tag{30}$$

It should not come as a surprise that permittivity and permeability cannot be independent, as by virtue of the definition of the relativistically covariant tensors $F_{\mu\nu}$ and $H_{\mu\nu}$ such transformations cannot act independently on $\vec E$ and $\vec B$ or $\vec{\mathcal D}$ and $\vec{\mathcal H}$, respectively. A possible route to relax this restriction is discussed in Sec. VI.

(b) Permittivity and permeability need no longer be symmetric. Therefore it is possible to describe nonreciprocal materials, or, in the language of Eq. (A23), the skewon part need not vanish. This happens if the mapping between the two electromagnetic spaces, $\partial\bar x^\mu/\partial\bar{\bar x}^\nu$, is not symmetric in μ and ν, e.g., for a material with mapping $\bar x=x-z$, $\bar{\bar x}=x+z$.

(c) The generalized transformations yield many more possibilities considering the signs of the eigenvalues of permittivity and permeability. Within the method of Ref. [3], μ and ϵ are essentially determined by the spatial metric of the electromagnetic space [cf. Eqs. (15) and (16) and recall the relation $g^{ij}=\gamma^{ij}$]. However, a spatial metric by definition must have three positive eigenvalues, a characteristic that cannot be changed by any diffeomorphism. Thus it follows that in any medium of this type the eigenvalues of ϵ and μ are all of the same sign.

Within the generalized setup of "triple spacetime metamaterials," however, the signs of the eigenvalues in ϵ can be chosen freely, as the metric is multiplied by a transformation matrix,

$$g^{\bar{i}\bar{\bar{j}}} = \bar{\bar{g}}^{i\mu}\frac{\partial \bar{x}^{j}}{\partial \bar{\bar{x}}^{\mu}}, \tag{31}$$

and no restrictions on the signs of the eigenvalues of the transformation matrix exist. In this way indefinite materials [6,7] can be designed as a result of different space inversions in the two different mappings. As an example the mapping $\bar{z}=-z$, $\bar{\bar{z}}=z$ (with all other directions mapped trivially) yields $\epsilon^{ij}=\mathrm{diag}(-1,-1,1)$, $\mu^{ij}=\mathrm{diag}(1,1,-1)$.

Furthermore the relative sign between the eigenvalues of ϵ and those of μ can be chosen as a consequence of the factor $\bar{\bar{s}}$ in Eq. (30). This change in the relative sign may be interpreted as a partial reversal of time as can be seen in the following list (space maps trivially here and all media are assumed to be homogeneous):

	$\bar{t}$	$\bar{\bar{t}}$	ϵ	μ
I	t	t	1	1
II	t	$-t$	-1	1
III	$-t$	t	1	-1
IV	$-t$	$-t$	-1	-1

We note that all eight classes of materials discussed in Ref. [6] allow a geometric interpretation within the setup of "triple spacetime metamaterials."

(d) More complicated than permittivity and permeability are the bianisotropic couplings. With the standard assumption of $g^{0i}=0$ in laboratory space it follows from

$$\xi^{ij} = -\bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}} g^{\bar{0}\bar{\bar{l}}}\epsilon_{klm} g^{\bar{m}\bar{\bar{j}}}, \tag{32}$$

$$\kappa^{ij} = \bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}}\epsilon_{klm} g^{\bar{l}\bar{\bar{0}}} g^{\bar{m}\bar{\bar{j}}}, \tag{33}$$

similarly to Eq. (15) that all electric-magnetic couplings vanish if the transformation does not mix space and time. In this case the crucial components $g^{\bar{0}\bar{\bar{l}}}$ and $g^{\bar{\bar{0}}\bar{l}}$ may be written as

$$g^{\bar{0}\bar{\bar{l}}} = \frac{\partial \bar{x}^{0}}{\partial x^{0}} g^{00}\frac{\partial \bar{\bar{x}}^{l}}{\partial x^{0}} + \frac{\partial \bar{x}^{0}}{\partial x^{i}} g^{ij}\frac{\partial \bar{\bar{x}}^{l}}{\partial x^{j}}, \tag{34}$$

$$g^{\bar{\bar{0}}\bar{l}} = \frac{\partial \bar{\bar{x}}^{0}}{\partial x^{0}} g^{00}\frac{\partial \bar{x}^{l}}{\partial x^{0}} + \frac{\partial \bar{\bar{x}}^{0}}{\partial x^{i}} g^{ij}\frac{\partial \bar{x}^{l}}{\partial x^{j}}. \tag{35}$$

Most importantly it is found from these expressions that one of the two bianisotropic couplings may vanish while the other one is nonvanishing, which is impossible within the context of diffeomorphism transforming media. Moreover, in the latter case the bianisotropic couplings must be symmetric matrices, which need no longer be the case in the present context.

(e) Finally, the result (28) and (29) reduces to the relations (15) and (16) if $g_{\bar{\mu}\bar{\bar{\nu}}}$ is a symmetric matrix of signature (3, 1), and, in addition, $\sqrt{-\bar{g}}=\sqrt{-\bar{\bar{g}}}$. This does not necessarily imply $\bar{x}^{\mu}=\bar{\bar{x}}^{\mu}$ but rather that there exists yet a different space which describes the same media properties in terms of the transformations of Sec. II.

A. Electric-magnetic duality and rotation

Finally, we should ask whether Eqs. (28) and (29) indeed describe the most general media fulfilling Definition 2. Taken separately, the two sets of equations in (1) and (2) do not exhibit more symmetries than diffeomorphisms. However, there exists the possibility of transformations that mix $F_{\mu\nu}$ and $H^{\mu\nu}$. Indeed a transformation of this type is known as electric-magnetic duality, which has important implications in modern theoretical high-energy physics [17]. It represents the fact that under the exchange

$$F_{\mu\nu} \leftrightarrow \tfrac{1}{4}\varepsilon_{\mu\nu\rho\sigma}H^{\rho\sigma} \tag{36}$$

or in terms of space vectors

$$B^{i} \rightarrow -\mathcal{D}^{i}, \quad \mathcal{H}_{i} \rightarrow -E_{i}, \tag{37}$$

$$E_{i} \rightarrow \mathcal{H}_{i}, \quad \mathcal{D}^{i} \rightarrow B^{i}, \tag{38}$$

the source-free equations of motion do not change (the action changes by an overall sign.) Of course, this duality transformation is problematic when applied to a solution with sources, as it transforms electric charges and currents into magnetic charges and currents and vice versa. In the remainder of this section we thus restrict to source-free solutions or should allow the possibility of artificial magnetic monopoles. Then it can be checked straightforwardly that electric-magnetic duality applied to the result (20), or (28) and (29), does not yield media relations not yet covered by diffeomorphisms alone.

However, as far as the equations of motion (1) and (2) are concerned, electric-magnetic duality can be promoted to a continuous U(1) symmetry with transformation[5]

$$\tilde{B}^{i} = \cos\alpha B^{i} - \sin\alpha\mathcal{D}^{i}, \quad \tilde{\mathcal{D}}^{i} = \cos\alpha\mathcal{D}^{i} + \sin\alpha B^{i}, \tag{39}$$

$$\tilde{E}_{i} = \cos\alpha E_{i} + \sin\alpha\mathcal{H}_{i}, \quad \tilde{\mathcal{H}}_{i} = \cos\alpha\mathcal{H}_{i} - \sin\alpha E_{i}. \tag{40}$$

These transformations comply with Definition 2 and thus their action onto a medium with general constitutive relation (A13) should be studied. The result

$$\tilde{\mathcal{D}}^{i} = [\cos^2\alpha\epsilon + \sin^2\alpha\mu + \sin\alpha\cos\alpha(\kappa+\xi)]^{ij}\tilde{E}_{j} + [\cos^2\alpha\kappa - \sin^2\alpha\xi + \sin\alpha\cos\alpha(\mu-\epsilon)]^{ij}\tilde{\mathcal{H}}_{j}, \tag{41}$$

$$\tilde{B}^{i} = [\cos^2\alpha\mu + \sin^2\alpha\epsilon - \sin\alpha\cos\alpha(\kappa+\xi)]^{ij}\tilde{\mathcal{H}}_{j} + [\cos^2\alpha\xi - \sin^2\alpha\kappa + \sin\alpha\cos\alpha(\mu-\epsilon)]^{ij}\tilde{E}_{j} \tag{42}$$

shows that the transformation acts trivially if $\epsilon=\mu$ and

[5]Notice that under the continuous transformation the action behaves as $\mathcal{S}\rightarrow(\cos^2\alpha-\sin^2\alpha)\mathcal{S}$ and thus for $\alpha=\pi/4$ transforms to zero. Therefore at the level of the action only the discrete duality transformation can be considered.

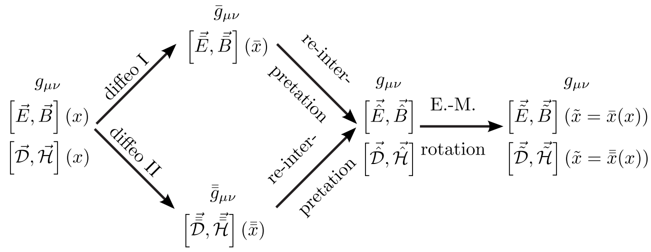

FIG. 3. Illustration and notation of the generalized "triple spacetime metamaterials" complemented by electric-magnetic rotation. The electric-magnetic rotation must act after the transformation of spacetime as these two steps do not commute.

$\xi=-\kappa$, in particular *in vacuo* and consequently for all diffeomorphism transforming media (15). However, they yield new media relations when acting on a solution of the type (28) and (29). Therefore these new relations are part of the materials covered by Definition 2. They are derived here for completeness, though their geometric interpretation is not immediate. The coordinate lines $\bar{x}^\mu(x)$ and $\bar{\bar{x}}^\mu(x)$ could be understood as the electromagnetic spaces of the linear combinations $(\bar{\bar{\vec{E}}},\bar{\bar{\vec{B}}})$ and $(\bar{\bar{\vec{\mathcal{D}}}},\bar{\bar{\vec{\mathcal{H}}}})$ as given in (39) and (40), respectively. Still, one should be careful with this interpretation: as the transformation of spacetime does not commute with the electric-magnetic rotation one cannot modify the situation in Fig. 3 in such a way that the two electromagnetic spaces $\bar{x}^\mu$ and $\bar{\bar{x}}^\mu(x)$ are identified with certain linear combinations of $(\vec{E},\vec{B})$ and $(\vec{\mathcal{D}},\vec{\mathcal{H}})$, respectively; rather the electric-magnetic rotation acts upon the fields after the transformation of spacetime.

IV. PERFECT LENS FROM INDEFINITE MATERIAL: AN EXAMPLE

To provide a better understanding of the formalism developed in the previous section a concrete example is demonstrated. To keep things simple we show how a proposal taken from the literature can be given a geometric interpretation.

In Ref. [6] it has been pointed out that two slabs of indefinite material (media with strong anisotropy) can form a perfect lens. Since, in constrast to standard diffeomorphism transforming media, strong anisotropy is available in triple spacetime metamaterials the question appears whether a geometric interpretation of the lens proposed in Ref. [6] can be given (see Ref. [18] for a related discussion.) We consider the lens to be an infinite slab in the x-y plane with a certain thickness in the z direction. In its simplest form the lens consists of two slabs of equal thickness d, where the media properties of the first slab are

$$\epsilon^{ij} = \mu^{ij} = \mathrm{diag}(1,1,-1), \tag{43}$$

while in the second slab

$$\epsilon^{ij} = \mu^{ij} = \mathrm{diag}(-1,-1,1). \tag{44}$$

To provide a geometric interpretation we start with the observation that a standard perfect lens with $\epsilon=\mu=-1$ may be produced by two different transformations, either a space

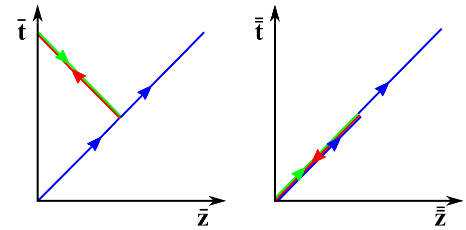

FIG. 4. (Color online) Mapping of the world-line $s(t)=(t,z(t))$ in the original space onto the deformed spaces by means of the two different transformations. Blue lines indicate the trajectory outside the lens (trivial mapping of all x^μ), the red line represents the first slab, and the green line the second slab. The parametrization of the world-line in the original space is assumed to obey $z(t)=t$.

inversion $\bar{z}=-z$, or a time reversal $\bar{t}=-t$. From Eqs. (15) and (16) it follows straightforwardly that these two transformations yield the same media properties. Within triple spacetime metamaterials we now may ask the question of what happens if space inversion is applied to one set of the fields, while time reversal is applied to the other set. For concreteness, space inversion is applied to the fields $\vec{E}$ and $\vec{B}$ and thus

$$\bar{z} = -z + Z_1, \tag{45}$$

where Z_1 is an unimportant constant necessary to meet the boundary conditions. All other fields $\bar{x}^\mu$ are mapped trivially. The second set of fields, $\vec{\mathcal{D}}$ and $\vec{\mathcal{H}}$, transform according to

$$\bar{\bar{t}} = -t + T_1 \tag{46}$$

with all other fields transformed trivially. Consider now these two transformations in Eqs. (28) and (29). From (19) one finds

$$g^{\bar{i}\bar{j}} = g^{\bar{\bar{i}}\bar{\bar{j}}} = \mathrm{diag}(1,1,-1). \tag{47}$$

Furthermore, $\bar{s}=\bar{\bar{s}}=-1$ as both tranformations are orientation changing. Furthermore, $g_{\bar{0}\bar{\bar{0}}}=1$ (remember our convention $g_{\bar{0}}^{\bar{0}}=-1$), such that indeed the media properties (43) are found in this slab.

This single slab of indefinite material does not establish a perfect lens, as can be seen easily when studying how a world-line $s(\tau)=(t(\tau),x(\tau),y(\tau),z(\tau))$ is mapped onto the two deformed spaces. For simplicity time may be interpreted with the parametrization variable $t(\tau)=\tau$ and furthermore we can assume without loss of generality $z(\tau)=\tau=t$.[6] The situation is illustrated in Fig. 4. As can be seen the mappings do not agree after the first slab, both trajectories are at the same point $z=0$, but they differ in time. This must be corrected in the second slab. In our example we have chosen a completely trivial mapping for $\vec{\mathcal{D}}$ and $\vec{\mathcal{H}}$, so these fields propagate in the second slab as in free space. $\vec{E}$ and $\vec{B}$, however, are transformed as

[6]"Time" t here is just a variable to parametrize the world-line, this choice does not make any statements about the speed of light.

$$\bar{t} = -t + T_2, \quad \bar{z} = -z + Z_2, \tag{48}$$

which actually reverses the transformation (45) and at the same time applies (46). Not surprisingly, the two trajectories now meet at the same point again and the perfect lens is established. Again it is immediate that this transformation establishes the media relations (44). Therefore, triple spacetime metamaterials indeed can provide a geometric interpretation of the lens of Ref. [6]. It should be noted, that this specific lens has focal length zero, it shrinks the effective width of the device from $2D$ to zero, but not to a negative value as is necessary for a real lens.

V. ENERGY, MOMENTUM, AND WAVE VECTOR

So far we have studied solutions of Maxwell's equations which—up to rescalings—are equivalent to certain vacuum solutions. Still we did not ask up to what point these transformation materials really are "media that look like empty space." To do so it is not sufficient to consider the transformation of the fields and sources, but equally well we should look at the conservation laws, summarized in the conservation of the stress-energy-momentum (SEM) tensor. While in the generic situation of electrodynamics in media, the definition of the "SEM tensor of electrodynamics" is not unique [19,20], we do not have to deal with these subtleties in the present situation as our (idealized) media are lossless and dispersion free and thus allow for a definition of a complete action [cf. Eq. (7)] without any reference to "matter." Therefrom we immediately derive the covariant SEM tensor

$$T^{\mu\nu} = -\frac{1}{4\pi}\left(F_{\rho\sigma}g^{\sigma\{\mu}H^{\nu\}\rho} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}H^{\rho\sigma}\right), \tag{49}$$

$$D_\mu T^{\mu\nu} = 0. \tag{50}$$

The advantage of this tensor over the canonical SEM tensor is the simple behavior under diffeomorphisms: being a real tensor field, $T_{\mu\nu}$ transforms exactly in the same way as the metric.

Let us now look at the materials as described in Sec. II. Thanks to its transformation properties the SEM tensor in the electromagnetic space follows immediately as

$$\bar{T}^{\mu\nu} = -\frac{1}{4\pi}\left(\bar{F}_{\rho\sigma}\bar{g}^{\sigma\{\mu}\bar{H}^{\nu\}\rho} + \frac{1}{4}\bar{g}^{\mu\nu}\bar{F}_{\rho\sigma}\bar{H}^{\rho\sigma}\right). \tag{51}$$

But how about $\tilde{T}_{\mu\nu}$? Of course one could define an "induced SEM tensor" from the electromagnetic space as [cf. Eqs. (12) and (13)]

$$\tilde{T}_I^{\mu\nu} = \mp\frac{\sqrt{-g}}{\sqrt{-\bar{g}}}\frac{1}{4\pi}\left(\tilde{F}_{\rho\sigma}\bar{g}^{\sigma\{\mu}\tilde{H}^{\nu\}\rho} + \frac{1}{4}\bar{g}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{H}^{\rho\sigma}\right), \tag{52}$$

but obviously this tensor is not conserved in laboratory space, $D_\mu\tilde{T}_I^{\mu\nu}\neq 0$, since it depends explicitly on the metric $\bar{g}_{\mu\nu}$. In other words, the crucial trick to re-interpret in the dynamical equations the coordinates in electromagnetic space, $\bar{x}^\mu$, as those in laboratory space, x^μ, works in the equations of motion (1) and (2), but does not work for the SEM tensor and its conservation.

Of course, the correct SEM tensor in laboratory space immediately follows from (49) as

$$\tilde{T}^{\mu\nu} = -\frac{1}{4\pi}\left(\tilde{F}_{\rho\sigma}g^{\sigma\{\mu}\tilde{H}^{\nu\}\rho} + \frac{1}{4}g^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{H}^{\rho\sigma}\right). \tag{53}$$

Clearly, requiring equivalence of the two tensors would not even allow for conformal transformations. But even when looking at integrated quantities (total energy and momentum flux in the material),

$$P^\mu = \int d^3x\sqrt{\gamma}T^{0\mu}, \tag{54}$$

the induced tensor does not yield the correct quantity in laboratory space. Of course, the situation is even more complicated for triple spacetime metamaterials: since the transformation of the explicit metrics appearing in Eq. (49) is not defined, an induced SEM tensor cannot even be defined.

Instead of the correct, directly evaluated SEM tensor (53), a slightly different tensor is considered in the following. To see its advantage we make the standard assumption that our laboratory space metric has $g_{00}=-1$ (x^0 is our laboratory time) and $g_{0i}=0$ (the measure of distances is time independent.) Then it is straightforward that the quantity

$$M^{\mu\nu} = -g^{\mu\rho}F_{\rho\sigma}H^{\sigma\nu} \tag{55}$$

contains the Poynting vector and the direction of the wave vector,

$$S^i = M^{0i} = \epsilon^{ijk}E_j\mathcal{H}_k, \tag{56}$$

$$n^i = M^{i0} = \gamma^{ij}\epsilon_{jkl}\mathcal{D}^k B^l \parallel k^i. \tag{57}$$

We recall that the original fields obey the constitutive relations of vacuous space and thus trivially $M^{0i}=M^{i0}$. From the transformation rules (21) the transformed tensor is found as

$$\tilde{M}^{\mu\nu} = -\bar{s}\frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}}g^{\mu\rho}\bar{F}_{\rho\sigma}\bar{\bar{H}}^{\sigma\nu}$$

$$= -\bar{s}\frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}}g^{\mu\rho}\frac{\partial x^\lambda}{\partial\bar{x}^\rho}F_{\lambda\tau}\frac{\partial x^\tau}{\partial\bar{x}^\sigma}\frac{\partial\bar{\bar{x}}^\sigma}{\partial x^\alpha}H^{\alpha\beta}\frac{\partial\bar{\bar{x}}^\nu}{\partial x^\beta}. \tag{58}$$

The transformation law of $M^{\mu\nu}$ encodes in a geometric language how energy flux and phase velocity behave in a medium. For simplicity let us now concentrate on media without bianisotropic couplings, in other words we allow for general spatial transformations as well as stretchings and reversal of time, but keep $\bar{g}_{0i}=\bar{\bar{g}}_{0i}=0$. Then we find for the transformed space vectors [see Eqs. (A9)–(A12)]:

$$\tilde{S}^i = \tilde{M}^{0i} = -\bar{s}\bar{\bar{s}}\bar{\sigma}\frac{\sqrt{-\bar{\bar{g}}_{00}}}{g_{00}}\frac{\partial x^0}{\partial\bar{x}^0}\epsilon^{ijk}\frac{\partial x^m}{\partial\bar{x}^j}E_m\frac{\partial x^n}{\partial\bar{\bar{x}}^k}\mathcal{H}_n, \tag{59}$$

$$\tilde{n}^i = \tilde{M}^{i0} = \bar{\sigma}\frac{\sqrt{-\bar{\bar{g}}}\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\frac{\partial\bar{\bar{x}}^0}{\partial x^0}\gamma^{ij}\epsilon_{jkl}\frac{\partial\bar{\bar{x}}^k}{\partial x^m}\mathcal{D}^m\frac{\partial\bar{x}^l}{\partial x^n}B^n. \tag{60}$$

The transformation of the Poynting vector may be abbreviated as

$$\tilde{S}^i = T^{ijk}E_j\mathcal{H}_k, \tag{61}$$

and it is then easily seen that n_i transforms as

$$\tilde{n}_i = \bar{\sigma}\bar{s}\bar{\bar{\sigma}}\bar{\bar{s}}\frac{\sqrt{\bar{\bar{\gamma}}\bar{\gamma}}}{2\gamma}g_{00}\frac{\partial\bar{\bar{x}}^0}{\partial x^0}\frac{\partial\bar{x}^0}{\partial x^0}U_{ikj}D^jB^k, \tag{62}$$

where U_{ijk} is the inverse of T^{ijk} in the sense of

$$U_{ijk}T^{ljk} = \delta^l_i. \tag{63}$$

While these formulas might look cumbersome, their geometric interpretation actually is quite straightforward. In the case of diffeomorphism transforming materials, $\bar{g}_{\mu\nu}=\bar{\bar{g}}_{\mu\nu}$, Eq. (59) states that S^i behaves under purely spatial transformations as a covector [1], while n_i from Eq. (60) behaves as a vector:

$$\tilde{S}^i = \bar{s}\frac{\sqrt{-\bar{g}}}{\sqrt{-g_{00}}\sqrt{-g}}\frac{\partial x^0}{\partial\bar{x}^0}\frac{\partial\bar{x}^i}{\partial x^j}S^j, \tag{64}$$

$$\tilde{n}_i = \bar{s}\sqrt{-\bar{g}}\frac{\partial\bar{x}^0}{\partial x^0}\frac{\partial x^j}{\partial\bar{x}^i}n_j. \tag{65}$$

Of course, the relative orientation of S^i and n^i is preserved under the diffeomorphisms, but this is no longer true for $\tilde{S}^i$ and $\tilde{n}^i$, since indices are raised or lowered by the space metric γ_{ij} in laboratory space as opposed to $\bar{\gamma}_{ij}$ in electromagnetic space.

For triple spacetime metamaterials no linear transformation $\tilde{S}^i=T^i{}_jS^j$ exists. This makes the interpretation a little bit more complicated, but at the same time is the source of the numerous additional possibilities within this generalized setup. In general, the value of the element T^{ijk} defines the component of the Poynting vector in direction x^i as generated by electric and magnetic fields that point in the original space in the directions x^j and x^k, respectively. In this way it is easy to engineer the direction of the Poynting vector in the medium for a given polarization of the incoming wave in vacuum. Similar conclusions apply for the transformation matrix U_{ijk}, with the notable restriction that $\tilde{n}^i$ can be parallel or antiparallel to k^i. Whether $(\tilde{\mathcal{D}}^i,\tilde{B}^j,\tilde{k}^l)$ form a right- or left-handed triple can be deduced from

$$\epsilon_{ijk}\tilde{\mathcal{D}}^j\tilde{B}^k = \frac{\tilde{k}_i}{\tilde{\omega}}\tilde{E}_j\epsilon^{jk}\tilde{E}_k = -\bar{s}\frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{\gamma g_{\bar{\bar{0}}\bar{\bar{0}}}}}\frac{\tilde{k}_i}{\tilde{\omega}}\tilde{E}_j\gamma^{\bar{\bar{j}}k}\tilde{E}_k. \tag{66}$$

A. Wave vector and dispersion relations

While the above relations correctly reproduce the direction of the Poynting and the wave vector, they cannot distinguish between propagating and evanescent modes. Consider as an example the following transformation:

$$\bar{x}^0 = x^0, \quad \bar{x}^i = x^i, \quad \bar{\bar{x}}^0 = -x^0, \quad \bar{\bar{x}}^i = x^i. \tag{67}$$

From Eqs. (15) and (16) it is found that this is a homogeneous material with $\epsilon=-1$ and $\mu=1$. As fields *in vacuo*, $\vec{E}=\vec{\mathcal{D}}$ and $\vec{B}=\vec{\mathcal{H}}$, we consider a monochromatic wave

$$\vec{E} = \vec{e}e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}, \quad \vec{k}\cdot\vec{e} = 0, \tag{68}$$

$$\vec{B} = \vec{b}e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}, \quad \vec{b} = \frac{1}{\omega}\vec{k}\times\vec{e}. \tag{69}$$

After the transformation the fields $\vec{\bar{\bar{E}}},\vec{\bar{\bar{B}}}$ and $\vec{\bar{\bar{D}}},\vec{\bar{\bar{H}}}$ refer to the original fields at different time instances:

$$\vec{\bar{\bar{E}}}(\bar{x}^\mu = \bar{x}^\mu(x)) = \vec{E}(\vec{x},t), \tag{70}$$

$$\vec{\bar{\bar{B}}}(\bar{x}^\mu = \bar{x}^\mu(x)) = \vec{B}(\vec{x},t), \tag{71}$$

$$\vec{\bar{\bar{\mathcal{D}}}}(\bar{x}^\mu = \bar{\bar{x}}^\mu(x)) = -\vec{\mathcal{D}}(\vec{x},-t), \tag{72}$$

$$\vec{\bar{\bar{\mathcal{H}}}}(\bar{x}^\mu = \bar{\bar{x}}^\mu(x)) = -\vec{\mathcal{H}}(\vec{x},-t). \tag{73}$$

Of course, Maxwell's equations are satisfied by the new fields by construction. Still, the partial exchange of positive and negative angular frequencies has important implications in the dispersion relation as any propagating wave in vacuo becomes evanescent in the medium, and vice versa.

Although this behavior may not appear immediate when transforming the monochromatic wave (68) and (69) with (70)–(73), it can be made explicit from geometric quantities as well. Indeed, from the relativistic wave equation [12]

$$D_\nu\chi^{\mu\nu\rho\sigma}D_\rho A_\sigma = -J^\nu \tag{74}$$

it follows straightforwardly that "triple-space metamaterials" in the absence of charges and currents and in the limit of approximate homogeneity obey the dispersion relation

$$g^{\bar{\bar{\mu}}\bar{\nu}}k_\mu k_\nu = 0, \quad k_\mu = (\omega,\vec{k}). \tag{75}$$

In our example the partial reversal of time yields $g^{\bar{\bar{0}}\bar{0}}=1$ and thus $\omega^2+\vec{k}^2=0$.

VI. NONINVARIANT TRANSFORMATIONS

Within the approaches to transformation media discussed so far invariant transformations of the equations of motions were used exclusively. This means that the transformations do not introduce charges or currents; in other words, the transformation medium based on a source-free vacuum solution will be source-free as well. What happens if this restriction is abandoned? Still insisting on a constitutive relation of the form (5) this suggests the following definition.

Definition 3. A transformation medium is defined by an arbitrary linear transformation applied to a (not necessarily source-free) vacuum solution of Maxwell's equations. The linear transformation constitutes the media properties as well as charges and currents of the transformation medium.

Although not in its most general form, this approach was proposed in [10,11]. Starting from the vacuum relation (6) the most general linear relation can be achieved by the field transformations[7]

$$\bar{H}^{\mu\nu} = \Omega^{\mu\nu}{}_{\rho\sigma} H^{\rho\sigma}, \quad F_{\mu\nu} = \Psi_{\mu\nu}{}^{\rho\sigma} \bar{F}_{\rho\sigma}, \tag{76}$$

with the transformed $\bar{\chi}$,

$$\bar{\chi}^{\mu\nu\rho\sigma} = (\Omega\chi\Psi)^{\mu\nu\rho\sigma}. \tag{77}$$

The original fields $F_{\mu\nu}$ and $H^{\mu\nu}$ by assumption are solutions to the equations of motion. If we allow besides the standard electric four-current J^μ also a magnetic four-current J^μ_M, any transformation of the type (76) can be mapped on a solution of the new equations [$\hat{\Psi}_{\mu\nu}{}^{\rho\sigma}$ is the inverse matrix $\hat{\Psi}_{\mu\nu}{}^{\lambda\tau}\Psi_{\lambda\tau}{}^{\rho\sigma} = (\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu)/2$]

$$D_\mu \bar{H}^{\mu\nu} = \bar{J}^\nu = D_\mu(\Omega^{\mu\nu}{}_{\rho\sigma} H^{\rho\sigma}), \tag{78}$$

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu \bar{F}_{\rho\sigma} = \bar{J}^\mu_M = \epsilon^{\mu\nu\rho\sigma}\partial_\nu(\hat{\Psi}_{\rho\sigma}{}^{\tau\lambda} F_{\tau\lambda}), \tag{79}$$

provided appropriate currents are introduced. These transformations in general are not symmetry transformations and accordingly a source free solution is no longer mapped on another source free solution. The transformations (76) and the ensuing equations of motion (78) and (79) are the relativistic form of the transformations proposed in Refs. [10,11].

We recover the invariant transformations of Sec. III by introducing the restrictions

$$\Omega^{\mu\nu}{}_{\rho\sigma} = S^\mu{}_\rho S^\nu{}_\sigma, \quad \Psi_{\mu\nu}{}^{\rho\sigma} = T_\mu{}^\rho T_\nu{}^\sigma. \tag{80}$$

Let us first count the degrees of freedom in the transformations. $\chi^{\mu\nu\rho\sigma}$ is a rank-4 tensor, antisymmetric in (μ,ν) and (ρ,σ), and thus has 36 independent components (20 components of the principal part, 15 of the skewon part, and one axion coupling). The same applies to the transformation matrices Ω and Ψ. Restriction to diffeomorphisms according to Eq. (80) reduces this number to six parameters for each Ω and Ψ; the electric-magnetic rotation of Sec. III A adds another parameter in the form of a rotation angle. Here, another important difference between a transformation material according to Definitions 2 and 3 emerges. In both cases the transformation yielding certain media properties is not unique. In the former case, however, different transformations are physically equivalent as they are connected by symmetry transformations (isometries of the laboratory metric $g_{\mu\nu}$). In the more general case of Eq. (76) the different transformations need not be physically equivalent. As is immediate from Eq. (76) a certain medium exhibiting sources due to noninvariant transformations can be designed using electric charges and currents, magnetic charges and currents, or both, which clearly characterizes physically different situations with the same media properties χ.

Does there exist the possibility of a geometric interpretation of Eq. (76)? If this shall be possible space must be transformed differently for different components of $F^{\mu\nu}$ and $H_{\mu\nu}$. In fact, the most general linear transformation can be interpreted as a separate transformation of spacetime for each component of the two tensors. Let us provide a simplified example, where independent spatial transformations are applied to $\vec{E}$, $\vec{B}$, $\vec{\mathcal{D}}$, and $\vec{\mathcal{H}}$. Laboratory space is denoted by x^i, the electromagnetic spaces by x^i_E, x^i_B, x^i_D, and x^i_H, respectively. Under time-independent spatial transformations all four fields transform as (co)vectors and thus Eqs. (A9)–(A12) suggest the interpretations

$$\tilde{E}_i = s_E \frac{\partial x^j}{\partial x^i_E} E_j, \quad \tilde{B}^i = \frac{\sqrt{\gamma_B}}{\sqrt{\gamma}} \frac{\partial x^i_B}{\partial x^j} B^j, \tag{81}$$

$$\tilde{\mathcal{D}}^i = \frac{\sqrt{\gamma_D}}{\sqrt{\gamma}} \frac{\partial x^i_D}{\partial x^j} \mathcal{D}^j, \quad \tilde{\mathcal{H}}_i = s_H \frac{\partial x^j}{\partial x^i_H} \mathcal{H}_j, \tag{82}$$

yielding for the permittivity and permeability

$$\epsilon^{ij} = s_E \frac{\sqrt{\gamma_D}}{\sqrt{\gamma}} \frac{\partial x^i_D}{\partial x^k} \gamma^{kl} \frac{\partial x^j_E}{\partial x^l}, \tag{83}$$

$$\mu^{ij} = s_H \frac{\sqrt{\gamma_B}}{\sqrt{\gamma}} \frac{\partial x^i_B}{\partial x^k} \gamma^{kl} \frac{\partial x^j_H}{\partial x^l}. \tag{84}$$

As is seen from (1) and (2), Gauss's law for $\tilde{B}^i$ and $\tilde{\mathcal{D}}^i$ remains unchanged, while Faraday's and Ampère's laws are changed according to

$$\frac{\sqrt{\gamma_E}}{\sqrt{\gamma_B}} \frac{\partial x^i_E}{\partial x^j_B} \nabla_0 \tilde{B}^j + \epsilon^{ijk}\partial_j \tilde{E}_k = 0, \tag{85}$$

$$\epsilon^{ijk}\partial_j \tilde{\mathcal{H}}_k - \frac{\sqrt{\gamma_H}}{\sqrt{\gamma_D}} \frac{\partial x^i_H}{\partial x^j_D} \nabla_0 \tilde{\mathcal{D}}^j = 0, \tag{86}$$

making the electric and magnetic currents

$$j^i = -\left(\delta^i_j - \frac{\sqrt{\gamma_H}}{\sqrt{\gamma_D}} \frac{\partial x^i_H}{\partial x^j_D}\right) \nabla_0 \tilde{\mathcal{D}}^j, \tag{87}$$

$$j^i_M = \left(\delta^i_j - \frac{\sqrt{\gamma_E}}{\sqrt{\gamma_B}} \frac{\partial x^i_E}{\partial x^j_B}\right) \nabla_0 \tilde{B}^j \tag{88}$$

necessary. For completeness it should be mentioned that the transformations (81) and (82) allow a straightforward interpretation since each of the four Maxwell's equations still can be transformed as a whole. Taking even more general transformations, e.g., transforming each component of the electric and magnetic fields separately, no longer allows this manipulation in a simple way and thus will make the derivation of the necessary media parameters more complicated.

[7]Alternatively, one could start from a relation of the form

$$\bar{H}^{\mu\nu} = A^{\mu\nu}{}_{\rho\sigma} H^{\rho\sigma} + B^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

but the form (76) appears more transparent to us.

VII. CONCLUSIONS

In this paper we have introduced a generalization of the concept of diffeomorphism transforming media, the basis of transformation optics [1–3]. As the basic idea we have found that spacetime can be transformed differently for the field strength tensor (containing $\vec{E}$ and $\vec{B}$) and the excitation tensor (encompassing $\vec{\mathcal{D}}$ and $\vec{\mathcal{H}}$). This extension allows design of nonreciprocal media, in particular the permittivity and permeability need not longer be symmetric. Furthermore, this approach permits a geometric interpretation of indefinite media [6,7].

Diffeomorphism transforming media are motivated by the wish to produce a medium that looks like a transformed but empty space. The basis of this interpretation is Fermat's principle applied to these media [9]: indeed, it is found that in a transformation medium the light rays travel along trajectories as if the medium was a transformed, empty space. Still, the transformation medium in general is quite different from transformed empty space, if the conservation laws from the stress-energy-momentum tensor are considered. This aspect is even more important within the extension proposed here, as there exist two different transformed (electromagnetic) spaces and light rays do not follow the geodesics of any of them. We have shown that one can make a virtue out of necessity: the geometric approach does not just provide a tool to design the path of light in a medium, but equally well it may be used to design the behavior of (parts of) the stress-energy-momentum tensor, e.g., the direction of the Poynting vector, and/or the behavior of the wave vector. Here the proposed generalization offers many more possibilities compared to the known diffeomorphism transforming media. In particular, we have derived the geometric relations that describe the transformation of the Poynting vector and of the direction of the wave vector as well as the dispersion relation.

Finally we have commented on a different route to generalize the notion of transformation media [10,11]. These field-transforming media are not based on invariant transformations of the equations of motion, and consequently source-free solutions of the original configuration are not mapped onto source-free solutions of the new medium. We have shown that also this approach may be covered by a generalized concept of coordinate transformations. Still, there remains a fundamental difference between the approach of Refs. [10,11] and the one discussed here: While in the former case the transformations are ultralocal (the transformed fields at the point x^μ are defined in terms of the original fields at this point), in the latter they are essentially nonlocal, as the transformed fields at $\tilde{x}^\mu$ are related to the original fields at some $x^\mu \neq \tilde{x}^\mu$. The preferable approach depends on the specific problem at hand; also a combination of the two is conceivable.

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APPENDIX: COVARIANT FORMULATION

In this appendix we present our notations and conventions regarding the covariant formulation of Maxwell's equations on a possibly curved manifold. For a detailed introduction to the topic we refer to the relevant literature, e.g., [12,13]. Throughout the whole paper natural units with $\epsilon_0 = \mu_0 = c = 1$ are used.

Greek indices $\mu, \nu, \rho, \ldots$ are spacetime indices and run from 0 to 3, latin indices $i, j, k, \ldots$ space indices with values from 1 to 3. For the metric we use the "mostly plus" convention, so the standard flat metric is $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. If we interpret $x^0 = t$ with (laboratory) time the space metric can be obtained as [13]

$$\gamma^{ij} = g^{ij}, \quad \gamma_{ij} = g_{lk} - \frac{g_{0i} g_{0j}}{g_{00}}, \quad \gamma^{ij}\gamma_{jk} = \delta^i_k. \tag{A1}$$

This implies the relation between the determinant of the spacetime metric, g, and that of the space metric, γ,

$$-g = -g_{00}\gamma. \tag{A2}$$

The four-dimensional Levi-Cività tensor is defined as

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}[\mu\nu\rho\sigma], \quad \epsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}[\mu\nu\rho\sigma], \tag{A3}$$

with $[0123]=1$. Therefore the reduction of the four-dimensional to the three-dimensional tensor reads

$$\epsilon_{0ijk} = \sqrt{-g_{00}}\epsilon_{ijk}, \quad \epsilon^{0ijk} = -\frac{1}{\sqrt{-g_{00}}}\epsilon^{ijk}. \tag{A4}$$

An additional complication when in the definition of $\bar{\epsilon}_{ijk}$ and $\bar{\bar{\epsilon}}_{ijk}$, since the orientation of the spacetime manifold may change without changing the orientation of space (e.g., by a mapping $\bar{t} = -t$.) Therefore the corresponding relations should be written as

$$\bar{\epsilon}_{0ijk} = \bar{\sigma}\sqrt{-\bar{g}_{00}}\bar{\epsilon}_{ijk}, \quad \bar{\bar{\epsilon}}_{0ijk} = \bar{\bar{\sigma}}\sqrt{-\bar{\bar{g}}_{00}}\bar{\bar{\epsilon}}_{ijk}, \tag{A5}$$

where $\bar{\sigma} = +1$ if space and spacetime have the same orientation, and $\bar{\sigma} = -1$ otherwise.

The field strength tensor $F_{\mu\nu}$ encompasses the electric field and the magnetic induction, the excitation tensor $H^{\mu\nu}$ the displacement vector and the magnetic field with the identification

$$E_i = F_{0i}, \quad B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}, \tag{A6}$$

$$\mathcal{D}^i = -\sqrt{-g_{00}}H^{0i}, \quad \mathcal{H}_i = -\frac{\sqrt{-g_{00}}}{2}\epsilon_{ijk}H^{jk}. \tag{A7}$$

Finally, electric charge and current are combined into a four-current $J^\mu = (\rho/\sqrt{-g_{00}}, j^i/\sqrt{-g_{00}})$. $F_{\mu\nu}$ and $H^{\mu\nu}$ are tensors, thus under the transformations of Sec. III they behave as

$$\bar{F}_{\mu\nu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} F_{\rho\sigma} \frac{\partial x^\sigma}{\partial \bar{x}^\nu}, \quad \bar{\bar{H}}^{\mu\nu} = \frac{\partial \bar{\bar{x}}^\mu}{\partial x^\rho} H^{\rho\sigma} \frac{\partial \bar{\bar{x}}^\nu}{\partial x^\sigma}. \tag{A8}$$

This implies for the transformed space vectors in laboratory space

$$\tilde{E}_i = \bar{s}\left[\left(\frac{\partial x^0}{\partial \bar{x}^\mu}\frac{\partial x^j}{\partial \bar{x}^i} - \frac{\partial x^0}{\partial \bar{x}^i}\frac{\partial x^j}{\partial \bar{x}^0}\right)E_j - \frac{\partial x^j}{\partial \bar{x}^0}\frac{\partial x^k}{\partial \bar{x}^i}\epsilon_{jkl}B^l\right], \tag{A9}$$

$$\tilde{B}^i = \bar{\sigma}\frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\frac{\partial \bar{x}^i}{\partial x^j}B^j - \bar{s}\epsilon^{ijk}\frac{\partial x^0}{\partial \bar{x}^j}\frac{\partial x^l}{\partial \bar{x}^k}E_l, \tag{A10}$$

$$\tilde{\mathcal{D}}^i = \frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}}\left[\left(\frac{\partial \bar{\bar{x}}^0}{\partial x^0}\frac{\partial \bar{\bar{x}}^i}{\partial x^j} - \frac{\partial \bar{\bar{x}}^0}{\partial x^j}\frac{\partial \bar{\bar{x}}^i}{\partial x^0}\right)\mathcal{D}^j + \frac{\partial \bar{\bar{x}}^0}{\partial x^j}\frac{\partial \bar{\bar{x}}^i}{\partial x^k}\epsilon^{jkl}\mathcal{H}_l\right], \tag{A11}$$

$$\tilde{\mathcal{H}}_i = \bar{\bar{s}}\bar{\bar{\sigma}}\frac{\sqrt{-\bar{\bar{g}}_{00}}}{\sqrt{-g_{00}}}\frac{\partial x^j}{\partial \bar{\bar{x}}^i}\mathcal{H}_j + \frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}}\epsilon_{ijk}\frac{\partial \bar{\bar{x}}^j}{\partial x^0}\frac{\partial \bar{\bar{x}}^k}{\partial x^l}\mathcal{D}^l. \tag{A12}$$

We characterize the general linear, lossless media usually by means of the Tellegen relations

$$\mathcal{D}^i = \epsilon^{ij}E_j + \kappa^{ij}\mathcal{H}_j, \quad B^i = \mu^{ij}\mathcal{H}_j + \xi^{ij}E_j. \tag{A13}$$

In terms of field strength and excitation tensor the media relations become the Boys-Post relation

$$H^{\mu\nu} = \frac{1}{2}\chi^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{A14}$$

where χ must be invertible with inverse

$$\hat{\chi}_{\mu\nu\lambda\tau}\chi^{\lambda\tau\rho\sigma} = (\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu). \tag{A15}$$

By virtue of Eqs. (A6) and (A7), Eq. (A14) may be written as

$$\begin{pmatrix}\mathcal{H}_i \\ \mathcal{D}^i\end{pmatrix} = \begin{pmatrix}\mathfrak{C}_i^{\ j} & \mathfrak{B}_{ij} \\ \mathfrak{A}^{ij} & \mathfrak{D}^i_{\ j}\end{pmatrix}\begin{pmatrix}E_j \\ B^j\end{pmatrix}, \tag{A16}$$

with

$$\mathfrak{A}^{ij} = -\sqrt{-g_{00}}\chi^{0i0j}, \tag{A17}$$

$$\mathfrak{B}_{ij} = \frac{1}{8}\sqrt{-g_{00}}\epsilon_{ikl}\epsilon_{jmn}\chi^{klmn}, \tag{A18}$$

$$\mathfrak{C}_i^{\ j} = -\frac{1}{2}\sqrt{-g_{00}}\epsilon_{ikl}\chi^{kl0j}, \tag{A19}$$

$$\mathfrak{D}^i_{\ j} = \frac{1}{2}\sqrt{-g_{00}}\epsilon_{jkl}\chi^{0ikl}. \tag{A20}$$

The Tellegen and Boys-Post formulations are related by

$$\epsilon^{ij} = (\mathfrak{A} - \mathfrak{D}\mathfrak{B}^{-1}\mathfrak{C})^{ij}, \quad \kappa^{ij} = (\mathfrak{D}\mathfrak{B}^{-1})^{ij}, \tag{A21}$$

$$\mu^{ij} = (\mathfrak{B}^{-1})^{ij}, \quad \xi^{ij} = -(\mathfrak{B}^{-1}\mathfrak{C})^{ij}. \tag{A22}$$

Finally, we mention that the rank-4 tensor $\chi^{\mu\nu\rho\sigma}$ may be decomposed as [15,16]

$$\chi^{\mu\nu\rho\sigma} = {}^{(1)}\chi^{\mu\nu\rho\sigma} + \epsilon^{\mu\nu\lambda[\rho}S_\lambda^{\ \sigma]} - \epsilon^{\rho\sigma\lambda[\mu}S_\lambda^{\ \nu]} + \alpha\epsilon^{\mu\nu\rho\sigma}, \tag{A23}$$

where the principal part ${}^{(1)}\chi^{\mu\nu\rho\sigma}$ has no part completely antisymmetric in its indices and is symmetric under the exchange $(\mu,\nu) \leftrightarrow (\rho,\sigma)$. The principal part has been discussed extensively in [12]. S_μ^ν was introduced in Refs. [15,16] as the skewon part (related to chiral properties of the material [16,21]), while α represents the well-known axion coupling [22].

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