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Linear and nonlinear optics in curved space

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Starting from Maxwell equations in curved space, we derive the field equations of light when constrainted to a curved surface via a possible nonlinear waveguide. We show that surfaces with constant extrinsic curvature, in particular minimal surfaces, allow solutions with a well-defined polarization. We derive a generalized nonlinear Schrödinger equation and solve the linear propagation for surfaces of revolution with constant positive and negative Gaussian curvature.

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I. INTRODUCTION

Recent years have witnessed extensive theoretical studies on analog models of general relativity (GR) [1–7]. Moreover, there are experimental attempts to find analogous effects, for example, the Hawking effect in the laboratory. Promising candidates for the experimental realizations of these analog models focus mainly on Bose-Einstein condensates or optics [5].

A few groups have considered also the case of a quantum mechanical particle, which is confined on a curved surface [8–11]. These works show some connections to a particle in a two-dimensional curved space if the effective model is used that was developed in [8]. As shown in [9,10], this effective model shows quantitative agreement with the exact results. One can also consider these as analog models of GR. These studies are not only of theoretical interest, since one can construct complex nanostructures with new properties [12,13] due to localization of electrons or light around a curved surface.

In optics we have the possibility to study the effects of curved space if we abandon one spatial dimension and confine light simply on a curved surface. Experimentally, this can be achieved by a film waveguide on such a surface. As long as the principal curvatures are not too large compared to the wave number in the waveguide, light will remain on the surface. As an example, it was theoretically shown [14,15] that one can find topological Bloch oscillations, where the change of the refractive index is given by the change of geometry.

In the present work, starting from the coordinate-independent Maxwell equations, we develop an effective theory that describes the propagation of light on a general curved surface. As discussed, the confinement could be achieved by a film waveguide which is attached to the curved surface of a three-dimensional body. Similarly to the flat case, we derive a generalized nonlinear Schrödinger equation which describes the propagation of light in a curved nonlinear waveguide. Furthermore, we study the linear propagation of an initially Gaussian profile on surfaces of revolution with constant Gaussian curvature.

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II. FIELD EQUATIONS

Let us consider the propagation of a monochromatic wave with frequency $\omega=k_0c$ in a general curved waveguide, i.e., a film waveguide on a curved surface S. To describe the full evolution of the electromagnetic field on such a surface, we consider Maxwell's equations written in general coordinates [16],

$$\eta^{ijk}\partial_j E_k + \frac{1}{c}\frac{\partial B^i}{\partial t} = 0, \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} B^i = 0, \tag{1}$$

and

$$\eta^{ijk}\partial_j H_k - \frac{1}{c}\frac{\partial D^i}{\partial t} = 0, \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} D^i = 0. \tag{2}$$

The indices i,j,k run from 1 to 3 and $\sqrt{g}\,\eta^{ijk}=\epsilon^{ijk}$ is the covariant generalization of the completely antisymmetric Levi-Cività tensor $\epsilon^{123}=1$. The electric displacement field is given through $D^i=E^i+P^i$, where the polarization P^i can also be nonlinear in the electric field E^i. Here g_{ij} are the coefficients of the induced metric

$$ds^2 = g_{ij}dx^i dx^j \tag{3}$$

with respect to the curvilinear coordinates x^1, x^2, x^3, and $g=\det(g_{ij})$. From this we can derive the wave equation for the electric or magnetic field. We focus on the electric field, since for a monochromatic wave the magnetic field can always be derived by using the first equation of (1). The wave equation takes the form

$$-\Delta_g E^i + \frac{1}{\sqrt{g}}\partial_j \sqrt{g} g^{ik}\partial_k E^j + \frac{1}{c^2}\frac{\partial^2 E^i}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 P^i}{\partial t^2}, \tag{4}$$

where Δ_g is the well-known covariant Laplace operator

$$\Delta_g = \frac{1}{\sqrt{g}}\partial_j \sqrt{g} g^{jk}\partial_k \tag{5}$$

with respect to the metric tensor [17]. The second term in (4) is an extra term, which is not present in the case of a scalar field [8–10,14,15]. In Cartesian coordinates this is the usual divergence term, which is zero for homogeneous materials due to the Gaussian constraint. This is not the case in curved coordinates, since curved space acts like an effectively inhomogeneous medium [16].

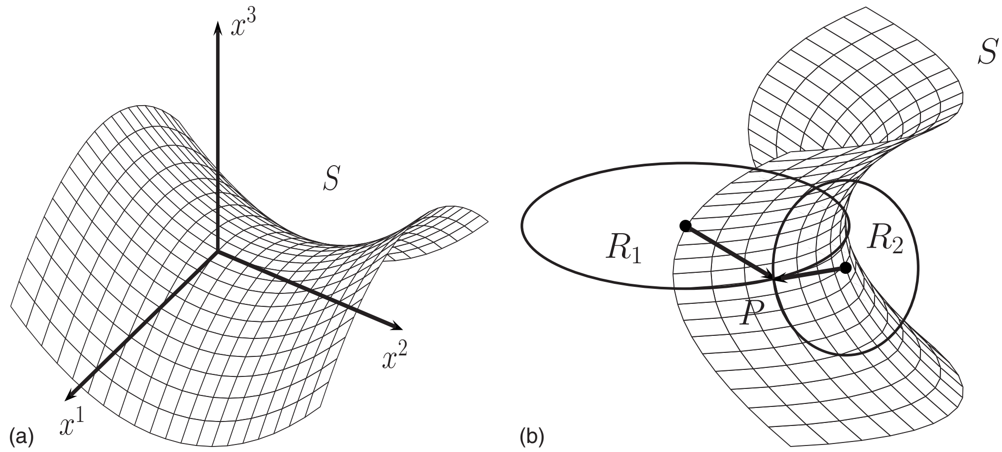

FIG. 1. (a) illustrates the chosen coordinates (6) for a general surface S which is parametrized by $\boldsymbol{r}=\boldsymbol{f}(x^1,x^2)$. (b) illustrates the definition of the intrinsic and extrinsic curvature. With $|\kappa_1|=1/R_1$ and $|\kappa_2|=1/R_2$ we have $K=\kappa_1\kappa_2$ and $H=(1/2)(\kappa_1+\kappa_2)$ for an arbitrary point P on a surface S. If the corresponding tangent circle bends away from the normal vector, the sign of κ_1 and κ_2 is defined as positive. Note that in the case shown the signs of κ_1 and κ_2 are different.

Now, following [8] we introduce a local set of coordinates in the vicinity of the surface S given by

$$\boldsymbol{r}(x^1,x^2,x^3)=\boldsymbol{f}(x^1,x^2)+x^3\boldsymbol{\nu}(x^1,x^2), \tag{6}$$

with $\boldsymbol{r}=(X,Y,Z)^T$ and $\boldsymbol{\nu}$ the normal vector to the surface. Hence, x^1 and x^2 are parametrizing the surface S where x^3 defines the distance to it [see Fig. 1(a)]. In these coordinates the coefficients of the metric are readily found to be

$$g_{ab}=\gamma_{ab}-2h_{ab}x^3+h_{ac}h_b^c(x^3)^2,$$

$$g_{33}=1, \quad g_{3a}=0. \tag{7}$$

The latin indices $a,b,c,\ldots$ take the values 1 and 2. The metric coefficients are expressed in terms of the coefficients γ_{ab} of the intrinsic metric and h_{ab} the coefficients of the second fundamental form of the surface [17,18]. Also one needs the determinant of the metric

$$\sqrt{g}=\sqrt{\gamma}\Omega, \quad \Omega=1-2Hx^3+K(x^3)^2, \tag{8}$$

which depends on the Gaussian or intrinsic curvature K and the extrinsic curvature H of the surface [see Fig. 1(b)]. Next we assume that a dielectric waveguide is attached to the curved surface. Furthermore, the index of the guiding layer n_0 is constant and homogeneous with respect to the coordinates x^1, x^2, and x^3.

Now we are able to derive the wave equations in the wave guiding layer. To find these equations we express the wave equation and the divergence constraint in these coordinates. Since light is confined near S we consider the narrow layer limit $x^3\to 0$. Note that this limit has to be taken at the end of the calculation [8]. Further more we introduce the new field $\hat{E}^i=\sqrt{\Omega}E^i$. This is necessary to get the right volume measure on the surface after we separated the x^3-dependent part of the electric field [8]. Following this procedure, we find the Gaussian law

$$\frac{1}{\sqrt{\gamma}}\partial_a\sqrt{\gamma}\hat{E}^a+\partial_3\hat{E}^3-H\hat{E}^3=0. \tag{9}$$

The Laplace operator is found to be of the form [8]

$$-\Delta_g E^i=-\Delta_\gamma\hat{E}^i-\partial_3^2\hat{E}^i-(H^2-K)\hat{E}^i. \tag{10}$$

Using (9) and (10) we can deduce the Helmholtz equation

$$-\Delta_\gamma\hat{E}^3-\partial_3^2\hat{E}^3+H^2\hat{E}^3=k_0^2n_0^2\hat{E}^3-(\partial_bH)\hat{E}^b-\frac{1}{c^2}\frac{\partial^2\hat{P}^3_{\mathrm{NL}}}{\partial t^2} \tag{11}$$

for the component perpendicular to surface S. Here Δ_γ is the Laplace operator (5) now with respect to the intrinsic surface metric γ_{ab}. In a similar way, using again (9) and (10) the following equations for the components tangential to S can be derived:

$$\begin{aligned}&-\Delta_\gamma\hat{E}^a-\partial_3^2\hat{E}^a-(H^2-K)\hat{E}^a+(\partial_c\gamma^{ab})\partial_b\hat{E}^c\\&\quad-\gamma^{ab}(\partial_b\partial_c\ln\sqrt{\gamma})\hat{E}^c\\&\quad=k_0^2n_0^2\hat{E}^a-2[h^{ab}\partial_b\hat{E}^3+\gamma^{ab}(\partial_bH)\hat{E}^3]-\frac{1}{c^2}\frac{\partial^2\hat{P}^a_{\mathrm{NL}}}{\partial t^2}.\end{aligned} \tag{12}$$

Equation (11) and (12) are the final equations which describe the propagation of light in a nonlinear waveguide confined on a general curved surface. In all these equations the narrow layer limit $x^3\to 0$ is assumed to be valid. Here we follow [9,10] and consider these equations as an approximation to the full three-dimensional problem.

Note that the normal component does not couple in the usual way to the surface (11), since the additional surface potential is only H^2 instead of $-H^2+K$ [8].

The coupling between the normal and tangential components is mainly given by the change of the extrinsic curvature along the surface. For a special class of surfaces it is possible to decouple (12) and (11). Let us consider surfaces

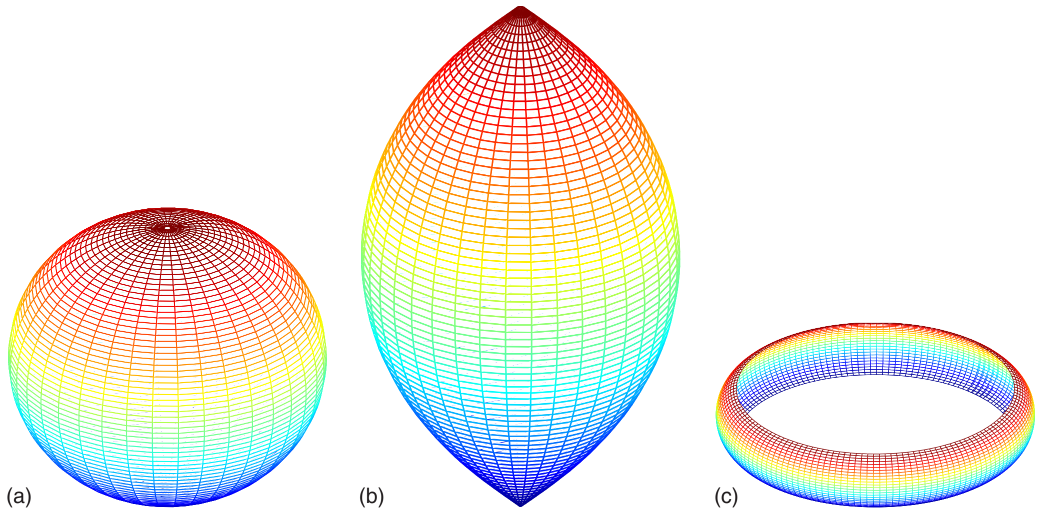

FIG. 2. (Color online) Surfaces of constant positive Gaussian curvature $K=1/R^2$: the sphere (a), the spindle type (b), and the bulge type (c). The color (online) or grayscale (print) illustrates the value of the arclength in the z direction, from $-\pi R/2$ to $\pi R/2$ or $-R\arcsin(R/R_0)$ to $R\arcsin(R/R_0)$.

with $H=0$. A surface with $H=0$ is called a minimal surface, since it minimizes for fixed boundary the volume or area of all possible surfaces when varied in normal direction. An example of such a surface would be the famous catenoid [18]. For these special class the tangential components of the TE field $\hat{\boldsymbol{E}}=(\hat{E}^1,\hat{E}^2,0)$ satisfy

$$-\Delta_\gamma \hat{E}^a-\partial_3^2\hat{E}^a+K\hat{E}^a+(\partial_c\gamma^{ab})\partial_b\hat{E}^c-\gamma^{ab}(\partial_b\partial_c\ln\sqrt{\gamma})\hat{E}^c$$
$$=k_0^2n_0^2\hat{E}^a-\frac{1}{c^2}\frac{\partial^2\hat{P}^a_{\mathrm{NL}}}{\partial t^2}, \tag{13}$$

where again $a=1,2$.

Now for simplification we assume that also for a general surface $\partial_a H$ is small when compared to $k_0^2n_0^2$ and can therefore be neglected. This is a valid approximation as long as the extrinsic curvature H varies on scales large compared with the wavelength only. For most cases this is well satisfied. Furthermore, we will focus on wave numbers of the order $k_0^2\sim10^{-4}\ \mathrm{nm}^{-2}$ whereas $K\sim H^2\sim10^{-14}\ \mathrm{nm}^{-2}$. This means the wavelengths are in the visible range and the curvature of the considered surface is of the order of centimeters. Therefore we can neglect the term $-H^2+K$ and $\partial_a H$ in (12) and find for TE polarization $\hat{\boldsymbol{E}}\approx(\hat{E}^1,\hat{E}^2,0)$ the equations

$$-\Delta_\gamma \hat{E}^a-\partial_3^2\hat{E}^a+(\partial_c\gamma^{ab})\partial_b\hat{E}^c-\gamma^{ab}(\partial_b\partial_c\ln\sqrt{\gamma})\hat{E}^c$$
$$=k_0^2n_0^2\hat{E}^a-\frac{1}{c^2}\frac{\partial^2\hat{P}^a_{\mathrm{NL}}}{\partial t^2}\quad(a=1,2). \tag{14}$$

Note that these equations are still coupled due to the coordinate dependence of the metric. For the decoupling of (14) we consider surfaces with an additional symmetry, namely, surfaces of revolution. To fix the notation we briefly review some important properties of these surfaces.

III. SURFACES OF REVOLUTION

A possible and convenient parametrization of surfaces of revolution is given by

$$\boldsymbol{f}(z,\varphi)=(r(z)\cos(\varphi),r(z)\sin(\varphi),h(z)), \tag{15}$$

where $x^1=z$ and φ is the angle of rotation. It is always possible to choose a parametrization such that $(dr/dz)^2+(dh/dz)^2=1$. For this parametrization we find the metric

$$ds^2=dz^2+r^2(z)d\varphi^2=\gamma_{zz}dz^2+\gamma_{\varphi\varphi}d\varphi^2. \tag{16}$$

We will call these coordinates warped coordinates, because the metric has the form of a warped product. As usual the elements of the inverse metric are denoted by $\gamma^{zz}=\gamma_{zz}^{-1}$ and $\gamma^{\varphi\varphi}=\gamma_{\varphi\varphi}^{-1}$. A special class of surfaces of revolution are surfaces with constant Gaussian curvature K. These will be considered next.

A. Surfaces with constant $K>0$

In this case there are three different surfaces (Fig. 2). They all have the same intrinsic curvature but different extrinsic curvature. These are parametrized by $r(z)=R_0\cos(z/R)$. For $R=R_0$ one gets the sphere, for $R_0<R$ a spindle type, and for $R_0>R$ a bulge type surface. The Gaussian or intrinsic curvature is then given by $K=1/R^2$. If we define the $x^2=x$ variable through $x=R_0\varphi$, one gets for these surfaces the metric

$$ds^2=dz^2+\cos^2\left(\frac{z}{R}\right)dx^2,\quad x\in[-\pi R_0,\pi R_0). \tag{17}$$

For $R_0\leqslant R$ we have $|z|<\pi R/2$, whereas in the other case z varies only in the limited range $|z|<R\arcsin(R/R_0)$. We have $\gamma_{zz}=1$ and $\gamma_{xx}=\cos^2(z/R)$ and we find the Laplace operator

$$\Delta_\gamma=\partial_z^2-\frac{1}{R}\tan\left(\frac{z}{R}\right)\partial_z+\frac{1}{\cos^2(z/R)}\partial_x^2. \tag{18}$$

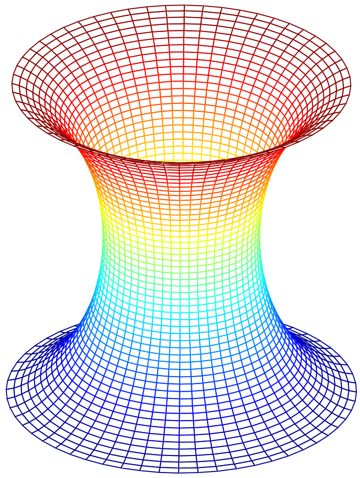

FIG. 3. (Color online) Hyperboloid type surface with negative Gaussian curvature $K=-1/R^2$. The color (online) or grayscale (print) illustrates the value of the arclength in the z direction, from $-R\,\mathrm{arcsinh}(R/R_0)$ to $R\,\mathrm{arcsinh}(R/R_0)$.

B. Surfaces with constant $K<0$

As in the positive case there are three distinguishable surfaces. In contrast to the case $K>0$ each surface has now a different parametrization. To show the similarities to the positive curvature case, we focus on the hyperboloid type surface given by $f(z)=R_0\cosh(z/R)$ with $K=-1/R^2$ (Fig. 3). Again we define $x=R_0\varphi$ for $R_0>0$. Therefore the metric has the form

$$ds^2 = dz^2 + \cosh^2\left(\frac{z}{R}\right)dx^2, \quad x \in [-\pi R_0, \pi R_0). \tag{19}$$

The coordinate z is also limited. It varies now in the range $|z|<R\,\mathrm{arcsinh}(R/R_0)$. In this case $\gamma_{zz}=1$ and $\gamma_{xx}=\cosh^2(z/R)$. The Laplace operator is then given by

$$\Delta_\gamma = \partial_z^2 + \frac{1}{R}\tanh\left(\frac{z}{R}\right)\partial_z + \frac{1}{\cosh^2(z/R)}\partial_x^2. \tag{20}$$

Note that all three surfaces are locally isometric [17,18]. This means that locally one can always find a coordinate transformation which brings the metric into the form (19).

C. Helmholtz equation for surfaces of revolution

In the warped coordinates (16) we are now able to decouple (14) if we choose the polarization $\hat{\boldsymbol{E}}=(0,\psi,0)$. This is due to the fact that the metric does not depend on the second coordinate x. Note that this polarization is exact in the case of minimal surfaces ($H=0$). Therefore we find that the component in the x direction ψ of the electric field satisfies the scalar equation

$$-\Delta_\gamma\psi - \partial_3^2\psi + V_{\mathrm{NL}}(\psi) = k_0^2 n_0^2 \psi. \tag{21}$$

Here V_{NL} is a nonlinear potential. For the standard Kerr nonlinearity it is given by $V_{\mathrm{NL}}(\psi)=-3k_0^2\chi^{(3,1)}|\psi|^2\psi$ [19]. For separating the normal part we use the ansatz $\psi(x,z,x^3)=\phi(x,z)\Xi(x^3)$, and following [20] we finally find

$$-\Delta_\gamma\phi + V_{\mathrm{NL}}(\phi) = k^2\phi, \tag{22a}$$

$$-\partial_3^2\Xi - k_0^2 n_0^2 \Xi = -k^2\Xi, \tag{22b}$$

where k^2 is the constant of separation and now $V_{\mathrm{NL}}(\phi)=-\lambda|\phi|^2\phi$. The effective nonlinear coefficient is defined as $\lambda=3k_0^2\int\chi^{(3,1)}|\Xi|^4 d(x^3)/\int|\Xi|^2 d(x^3)$. Note that only the second equation (22b) depends on the actual layer. Hence the mode structure and the propagation constant k have to be determined for the respective index structure.

IV. NONLINEAR SCHRöDINGER EQUATION IN CURVED SPACE

If we restrict ourselves to optical wavelengths we can assume an effective wave number of the order of $k^2 \sim 10^{-4}$ nm^{-2}. If we now consider surfaces where K and H^2 are much smaller than 10^{-4} nm^{-1}, we would expect that the envelope of the field varies slowly compared to the phase. As in flat space [20] we can try to find a paraxial or slowly varying envelope approximation of (22a). However, in a curved space this is not as straightforward as in flat space. We need to make sure that the fast oscillating phase, which is separated, is close to the actual phase of the field. Here we consider only surfaces of revolution in warped coordinates. The general case is discussed in Appendix A.

As a guiding road we review the flat case. In Euclidean space the phase is simply given by $S(z)=kz$, when propagation is in the z direction. Now $S(z)$ is just the flat Hamilton-Jacobi function, which satisfies in the flat case

$$(\partial_z S)^2 + (\partial_x S)^2 = k^2 \tag{23}$$

for $k_z=\partial_z S=k$ and $k_x=\partial_x S=0$. Thus, in comparison to this, it is natural to assume that the fast oscillating phase on a curved surface is given by the Hamilton-Jacobi function in this curved space, which now satisfies

$$\gamma^{ab}\partial_a S\partial_b S = \gamma^{zz}(\partial_z S)^2 + \gamma^{xx}(\partial_x S)^2 = k^2. \tag{24}$$

In general S is a function of both variables, but if we consider surfaces of revolution it separates. This agrees with the flat case and we can readily integrate (24),

$$S(x,z) = \int^z \sqrt{\gamma_{zz}(k^2 - \gamma^{xx}k_x^2)}dz' + k_x x. \tag{25}$$

Due to the rotational symmetry $k_x=\partial_x S$ is still constant. Now there are two possible cases: the propagation is in the z or x direction. Here we consider only propagation in the z direction. The other case is analyzed in Appendix A. For the first case we can simply set $k_x=0$. Therefore the phase is given by $S(z)=k\int^z\sqrt{\gamma_{zz}}dz'=kz$, which is k times the proper length in the z direction.

A. Nonlinear Schrödinger equation for constant $K>0$

Regarding the discussion in the above section we use the following ansatz:

$$\phi(x,z) = \frac{1}{\sqrt{\cos(z/R)}} u(x,z) e^{ikz}, \tag{26}$$

where the field is also rescaled. Due to this rescaling the first-order derivative term in (18) vanishes and the analogy to the standard flat nonlinear Schrödinger equation becomes explicit. Inserting (26) in (22a) and assuming that the envelope u is slowly varying with respect to the phase,

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \ll 2k \left|\frac{\partial u}{\partial z}\right|, \tag{27}$$

we finally get

$$2ik\frac{\partial u}{\partial z} = -\frac{1}{\cos^2(z/R)}\frac{\partial^2 u}{\partial x^2} - V_{\mathrm{eff}} u - \frac{\lambda |u|^2}{\cos(z/R)} u. \tag{28}$$

The effective potential is found to be

$$V_{\mathrm{eff}} = \frac{1}{4R^2}\left(1 + \frac{1}{\cos^2(z/R)}\right). \tag{29}$$

B. Nonlinear Schrödinger equation for constant $K<0$

This case is analogous to the positive curvature case. With the ansatz

$$\phi(x,z) = \frac{1}{\sqrt{\cosh(z/R)}} u(x,z) e^{ikz}, \tag{30}$$

we find the nonlinear Schrödinger equation

$$2ik\frac{\partial u}{\partial z} = -\frac{1}{\cosh^2(z/R)}\frac{\partial^2 u}{\partial x^2} - V_{\mathrm{eff}} u - \frac{\lambda |u|^2}{\cosh(z/R)} u. \tag{31}$$

Now the effective potential is given by

$$V_{\mathrm{eff}} = -\frac{1}{4R^2}\left(1 + \frac{1}{\cosh^2(z/R)}\right). \tag{32}$$

We see that in the flat limit $R \to \infty$ Eqs. (28) and (31) become the usual nonlinear Schrödinger equation. Note that these equations show some interesting similarities to dispersion management in optical fibers.

V. LINEAR PROPAGATION

In this section we consider the linear case $\lambda = 0$ and solve (28) and (31) in the case of an initially Gaussian profile.

A. Solution for $K>0$

As the initial profile we use

$$u(x,z=0) = u_0 \exp\left(-\frac{l^2(x,0)}{2\sigma_0^2}\right), \tag{33}$$

with the initial width σ_0. Note that the exponent depends on the proper length in the x direction $l(x,z) = \cos(z/R)x$ for constant $z=0$. Due to the translation symmetry in the x direction, we can set a possibly constant $x_0 = 0$. After separating the phase

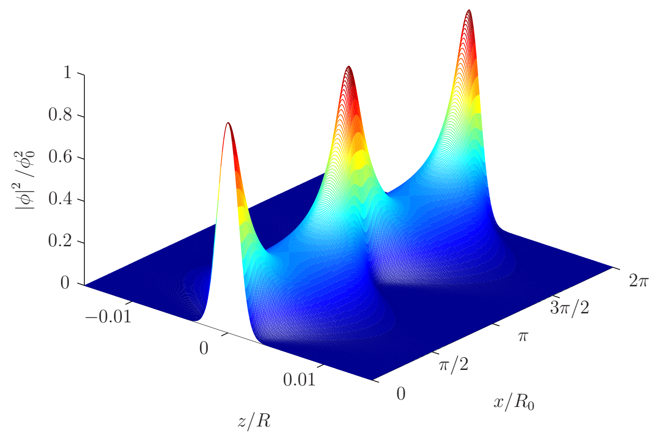

FIG. 4. (Color online) $|\phi|^2/\phi_0^2$ for $K>0$, $R=R_0$, $\sigma_0=\sigma_S/2$, and $\sqrt{Rk}=10^3/3$.

$$u(x,z) = v(x,z)\exp\left(\frac{i}{2k}\int_0^z V_{\mathrm{eff}}(z')dz'\right) \tag{34}$$

and performing the coordinate transformation $z \to \xi(z) = R\tan(z/R)$, we find that v satisfies the standard flat Schrödinger equation, which can be solved easily.

In an experiment we would, for example, measure the energy density on these surfaces, which is proportional to $|\phi|^2$. In the slowly varying envelope approximation we find

$$T_{00} \propto |\phi|^2 = \phi_0^2 \frac{\sigma_0}{\sigma_{(K>0)}(z)} \exp\left(-\frac{l^2(x,z)}{\sigma^2_{(K>0)}(z)}\right), \tag{35}$$

with the effective or space-dependent width for positive curvature

$$\sigma_{(K>0)}(z) = \sigma_0\sqrt{1 - \left(1 - \frac{\sigma_S^4}{\sigma_0^4}\right)\sin^2\left(\frac{z}{R}\right)}. \tag{36}$$

Obviously the width of the Gaussian beam oscillates (see Fig. 4). In the case of a sphere, the beam refocuses after each half of a round trip. The phase of the oscillations is determined by the initial width σ_0 and $\sigma_S = \sqrt{R/k}$. Only for $\sigma_0 = \sigma_S$ do the oscillations vanish. The diffraction and the effect of curvature compensate each other, and a stationary solution is formed. Note that in an experiment one would not measure the energy density in terms of the coordinate x when propagating in the z direction. This coordinate is not natural in the sense that it does not correspond to a proper length on the surface. A more natural possibility is to measure the energy density in terms of the proper length $l(x,z) = \cos(z/R)x$. Therefore $\sigma_{(K>0)}(z)$ is the actually measured width.

Another possibility in the experiment would be to choose coordinates such that the field is propagating along the equator $z=0$. In the vicinity of the equator both coordinates are nonsingular and valid. This can now be achieved using the symmetries of these surfaces. All three surfaces have the same metric as the sphere and therefore the same symmetries. We might use the rotational symmetry of the sphere to find a solution that is now propagating along the equator. The coordinate transformation which corresponds to such a rotation is

$$\sin(z/R) = \cos(z'/R)\sin(x'/R),$$

$$\sin(x/R) = \frac{\sin(z'/R)}{\sqrt{1 - \sin^2(x'/R)\cos^2(z'/R)}}. \quad (37)$$

In the slowly varying envelope approximation, the rotational symmetry is of course not preserved in a strict sense. However, one could argue that we found a solution in this limit and the rotated solution should also be a valid solution of the wave equation in the slowly varying envelope approximation. The validity of the transformation could be quantified by demanding

$$\int_{-\infty}^{\infty} |\phi(x,z)|^2 \sqrt{\gamma} dx = \int_{-\infty}^{\infty} |\phi'(x',z')|^2 \sqrt{\gamma'} dz', \quad (38)$$

where ϕ is now the total field, $x' = x'(x,z)$, and $z' = z'(x,z)$. This means that for every constant z slice or x' slice the partial energy density must remain the same. Therefore we need to have the right transformation law for the integral measure. In general the coordinate transformation (37) does not satisfy (38). However, for narrow light beams it is possible to expand (37) with respect to the new transverse direction z', i.e.,

$$z = x' + O\left(\frac{z'^2}{R^2}\right), \quad x = \frac{z'}{\cos(x'/R)} + O\left(\frac{z'^3}{R^3}\right). \quad (39)$$

Thus up to this order (38) is satisfied. Note that the coordinate transformation of x is the same as expressing x in terms of the proper length l, but now the proper length l is replaced with z', which is indeed the proper length in the new transverse direction.

After transforming the total field we finally get the expression

$$T'_{00} \propto |\phi'|^2 = \phi_0^2 \frac{\sigma_0}{\sigma_{(K>0)}(x')} \exp\left(-\frac{z'^2}{\sigma^2_{(K>0)}(x')}\right). \quad (40)$$

In terms of x' and z' this result agrees with (35) and (36). Concerning the energy density, the propagation of light is the same for all three surfaces and both directions when expressed in terms of proper lengths. The invariance with respect to the propagation direction reflects the fact that these surfaces are homogeneous and isotropic, i.e., maximal symmetric spaces [21]. Thus in the limit of the slowly varying envelope approximation this is preserved as well.

B. Solution for $K < 0$

Formally, this case is similar to that of positive Gaussian curvature. In warped coordinates we have $\xi(z) = R\tanh(z/R)$. The form of the energy density is the same as in (35) with the tangent being replaced by the hyperbolic one. As for $K > 0$ we express $|\phi|^2$ in terms of the proper length $l(x,z) = \cosh(z/R)x$ and find

$$T_{00} \propto |\phi|^2 = \phi_0^2 \frac{\sigma_0}{\sigma_{(K<0)}(z)} \exp\left(-\frac{l^2(x,z)}{\sigma^2_{(K<0)}(z)}\right), \quad (41)$$

with the effective width

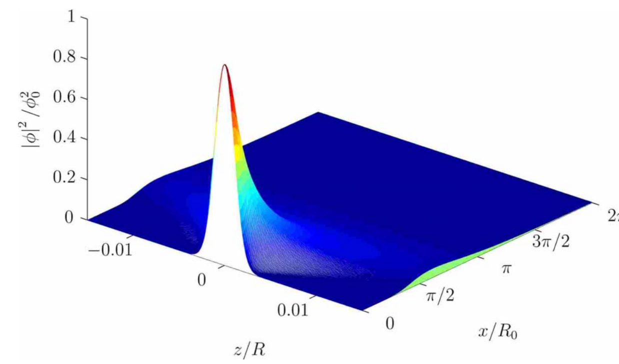

FIG. 5. (Color online) $|\phi|^2/\phi_0^2$ for $K < 0$, $R = R_0$, $\sigma_0 = \sigma_S/2$, and $\sqrt{R}k = 10^3/3$.

$$\sigma_{(K<0)}(z) = \sigma_0 \sqrt{1 + \left(1 + \frac{\sigma_S^4}{\sigma_0^4}\right)\sinh^2\left(\frac{z}{R}\right)}. \quad (42)$$

One can immediately see that there are neither oscillations nor a stationary solution (see Fig. 5). Also, the field spreads more if compared to the flat case. For $R \to \infty$ we find the usual flat solution

$$\sigma_{(K=0)}(z) = \sigma_0 \sqrt{1 + \frac{z^2}{\sigma_0^4 k^2}}, \quad (43)$$

and therefore $\sigma_{(K<0)}(z)/\sigma_{(K=0)}(z) \sim \exp(z/R)$, which grows exponentially with the propagation distance. Note that we also find (43) for $R \to \infty$ in the positive curvature case (36). For comparison of the three cases $K > 0$, $K = 0$, and $K < 0$, see Fig. 6.

For positive curved surfaces we used the rotational symmetry to find a solution that is propagating along the equator. The same can be done for the negative case if we recall that similar symmetries are present. The symmetry transformation which transforms the z direction into the new x' direction and vice versa is then

$$\sinh(z/R) = \cosh(z'/R)\sinh(x'/R),$$

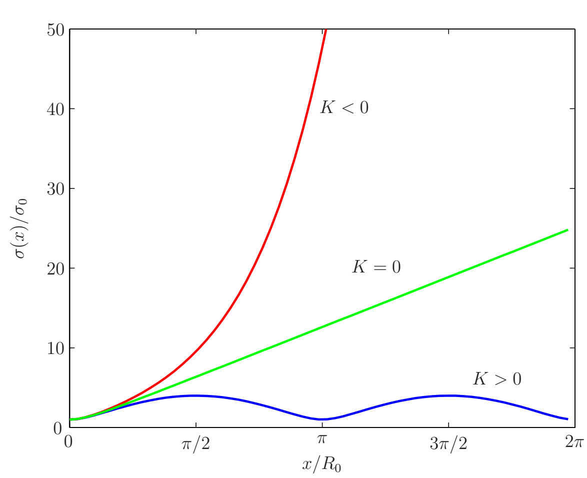

FIG. 6. (Color online) Evolution of the width for the three different cases $K > 0$, $K = 0$, and $K < 0$. The parameters are $R = R_0$, $\sigma_0 = \sigma_S/2$, and $\sqrt{R}k = 10^3/3$.

$$\sinh(x/R) = \frac{\sinh(z'/R)}{\sqrt{1+\sinh^2(x'/R)\cosh^2(z'/R)}}. \quad (44)$$

It can be shown easily that this is a symmetry of the metric (19). Furthermore, we see that there are no singular points and the coordinate transformation is well defined in the given parameter range. In the sense of (38) this is a valid coordinate transformation only as long as $z' \ll R$. However, now the field is always spreading. In this approximation we find for the propagation along the equator ($z'=0$) the energy density

$$T'_{00} \propto |\phi'|^2 = \phi_0^2 \frac{\sigma_0}{\sigma_{(K<0)}(x')} \exp\left(-\frac{z'^2}{\sigma^2_{(K<0)}(x')}\right), \quad (45)$$

now with the effective width

$$\sigma_{(K<0)}(x') = \sigma_0 \sqrt{1+\left(1+\frac{\sigma_S^4}{\sigma_0^4}\right)\sinh^2\left(\frac{x'}{R}\right)}. \quad (46)$$

For the negative curved surface we also find invariance with respect to the propagation direction.

VI. CONCLUSION

The problem of light confined to a general curved surface via a nonlinear film waveguide was described theoretically. For each component of the electromagnetic field a wave equation was derived. We showed that surfaces of constant extrinsic curvature, in particular minimal surfaces, allow decoupling of the tangential and normal components. We derived a generalized nonlinear Schrödinger equation for surfaces of revolution.

As an example, we analyzed the linear propagation on surfaces with constant Gaussian curvature, positive and negative. Furthermore, we showed that in the slowly varying envelope approximation the isotropy with respect to the propagation direction is preserved. In principle, the methods presented can be used for a general curved surface.

APPENDIX A: NONLINEAR SCHRÖDINGER EQUATION

We assume that the metric of a surface is of the form

$$ds^2 = \gamma_{zz}(z)dz^2 + \gamma_{xx}(z)dx^2. \quad (A1)$$

The Hamilton-Jacobi function satisfies

$$\gamma^{ab}\partial_a S \partial_b S = k^2. \quad (A2)$$

In this case we are able to readily integrate (A2) and find

$$S(x,z) = \int^z \sqrt{\gamma_{zz}(k^2 - \gamma^{xx}k_x^2)}dz' + k_x x, \quad (A3)$$

where k_x is constant.

1. Propagation along the z direction

For this propagation direction we simply set $k_x=0$. Therefore the phase is given by $S(z)=k\int^z\sqrt{\gamma_{zz}}dz'$, which is k times the proper length in the z direction.

In flat space the Hamilton-Jacobi function is proportional to the coordinate. To carry this forward to curved space, we now make a coordinate transformation and use $S(z)$ as the new coordinate, i.e., we use the proper length. Next we make the ansatz $\phi(x,z)=(pq)^{-1/2}u(x,z)\exp[iS(z)]$, where $q=\partial_z S$ and $p=\sqrt{\gamma\gamma^{zz}}$. Under the assumption that the envelope u is slowly varying with respect to the phase

$$\left|\frac{\partial^2 u}{\partial S^2}\right| \ll 2k\left|\frac{\partial u}{\partial S}\right|, \quad (A4)$$

we finally get

$$2ik\sqrt{\gamma^{zz}}\frac{\partial u}{\partial z} = -\gamma^{xx}\frac{\partial^2 u}{\partial x^2} - V_{\text{eff}}u - \lambda\sqrt{\gamma^{xx}}|u|^2 u, \quad (A5)$$

the nonlinear Schrödinger equation in curved space. Here we already transformed back to the original coordinate z. The effective potential is then found to be

$$V_{\text{eff}} = H^2 - K - \gamma^{zz}\left(\frac{1}{2}(\partial_z \ln p)^2 - p^{1/2}\partial_z^2 p^{-1/2} - q^{1/2}\partial_z^2 q^{-1/2}\right). \quad (A6)$$

2. Propagation along the x direction

For this case it is not possible to set $k_z=\partial_z S$ equal to zero and $k_x=k$. However, if we assume that in the given propagation area the metric does not change too quickly around some point z_0, we can expand $S(x,z)$ and find to first order $k_x \approx k\sqrt{\gamma_{xx}(z_0)}$. In this case, k_z is almost zero and the fast phase is given by $S(x)=k\sqrt{\gamma_{xx}(z_0)}x$. This is k times the proper length in the x direction for fixed $z=z_0$. Now here we can still use the coordinate x. Therefore we use the ansatz $\phi(x,z)=p^{-1/2}u(x,z)\exp[iS(x)]$ where $p=\sqrt{\gamma\gamma^{zz}}$. Following the same steps as before, we find for this direction

$$2ik_x\gamma^{xx}\frac{\partial u}{\partial x} = -\gamma^{zz}\frac{\partial^2 u}{\partial z^2} - V_{\text{eff}}u - \lambda\sqrt{\gamma_{zz}\gamma^{xx}}|u|^2 u, \quad (A7)$$

where the effective potential is now given by

$$V_{\text{eff}} = H^2 - K - \gamma^{zz}\left(\frac{1}{2}(\partial_z \ln p)^2 - p^{1/2}\partial_z^2 p^{-1/2}\right) - \gamma^{xx}k_x^2 + k^2. \quad (A8)$$

Interestingly, these nonlinear Schrödinger equations in curved space show some similarities to Bose-Einstein condensates in external potentials and dispersion management in optical fibers.

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