

## Preparation of entangled states by quantum Markov processes

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We investigate the possibility of using a dissipative process to prepare a quantum system in a desired state. We derive for any multipartite pure state a dissipative process for which this state is the unique stationary state and solve the corresponding master equation analytically. For certain states, such as the cluster states, we use this process to show that the jump operators can be chosen quasilocally, i.e. they act nontrivially only on a few, neighboring qubits. Furthermore, the relaxation time of this dissipative process is independent of the number of subsystems. We demonstrate the general formalism by considering arbitrary matrix-product states or projected entangled pair states. In particular, we show that the ground state of the Affleck-Kennedy-Lieb-Tasaki model can be prepared employing a quasi-local dissipative process.

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### I. INTRODUCTION

Preparation of entangled pure quantum states is of interest in the context of both quantum information and condensed matter physics. In quantum information, entangled states of qubits can act as a resource for quantum computing, e.g., cluster states [1] in measurement-based quantum computing [2], while in condensed matter physics entangled states represent ground states of strongly correlated systems.

A possible scenario for the preparation of the entangled states of interest is the cooling of the system to the ground state of an appropriate many-body Hamiltonian. Alternatively, we can generate a state of interest from an initial pure state, e.g., a product state, which can easily be prepared with available resources. This is achieved either by coherent evolution generated by a system Hamiltonian (i.e., a sequence of quantum gates), or, more generally, by applying the most general physical transformation, which is mathematically represented by a completely positive map. We will discuss here another route: the preparation of entangled states by designing dissipative processes, so that we drive the system via nonequilibrium dynamics to a pure entangled state of interest  $|\Psi\rangle$  for long times, for any initial mixed state represented by a system density operator  $\rho$ , i.e.,

$$\rho \xrightarrow{t \rightarrow \infty} |\Psi\rangle\langle\Psi|. \quad (1)$$

In particular, we will consider a situation where the time evolution of the system coupled to a reservoir can be described as a quantum Markov process with dynamics obeying a master equation,

$$\dot{\rho} = \mathcal{L}(\rho) \equiv -i[H, \rho] + \sum_{\ell} g_{\ell} (2c_{\ell}\rho c_{\ell}^{\dagger} - c_{\ell}^{\dagger}c_{\ell}\rho - \rho c_{\ell}^{\dagger}c_{\ell}). \quad (2)$$

Here,  $H$  represents a system Hamiltonian while the Liouvillian  $L(\rho) = \sum_{\ell} g_{\ell} (2c_{\ell}\rho c_{\ell}^{\dagger} - c_{\ell}^{\dagger}c_{\ell}\rho - \rho c_{\ell}^{\dagger}c_{\ell})$  can always be written in Lindblad form with  $c_{\ell}$  a set of “quantum jump operators” and dissipation rates  $g_{\ell} \geq 0$  [3]. Such a description in terms of a master equation is valid provided the system dynamics is slow on the time scale of the reservoir correlation time, as is the case for typical quantum optical systems.

Thus the goal of *dissipative entangled state preparation* is to design quantum reservoirs and system-reservoir couplings, and to identify necessary and sufficient conditions for the master equation (2), such that the desired pure state of a many-body system is obtained as the unique stationary state. Indeed, it will be shown below that for any given  $|\Psi\rangle$  there is a master equation which yields the required state as the unique pure state within a relaxation time  $T_{\text{relax}} \sim 1/\min g_k$ , independent of the number of qubits.

We will be particularly interested in a situation where qubits (or spin-1/2 particles) reside on a lattice. Thus it is natural to restrict dissipation represented by the Liouvillian  $L$  to *quasiloocal* jump operators acting only on a neighborhood of a given qubit, which raises the question of the class of states which can be prepared with these resources. We will show below that examples of states which can be generated include stabilizer states, matrix-product states (MPSs), or projected entangled pair states (PEPSs).

The results of the present paper are also of immediate relevance for a nonequilibrium condensed matter physics where pure many-body states and quantum phases are prepared as a result of a driven dissipative system dynamics. In a standard equilibrium situation of condensed matter and cold atom physics, states close to the ground state of the Hamiltonian  $H|G\rangle = E_G|G\rangle$  are prepared by cooling the system, where in particular for a finite system  $\rho \sim e^{-H/k_B T} \rightarrow |G\rangle\langle G|$  for temperature  $T \rightarrow 0$ . This has already led to the preparation of intriguing quantum phases [4–10]. In contrast, we obtain a pure state representing a nonequilibrium quantum phase as a result of the dynamics (1) with the master equation (2). In recent work we have provided examples of master equations, including the example of noninteracting bosons and paired interacting fermions corresponding to cold atoms moving in an optical lattice which are driven by coupling to quasiloocal reservoirs into pure states exhibiting long-range order. While the goal of Ref. [11] was to study nonequilibrium condensates, Luttinger liquids, and Kosterlitz-Thouless phases, in the present work we will provide the uniqueness proofs for the corresponding driven dissipative dynamics. Furthermore, the results of the present paper are of direct relevance for nonequilibrium spin models. As an

example we will discuss a master equation whose unique steady state is the ground state of the familiar Affleck-Kennedy-Lieb-Tasaki (AKLT) model [12].

The outline of the paper is as follows. First of all, we summarize some properties of the master equation. In Sec. III we characterize all stationary pure states. Since we are interested in unique stationary states, we derive a sufficient condition for the uniqueness of the stationary state. In Sec. IV we derive, for any multipartite state, a dissipative process that can be used to prepare this state. That is, we construct a dissipative process for which the desired pure state is the unique stationary state. For this process it is simple to solve the underlying master equation analytically and to show that the relaxation time of the system is independent of the number of subsystems. In Sec. V we finally show that for certain states, such as the two-dimensional (2D) cluster states [1], this construction can be used to choose jump operators quasilocally. We furthermore derive a quasilocal dissipative process which has a general PEPS [13] as the unique stationary pure state. Also in the context of PEPSs, we consider the ground state of the familiar AKLT model [12] and derive the dissipative process for which this state is the unique stationary state. We also prove the uniqueness of the driven noninteracting Bose-Einstein condensate (BEC) and the  $\eta$  condensate of paired fermions given specific dissipative processes, complementing the work done on these states in [11].

## II. PRELIMINARIES AND NOTATION

We are interested in the stationary solutions of the master equation  $\dot{\rho} = \mathcal{L}(\rho)$ , presented in Eq. (2). We write  $\mathcal{L}(\rho) = \mathcal{E}(\rho) - Q^\dagger \rho - \rho Q$ , where  $\mathcal{E}(\rho) = 2 \sum_l g_l 2c_l \rho c_l^\dagger$  is a completely positive map and  $Q = P - iH$ , with  $P = \sum_l g_l c_l^\dagger c_l$  a positive semidefinite operator. Sometimes we denote  $\mathcal{L}$ , as given in Eq. (2), by  $\mathcal{L}_{\{H, c_l\}}$  and by  $\mathcal{L}_{\{c_l\}}$  if we consider a purely dissipative process. Note that the partition in the Hamiltonian and the dissipative part are unique if the operators  $c_l$  are traceless and orthonormal [14].

Since the master equation is linear, the eigenvalue equation  $\mathcal{L}(\sigma) = \lambda \sigma$  can be written as a matrix equation. Due to the fact that  $\text{tr}[\mathcal{L}(\sigma)] = 0$  for any  $\sigma$ , the eigenvectors to eigenvalues different from zero must be traceless. The eigenvalues can be complex; however, the real part of the eigenvalues is not positive (see, for instance, [15]). Considering a purely dissipative process, with Hermitian quantum jump operators  $c_l$ , all the eigenvalues are real. This is due to the fact that in this case the matrix corresponding to the Liouvillian is Hermitian. However, in general the matrix is not Hermitian; it is not even diagonalizable and has therefore generalized eigenvectors  $\sigma$  [16]. Furthermore, since  $\mathcal{L}(\sigma^\dagger) = [\mathcal{L}(\sigma)]^\dagger$  the eigenvalues occur in pairs of the form  $\lambda, \lambda^*$  with the corresponding eigenvectors  $\sigma$  and  $\sigma^\dagger$ . The set of all proper and all generalized eigenvectors,  $\{\sigma_j\}$ , forms a basis in the operator space. Expanding  $\rho(0)$  in this basis,  $\rho(0) = \sum_i c_i \sigma_i$ , we obtain  $\rho(t) = \sum_i \sum_j c_j^i(t) \sigma_j e^{\lambda_j t}$  where  $c_j^i(t)$  are polynomials of degree less than the largest order of the Jordan block corresponding to the eigenvalue  $\lambda_j$  [16].

We are interested in the stationary states of the evolution. That is, we want to find the states  $\rho$  for which  $\mathcal{L}_{\{H, c_l\}}(\rho) = 0$ .

In order to do so, we use the following notation. By  $K(X)$  [ $R(X)$ ] we denote the kernel (range) of a Hermitian operator  $X$ , and  $r(X)$  denotes the rank of  $X$ . Note that the pure states, which are not affected at all by the dissipative process, coincide with the kernel of  $P$ ,  $K(P) = \{|\Psi\rangle \text{ such that } c_l |\Psi\rangle = 0 \forall l \in \mathbf{L}\}$ . Thus, if  $|\Psi\rangle \in D_{\{H, c_l\}} \equiv K(P) \cap ES(H)$ , where  $ES(H)$  denotes the eigenspace of  $H$ , then  $\mathcal{L}_{\{H, c_l\}}(|\Psi\rangle\langle\Psi|) = 0$ . This implies that any state  $\rho$  with  $R(\rho) = \text{span}\{|\phi_i\rangle\}$  such that  $\{|\phi_i\rangle\} \subseteq D_{\{H, c_l\}}$  is a stationary state. We call these states dark states. Whenever it is clear from the context, we omit the subscripts  $\{H, c_l\}$  and write, for instance, simply  $D$  for the subset of dark states.

## III. PURE STATIONARY STATES AND UNIQUENESS OF STATIONARY STATES

In this section we first of all characterize all pure stationary states. We show that a pure state is a stationary state of some dissipative process if and only if it is a dark state of some other. Then we derive a sufficient condition for the uniqueness of stationary states.

Let us first of all, consider a given dissipative process and characterize the pure states, which are stationary states of it.

*Theorem 1.* Let  $\mathcal{L}$  be defined as in Eq. (2). Then  $\mathcal{L}(|\Phi\rangle\langle\Phi|) = 0$  if and only if the following two conditions are satisfied:

- (1)  $|\chi\rangle \equiv Q^\dagger |\Phi\rangle = \lambda |\Phi\rangle$  for some  $\lambda \in \mathbb{C}$ ;
- (2)  $c_l |\Phi\rangle = \lambda_l |\Phi\rangle \forall l \in \mathbf{L}$ , for some  $\lambda_l \in \mathbb{C}$  with  $\sum_l g_l |\lambda_l|^2 = \text{Re}(\lambda)$ , where  $\text{Re}(x)$  denotes the real part of  $x$ .

*Proof.*  $\mathcal{L}(|\Phi\rangle\langle\Phi|) = 0$  if and only if

$$2 \sum_l g_l |\Psi_l\rangle\langle\Psi_l| = \mathcal{E}(|\Phi\rangle\langle\Phi|) = |\Phi\rangle\langle\chi| + |\chi\rangle\langle\Phi|, \quad (3)$$

where  $|\Psi_l\rangle = c_l |\Phi\rangle$ . Therefore, the operator  $A \equiv |\Phi\rangle\langle\chi| + |\chi\rangle\langle\Phi|$  must be positive semidefinite. It can be easily verified that  $A \geq 0$  if and only if  $r(A) \leq 1$ . Thus,  $|\chi\rangle = \lambda |\Phi\rangle$  for some  $\lambda \in \mathbb{C}$  and  $A = 2 \text{Re}(\lambda) |\Phi\rangle\langle\Phi|$ . The fact that all  $|\Psi_l\rangle$  are in the range of  $\mathcal{E}(|\Phi\rangle\langle\Phi|)$  implies then that Eq. (3) is satisfied if and only if  $|\Psi_l\rangle = \lambda_l |\Phi\rangle$  with  $\sum_l g_l |\lambda_l|^2 = \text{Re}(\lambda)$ . ■

Using this theorem it is now easy to characterize all pure states for which there exists a Liouvillian such that the state is a stationary state of the master equation. Defining  $c'_l = c_l - \lambda_l \mathbb{1}$  and  $H' = H - i \sum_l g_l \lambda_l (c'_l)^\dagger + i \sum_l g_l \lambda_l^* c'_l$  it is straightforward to show that the conditions above imply that (1)  $H' |\Phi\rangle = \lambda |\Phi\rangle$ , with  $\lambda \in \mathbb{R}$  and (2)  $c'_l |\Phi\rangle = 0 \forall l \in \{1, \dots, m\}$ . Thus, we have that there exists a Liouvillian  $\mathcal{L}$  such that  $\mathcal{L}(|\Phi\rangle\langle\Phi|) = 0$  if and only if there exists a set of operators  $\{c'_l\}_{l=1, \dots, m}$ , a Hamiltonian  $H'$ , and some  $\lambda \in \mathbb{R}$ , such that the following two conditions are satisfied: (1')  $c'_l |\Phi\rangle = 0 \forall l \in 1, \dots, m$ ; (2')  $H' |\Phi\rangle = \lambda |\Phi\rangle$ .

Note that the two conditions (1') and (2') are equivalent to  $|\Phi\rangle \in D_{\{H', c'_l\}}$ . Therefore,  $|\Phi\rangle$  is a stationary state, if and only if it is a dark state for some other physical process. Thus, in order to design a dissipative process, which leads to the desired pure state, we have to find a set of operators (or a single operator) which have only one common eigenstate; the corresponding eigenvalue can be chosen to be zero. Due to the results presented above, we know that the correspond-

ing dissipative process will have the desired state as the unique pure stationary state. However, since we want to use this process for state preparation, we have to guarantee that there exists no mixed stationary state. How this can be ensured will be shown next.

We consider the general master equation given in Eq. (2). Note that the corresponding set of dark states is in general a set, not a subspace. However, since we want to use the dissipative process for state preparation, or to drive the system to a certain (higher-dimensional) subspace, we will consider here the situation where  $D$  is a subspace, i.e., all the states in  $D$  correspond to the same eigenvalue of the Hamiltonian. We are going to show that if there exists a stationary state,  $\rho$ , which is not a dark state, i.e.,  $R(\rho) \not\subseteq D$ , then there must exist a subspace of the Hilbert space  $\mathcal{H}$  which is left invariant under the operators  $\{c_l\}$ .

*Theorem 2.* If there exists no subspace  $S \subseteq \mathcal{H}$  with  $S \perp D$  such that  $c_l S \subseteq S \forall c_l$ , then the only stationary states are the dark states.

*Proof.* We prove the statement by contradiction. That is, we assume that there exists a state  $\rho$ , with  $R(\rho) = S' \not\subseteq D$  such that  $\mathcal{L}(\rho) = 0$ , and show that this implies that there exists a subspace  $S \subseteq \mathcal{H}$  with  $S \perp D$  such that  $c_l S \subseteq S \forall c_l$ . Using the notation of Eq. (2) and the fact that  $\mathcal{E}$  is a completely positive map, we have that  $\mathcal{E}(\rho) = Q^\dagger \rho + \rho Q \geq 0$ . Since  $\mathcal{E}(\rho)$  is a positive semidefinite operator, we have that  $R(\mathcal{E}(\rho)) = \text{span}\{c_l|\xi\rangle, l \in \mathbf{L}, |\xi\rangle \in S'\}$ . We are going to show now that  $R(\mathcal{E}(\rho))$  must be within  $S'$ , which implies that  $c_l S' \subseteq S' \forall c_l$ . Since  $Q^\dagger \rho + \rho Q$  must be positive semidefinite,  $K(Q^\dagger \rho + \rho Q) = \{|\Phi\rangle\}$ , such that  $\langle \Phi | Q^\dagger \rho + \rho Q | \Phi \rangle = 0 \supseteq K(\rho) \cup D$ . Therefore we have that  $K(\mathcal{E}(\rho)) \supseteq K(\rho) \cup D$ , which implies that  $R(\mathcal{E}(\rho)) \subseteq R(\rho) \cap D^\perp$ , where  $D^\perp$  denotes the orthogonal complement of  $D$ . We have shown now that  $\{c_l|\xi\rangle, l \in \mathbf{L}, |\xi\rangle \in S'\} \subseteq S' \cap D^\perp$ . It remains to show that, if such an  $S' \not\subseteq D$  exists, then there exists a subspace  $S \subseteq D^\perp$  satisfying  $c_l S \subseteq S \forall c_l$ . Note that the set  $D' = \{|\Psi\rangle \text{ such that } Q^\dagger |\Psi\rangle = i\lambda |\Psi\rangle, \text{ with } \lambda \in \mathbb{R}\}$  is equal to  $D$ .<sup>1</sup> Using again that  $Q^\dagger \rho + \rho Q$  must be positive semidefinite, which implies that  $\langle \Phi | Q^\dagger \rho + \rho Q | \Phi \rangle = 0$  if and only if  $(Q^\dagger \rho + \rho Q)|\Phi\rangle = 0$ , we have that  $Q^\dagger \rho |\Phi\rangle = i\lambda \rho |\Phi\rangle \forall |\Phi\rangle \in D$ . Thus,  $\rho |\Phi\rangle \in D \forall |\Phi\rangle \in D$ , which implies that  $\rho$  can be decomposed as  $\rho = \rho_D + \rho_{D^\perp}$ , where  $R(\rho_D) \subseteq D$  and  $R(\rho_{D^\perp}) \subseteq D^\perp$ . Now, since  $\mathcal{L}(\rho) = \mathcal{L}(\rho_{D^\perp})$  we only have to choose  $S = R(\rho_{D^\perp})$ . ■

This proof shows that, if there exists a stationary state  $\rho$  which is not a dark state, i.e.,  $R(\rho) \not\subseteq D$ , then  $\rho = \rho_D + \rho_{D^\perp}$  with  $R(\rho_D) \subseteq D$  and  $R(\rho_{D^\perp}) \subseteq D^\perp$  such that  $R(\rho_{D^\perp})$  is invariant under the operators  $\{c_l\}$ .

In the next sections we use the results presented above to design dissipative processes which can be used for state preparation. That is, we derive the jump operators, such that the system is driven into the unique stationary state. Due to the results above this state can be chosen to be a dark state of the process.

Note that this goal can never be achieved using Hermitian jump operators  $c_l$ . The reason for that is the following. If all

operators  $c_l$  are Hermitian, then the master equation can be written as

$$\mathcal{L}(\rho) = \sum -i[H, \rho] + g_l[[c_l, \rho], c_l]. \quad (4)$$

Thus, any operator which commutes with  $\{c_l\}$  and  $H$  is a stationary state, for instance, the completely mixed state proportional to  $\mathbb{1}$  is stationary.

#### IV. CONSTRUCTION OF DISSIPATIVE PROCESSES LEADING TO THE DESIRED STATE

In this section we show that one can design the system-reservoir coupling in such a way that any multipartite state can be obtained as the unique stationary state of a dissipative process. Furthermore, we solve the corresponding master equation analytically by deriving the whole spectrum of the corresponding Liouvillian. Given this solution, it is then easy to compute any relevant quantity of the process, such as, for instance, the relaxation time, or any correlation function. In fact, we show that the relaxation time is independent of the number of subsystems. Even though these facts are not very surprising, we will use the constructed process to show that for certain states the jump operators can be chosen quasilocally. In general such a construction will not lead to a quasilocal dissipative process. Therefore, we demonstrate how the quasilocal operators can be constructed given a quasilocal description of the state at the end of this section. We start out by considering  $n$ -qubit states and generalize the formalism later to a  $d$ -level system.

For any  $n$ -qubit state  $|\Psi\rangle = U|0\rangle^{\otimes n}$ , we construct a set of operators  $\{c_k\}_{k=1}^n$  such that the unique stationary state of the dissipative process, described by  $\dot{\rho} = \mathcal{L}_{\{c_k\}}(\rho)$ , is  $|\Psi\rangle$ . Note that  $|0\rangle^{\otimes n} \equiv |0^n\rangle$  is the unique stationary state of the dissipative process with jump operators  $d_k = \sigma_-^{(k)} = \mathbb{1}_1 \otimes \cdots \otimes |0\rangle_k \langle 1| \otimes \mathbb{1}_n$ , i.e.,  $d_k$  acts nontrivially only on the  $k$ th qubit. This can be easily seen, since  $d_k|\phi\rangle = 0 \forall k$  if and only if  $|\phi\rangle = |0^n\rangle$ , which implies that  $|0^n\rangle$  is the unique pure stationary state (see Theorem 1). Furthermore, it is straightforward to show that for any  $|\chi\rangle \perp |0^n\rangle$  there exists a monomial of the jump operators,  $P(|\chi\rangle)$  such that  $\langle 0^n | P(|\chi\rangle) |\chi\rangle \neq 0$ . This shows that  $|0^n\rangle$  is the unique stationary state of  $\mathcal{L}_{\{d_k\}}$ .

We construct now the operators  $c_k$  which lead to the unique stationary state  $|\Psi\rangle = U|0^n\rangle$ . Defining  $c_k = U d_k U^\dagger$  we have  $c_k|\phi\rangle = 0$  if and only if  $|\phi\rangle = U|0^n\rangle = |\Psi\rangle$ . One can use the same arguments as above to show that this is the unique stationary state. This immediately implies that, if the jump operators  $c_k$  can be written as  $U(\mathbb{1}_{1, \dots, k-1} \otimes |0\rangle_k \langle 1| \otimes \mathbb{1}_{k+1, \dots, n})U^\dagger$  and  $|\Psi\rangle \equiv U|0^n\rangle$  is an eigenstate of the Hamiltonian  $H$ , then  $|\Psi\rangle$  is the unique stationary state of  $\mathcal{L}_{\{H, c_k\}}$ .

Note that the statements above would not be changed if we would use an invertible matrix  $X$  instead of a unitary, as long as this does not lead to more common eigenstates of the operators  $\{X c_k X^{-1}\}$ .

##### A. Analytic solution of the master equation

In this section we solve analytically the master equation  $\dot{\rho} = \mathcal{L}_{\{c_k\}}(\rho)$ , for any set of operators  $\{c_k\}$  which are unitarily

<sup>1</sup>This can be seen as follows. It is obvious that  $D \subset D'$  to show the inverse, we use the fact that  $|\Psi\rangle \in D'$  implies that  $\langle \Psi | P | \Psi \rangle + i \langle \Psi | H | \Psi \rangle = i\lambda$ . Since both  $P$  and  $H$  are Hermitian (and  $\lambda \in \mathbb{R}$ ), this last equation can only be satisfied if  $|\Psi\rangle \in D$ .

equivalent to the set  $\{\sigma_{-}^{(k)}\}$ , i.e.,  $c_k = U\sigma_{-}^{(k)}U^\dagger$  for some unitary  $U$ . In order to compute the eigenvalues and eigenvectors of the Liouvillian  $\mathcal{L}_{\{c_k\}}$  we note that  $\mathcal{L}_{\{U\sigma_{-}^{(k)}U^\dagger\}}(\sigma) = U\mathcal{L}_{\{\sigma_{-}^{(k)}\}}(U^\dagger\sigma U)U^\dagger$ . Therefore,  $\mathcal{L}_{\{U\sigma_{-}^{(k)}U^\dagger\}}(\sigma) = \lambda\sigma$  if and only if  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}(U^\dagger\sigma U) = \lambda U^\dagger\sigma U$ . Thus, computing the eigenvectors and eigenvalues of  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}$  gives us immediately the eigenvalues and eigenvectors of  $\mathcal{L}_{\{c_k\}}$ .

Since  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}$  describes the situation where  $n$  qubits are interacting with identical and independent reservoirs, we only have to find the eigenvectors and eigenvalues of  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}$  for one fixed  $k$ . This describes the situation where a single two-level system is interacting with a thermal bath. The solution to the eigenvalue problem is well known [17,18]. We find  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}(\sigma_i^{(k)}) = \lambda_i\sigma_i^{(k)}$  if and only if  $\sigma_i^{(k)} \in \Sigma_k \equiv \{\sigma_0^{(k)} = |0\rangle\langle 0|, \sigma_3^{(k)} = \sigma_z^{(k)}\} \cup \text{span}\{\sigma_1^{(k)} = \sigma_x^{(k)}, \sigma_2^{(k)} = \sigma_y^{(k)}\}$ ; the corresponding eigenvalues are  $\lambda_0 = 0, \lambda_3 = -2g_k$ , and the two-fold-degenerate eigenvalue is  $\lambda_1 = \lambda_2 = -g_k$ . Thus, a basis of the eigenvectors of the total Liouvillian is  $\{\sigma_{i_1}^{(1)} \otimes \cdots \otimes \sigma_{i_n}^{(n)}\}$  with  $\sigma_{i_k}^{(k)} \in \Sigma_k, i_k \in \{0,1,2,3\}$  and the corresponding eigenvalues are  $\lambda_{i_1, \dots, i_n} \equiv \sum_k \lambda_{i_k}$ . Note that the eigenvectors of  $\mathcal{L}_{\{c_k\}}$  are just the unitary transformation of the eigenvectors of  $\mathcal{L}_{\{\sigma_{-}^{(k)}\}}$  with the same eigenvalues. Thus, writing the initial state in the eigenbasis,  $\rho(0) = U \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} \sigma_{i_1}^{(1)} \otimes \cdots \otimes \sigma_{i_n}^{(n)} U^\dagger$ , we obtain

$$\rho(t) = e^{\mathcal{L}t}(\rho(0)) = U \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} e^{-\lambda_{i_1, \dots, i_n} t} \sigma_{i_1}^{(1)} \otimes \cdots \otimes \sigma_{i_n}^{(n)} U^\dagger. \quad (5)$$

Let us stress here the fact that there exists no purely imaginary eigenvalue. If such a pair (recall that complex eigenvalues occur in pairs; see the Introduction) existed, then the additional condition that  $\lim_{t \rightarrow \infty} \rho(t) = \rho_{\text{SS}}$ , where  $\rho_{\text{SS}}$  denotes the stationary state, would not be satisfied. In order to see how fast the system is driven into the stationary state, we compute the relaxation time  $T_{\text{relax}}$ , which is defined as the inverse of the minimum of the negative real part of  $\lambda_{i_1, \dots, i_n}$  different from zero. We find  $T_{\text{relax}} = 1 / \min_k g_k$  and is therefore given by the minimal coupling constant. Note that it is not very surprising that  $T_{\text{relax}} \geq 1 / \min_k g_k$ , since the system cannot be driven faster into the stationary state. The reason that this amount of time is already sufficient is because the evolution of the qubits can be decoupled. In other words, for fixed coupling constants, the relaxation time is constant in the number of qubits.

It is straightforward to generalize this formalism to  $d$ -level systems. One simply has to replace the operators  $|0\rangle\langle 1|$  by a matrix  $J_d = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3| + \cdots + |d-1\rangle\langle d|$ , i.e., a Jordan matrix with only one eigenstate (here to eigenvalue zero).<sup>2</sup> Only the state  $|0\rangle$  is a proper eigenstate of  $J_d$ . All the other computational basis states are generalized eigenvectors, which means that for any  $k \neq 0$  there exists an  $i$  such that  $J_d^i |k\rangle = |0\rangle$ . This implies that for any state  $|\phi\rangle$  there exists

a polynomial of the operators  $J_d, P(J_d)$ , such that the overlap  $\langle 0|P(J_d)|\phi\rangle$  is not vanishing, which is exactly the property that we need in order to prove that the stationary state  $|0^n\rangle$  or, respectively,  $U|0^n\rangle$  is the unique stationary state. Since in this case also the master equation decouples for the different subsystems, it is straightforward to solve it analytically.

### B. Quasiloca l dissipative processes

If one wants to prepare the state  $|\Psi\rangle$  using a dissipative process, then one will be interested in a simple physical interaction between the reservoir and the system. One requirement, for instance, could be that the operators  $c_k$  are quasilocal, which means that they act nontrivially only on a small number of qubits. Depending on the state  $|\Psi\rangle$  the operators  $c_k$  might be chosen to be quasilocal, as we are going to show next. Note, however, that this cannot be true for any state. The reason for this is the following.

As shown before, for any  $n$ -qubit state  $|\Psi\rangle$  one can find  $n$  operators  $c_k$  which uniquely define the state  $|\Psi\rangle$ , in the sense that  $|\Psi\rangle$  is the only state that is a (right) eigenstate to eigenvalue 0 of all operators  $c_k$ . Thus, a description of the state  $|\Psi\rangle$  is the set of operators  $\{c_k\}_{k=1}^n$ . In general one might need more than one operator per subsystem. We denote the corresponding set of operators by  $\{c_k^\alpha\}$ , where  $\alpha = 1, \dots, d$ , for some  $d$ , and call this set a quasilocal description of  $|\Psi\rangle$  if all  $c_k^\alpha$  are quasilocal. The quantum Kolmogorov complexity, i.e., the number of classical bits required to describe the state  $|\Psi\rangle$  equals the classical Kolmogorov complexity of the set  $\{c_k^\alpha\}$ . If all these operators are quasilocal, then the Kolmogorov complexity scales only polynomially with the number of qubits. It is known, however, that for any  $n$  there exists an  $n$ -qubit state, whose quantum Kolmogorov complexity scales exponentially with the number of qubits [19].

Before we discuss more generalized schemes we want to use the process discussed above to show that certain states can be prepared using quasilocal dissipative processes. We consider the state  $|\Psi\rangle = U|0\rangle^{\otimes n}$ . Now, suppose that  $U = U_1 U_2 \cdots U_n$ , where each of the unitaries  $U_k$  commute with each other and are quasilocal. Let us assume that  $U_k$  is acting on particles  $k-1, k, k+1$ . Then the jump operators  $c_k = U\sigma_k U^\dagger = U_{k-1} U_k U_{k+1} \sigma_k (U_{k-1} U_k U_{k+1})^\dagger$  are also quasilocal and the dissipative process corresponding to  $c_k$  has as a unique stationary state the state  $|\Psi\rangle$ . As shown before, for such a process the relaxation time is constant, i.e., independent of the number of subsystems.

For certain cases such a simple construction will not be possible. Therefore, we describe here a general method of deriving the jump operators, i.e., the dissipative process, which give rise to the desired state. Since the jump operators are not Hermitian (at least not all of them can be chosen to be Hermitian), they might not be diagonalizable. However, one can use the Jordan normal form to gain some insight into the necessary properties of these operators [20]. The jump operators must be chosen such that there exists only one common eigenstate to eigenvalue zero. The Jordan decomposition of a matrix  $c$  is

$$c = SJS^{-1}, \quad (6)$$

where  $J$  denotes the Jordan matrix of  $c$  and  $S$  is an invertible matrix.  $J$  is a block diagonal matrix with  $d_i \times d_i$  diagonal blocks,

<sup>2</sup>Note that any factor in front of the terms  $|j-1\rangle\langle j|$  would not change the argument. Thus  $J_d$  is like the annihilation operator for finite dimensions.

$$J_{d_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & & \\ 0 & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}. \quad (7)$$

The number of Jordan blocks is the number of linearly independent eigenvectors of  $c$ . Let us for simplicity consider here the case where  $S$  is a unitary,  $U$ . The proper eigenvectors of  $c$  are then  $U|e_{f(k)}\rangle$ , where  $f(k) = \sum_{i=0}^k d_i + 1$  and  $|e_k\rangle$  denotes the standard basis. A simple example is the operator  $\sigma_-$ . The only eigenstate is the state  $|0\rangle$ . Considering multipartite entangled states, the operators  $c_k$  must have more than a single eigenstate to eigenvalue 0 (as  $\mathbb{1} \otimes \sigma_-^k \otimes \mathbb{1}$  has). Therefore the matrices  $c_k$  will have Jordan matrices with several Jordan blocks and all eigenvalues 0. The unique eigenstate that is common to all operators is the state one wants to prepare,  $|\Psi\rangle$ . Apart from that, one has to assure that any other state can be mapped into some state having nonvanishing overlap with  $|\Psi\rangle$ .

One might also choose a single jump operator, which has only one proper eigenstate (which corresponds to eigenvalue zero) (see also [15]). In order to be more precise, let us denote by  $\{|\phi_i\rangle\}_{i=0}^{2^n-1}$  an orthonormal basis with  $|\phi_0\rangle = |\Psi\rangle$  being the state we want to prepare. The operator  $C = \sum_{i=0}^{2^n-1} |\phi_i\rangle\langle\phi_{i+1}|$  has only one proper eigenstate, namely,  $|\Psi\rangle$ . For the generalized eigenvectors  $|\phi_i\rangle$  for  $i > 0$  it holds that  $C^i|\phi_i\rangle = |\Psi\rangle$ . Thus, for any state  $|\chi\rangle$ , there exists a  $k$  such that  $\langle\Psi|C^k|\chi\rangle \neq 0$ . This shows that the single operator  $C$  leads, like the operators  $\{c_k\}$  described above, to the unique stationary state  $|\Psi\rangle$ . However, in contrast to the operators  $\{c_k\}$ , the operator  $C$  will act nontrivially on all subsystems and is therefore (in general) harder to implement experimentally.

In summary, a sufficient condition for the existence of a quasilocal dissipative process, which can be employed to prepare a certain state is the following. First of all, the state must have a quasilocal description, i.e., there exists a set of operators  $\{c_k^\alpha\}$  such that the only common eigenstate is  $|\Psi\rangle$ . This implies that there exists only one dark state for the corresponding dissipative process. Second, if there exist polynomials  $P_i(\{c_k^\alpha\})$  such that the states  $P_i^\dagger(\{c_k^\alpha\})|\Psi\rangle$  form a basis in the Hilbert space, then  $|\Psi\rangle$  is the unique stationary state. More generally, if there exist polynomials  $P_i(\{c_k^\alpha\})$  such that  $\mathcal{E}(|\Psi\rangle\langle\Psi|) = \sum_i P_i^\dagger(\{c_k^\alpha\})|\Psi\rangle\langle\Psi|P_i(\{c_k^\alpha\}) = X$  where  $X$  is invertible, then  $|\Psi\rangle$  is the unique stationary state. This can be seen as follows. If  $\mathcal{E}(|\Psi\rangle\langle\Psi|) = X$ , with  $X$  invertible, then we have, for any state  $|\chi\rangle \in \mathcal{H}$ ,  $\langle\chi|\mathcal{E}(|\Psi\rangle\langle\Psi|)|\chi\rangle \neq 0$ . Since  $\mathcal{E}(|\Psi\rangle\langle\Psi|)$  is a sum of positive semidefinite operators this implies that there exists at least one term  $P_i^\dagger(\{c_k^\alpha\})|\Psi\rangle\langle\Psi|P_i(\{c_k^\alpha\})$ , which has a nonvanishing overlap with the state  $|\chi\rangle\langle\chi|$ .

## V. EXAMPLES

In this section we illustrate the formalism described above by applying it to several examples. In Secs. V A and V D, we show that the dissipative processes which we analyzed in detail in [11] have unique stationary states, namely, the BEC

state and the  $\eta$  condensate, respectively. In Sec. V B we derive the processes for stabilizer states, and in Sec. V C we consider arbitrary PEPS states and show how, for instance, the ground state of the AKLT model can be generated with a quasilocal dissipative process.

### A. Example: Driven dissipative Hubbard dynamics of bosons on a lattice

In Ref. [11] we described a driven dissipative Hubbard dynamics of bosonic particles on a lattice. The corresponding dynamics was written in terms of a master equation (2) with  $H$  a Hubbard Hamiltonian, containing the coherent hopping of particles between the sites of the lattice, and their interaction, and where a Liouvillian with quasilocal jump operators was designed to drive the system into a nonequilibrium condensate of bosons or paired fermions. While Ref. [11] focused on nonequilibrium condensed matter aspects and in particular on the effect of interactions, and implementation of this master equation with cold atoms, we claimed—but did not prove—the uniqueness of the many-body dark states for this master equation. This missing proof will be provided here based on the theorems derived in the previous sections.

We consider  $N$  bosons hopping on a  $d$ -dimensional lattice with Hamiltonian

$$H_B = H_0 + V \equiv -J \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{1}{2} U \sum_i a_i^{\dagger 2} a_i^2, \quad (8)$$

which is written as the sum of a kinetic and an interaction energy term. Here  $a_i$  denotes the destruction operator for a boson on site  $i$ , and  $\langle i,j \rangle$  denotes adjacent lattices sites. Reference [11] suggested that for noninteracting bosons ( $U=0$ ) a master equation (2) with the above Hamiltonian and quasilocal dissipation

$$c_\ell \equiv c_{ij} = (a_i^\dagger + a_j^\dagger)(a_i - a_j) \quad (9)$$

has the unique steady state solution in the form of a pure Bose-Einstein condensate representing a state with long-range order

$$|\text{BEC}\rangle = a_{\mathbf{q}=\mathbf{0}}^{\dagger N} / \sqrt{N!} |\text{vac}\rangle.$$

Here  $a_{\mathbf{q}} = (1/\sqrt{M^d}) \sum_j a_j e^{i\mathbf{q}\cdot\mathbf{e}_j}$  is the destruction operator for quasimomentum  $\mathbf{q}$  on a lattice with  $M^d$  lattice sites, lattice vectors  $\mathbf{e}_\lambda$ , and spacing  $a$ . It is easy to see that the state  $|\text{BEC}\rangle$  satisfies

$$(i) \quad (a_i - a_j)|\text{BEC}\rangle = 0 \quad \forall \langle i,j \rangle$$

$$(ii) \quad H_0|\text{BEC}\rangle = N\epsilon_{\mathbf{q}=\mathbf{0}}|\text{BEC}\rangle$$

with  $\epsilon_{\mathbf{q}=\mathbf{0}}$  the single-particle Bloch energy for quasimomentum  $\mathbf{q}=\mathbf{0}$ , and thus is consistent with the conditions of a pure steady state of the master equation. We show now that this state is unique.

Before proceeding we find it convenient to go to a quasimomentum presentation. The dissipative part of the master equation then takes on the form (2) with jump operators

$$c_\ell \equiv c_{\mathbf{q},\lambda} = \frac{1}{\sqrt{M^d}} \sum_{\mathbf{k}} (1 + e^{i(\mathbf{k}-\mathbf{q}) \cdot \mathbf{e}_\lambda})(1 - e^{-i\mathbf{k} \cdot \mathbf{e}_\lambda}) a_{\mathbf{k}-\mathbf{q}}^\dagger a_{\mathbf{k}} \quad (10)$$

and coupling constants  $g_l$  independent of  $l$ .

To show uniqueness we show first off all that the BEC state is the unique dark state and that there exists no invariant subspace for our choice of jump operators. For a fixed particle number  $N$ , the first term in  $c_\ell$  is a creation operator and has no eigenvalues; in particular, no zero eigenvalues. Thus, in order to identify dark states  $|D\rangle$  with zero eigenvalue, we may restrict ourselves to the equation  $(a_i - a_j)|D\rangle = 0 \forall \langle i, j \rangle$ . Taking the Fourier transform, this translates to  $(1 - e^{i\mathbf{q} \cdot \mathbf{e}_\lambda}) a_{\mathbf{q}} |D\rangle = 0 \forall \mathbf{q}$ . Thus the BEC state with  $\mathbf{q} = \mathbf{0}$  is the only dark state. Next, we will construct for every state  $|\Phi\rangle$  in the Hilbert space a polynomial operator  $O(c_{\mathbf{q},\lambda})$ , where the jump operators  $c_{\mathbf{q},\lambda}$  are given in Eq. (10), such that  $\langle \text{BEC} | O | \Phi \rangle \neq 0$ . With the notation  $\mathbf{n} = \{\dots, n_{\mathbf{k}}, \dots\}$ , the states  $|\mathbf{n}\rangle = \prod_{\mathbf{q}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle$  form a basis in the Hilbert space; a general state can be written as  $|\Phi\rangle = \sum_{\mathbf{n}} f_{\mathbf{n}} |\mathbf{n}\rangle$ . We select a state  $|\mathbf{m}\rangle$  with  $f_{\mathbf{m}} \neq 0$  and the number of particles in the zero-momentum mode maximal. Furthermore, for each  $\mathbf{q} \neq \mathbf{0}$  we fix  $\lambda_{\mathbf{q}}$  such that  $\mathbf{e}_{\lambda_{\mathbf{q}}}$  is not orthogonal on  $\mathbf{q}$ , i.e.,  $\mathbf{e}_{\lambda_{\mathbf{q}}} \cdot \mathbf{q} \neq 0$ . Note that  $\langle \text{BEC} | (c_{\mathbf{q},\lambda_{\mathbf{q}}})^{n_{\mathbf{q}}} (a_{\mathbf{k}}^\dagger)^{m_{\mathbf{k}}} |0\rangle \neq 0$  only if  $n_{\mathbf{q}} = m_{\mathbf{k}}$  and  $\mathbf{q} = \mathbf{k}$ . Thus, applying the polynomial operator  $O = \prod_{\mathbf{q} \neq \mathbf{0}} (c_{\mathbf{q},\lambda_{\mathbf{q}}})^{m_{\mathbf{q}}}$  to  $|\Phi\rangle$  provides a finite overlap with the BEC, which implies that the BEC is the unique stationary state of the dissipative process.

## B. Preparation of stabilizer states

As a second example we show that stabilizer states can be obtained as the unique stationary states of a dissipative process involving only quasilocal interactions. We denote by  $X, Y, Z$  the standard Pauli operators. The Pauli group  $\mathcal{P}$  consists of all Pauli matrices  $X, Y, Z, 1$ , together with the multiplicative factors,  $\pm 1, \pm i$ .  $\mathcal{P}_n = \mathcal{P}^{\otimes n}$  defines the Pauli group on  $n$  qubits. An  $n$ -qubit state  $|\Psi\rangle$  is called a stabilizer state if it is uniquely defined as the only eigenstate to eigenvalue  $+1$  of a Hermitian subgroup (of order  $n$ ) of  $\mathcal{P}_n$ , called the stabilizer of  $|\Psi\rangle$  [21]. That is,  $|\Psi\rangle$  is the only state left invariant under the subgroup

$$S(|\Psi\rangle) = \{g \in \mathcal{P}_n : g|\Psi\rangle = |\Psi\rangle\}. \quad (11)$$

Let us denote the generators of this group by  $U_l$ ,  $l=1, \dots, n$ . A subset of the stabilizer states are the so-called graph states [22], which are associated with a mathematical graph which consists of vertices and edges. Whenever two vertices  $a, b$  are connected by an edge, we say that  $b$  is in the neighborhood of  $a$  and write  $b \in N_a$ . An  $n$ -qubit graph state,  $|\Psi_{0, \dots, 0}\rangle$ , can now be defined as the unique eigenstate of a set of independent commuting observables  $U_k = X_k Z_{N_k} = X_k \prod_{b \in N_k} Z_b$ , where  $W_k = \sigma_w^k$  for  $W \in X, Y, Z$  denotes the Pauli operator  $\sigma_w$  acting on qubit  $k$ . Note that these unitaries define a unique basis, the so-called graph state basis, which we denote by  $\{|\Psi_{i_1, \dots, i_n}\rangle\}_{i_j \in \{0,1\}}$ . It can be shown that  $|\Psi_{i_1, \dots, i_n}\rangle = Z_1^{i_1} \dots Z_n^{i_n} |\Psi_{0, \dots, 0}\rangle$  [22], where  $U_k |\Psi_{i_1, \dots, i_n}\rangle = (-1)^{i_k} |\Psi_{i_1, \dots, i_n}\rangle$ . In particular we have that  $|\Psi_{i_1, \dots, i_n}\rangle = Z_k^{i_k} |\Psi_{i_1, \dots, i_n}\rangle$ .

$= Z_k^{i_k} |\Psi_{i_1, \dots, i_n}\rangle$ . We consider now those graph states for which all the unitaries  $U_k$  are quasilocal. An example would be the linear cluster state, where  $U_k = Z_{k-1} X_k Z_{k+1}$ . An other example would be the 2D cluster state, which is a universal resource for quantum computations [2]. As shown in [2], once such a state is prepared, any quantum computation can be performed by means of local measurements only.

We define the operators  $c_k = \frac{1}{2}(1 + U_k)Z_k$ , which act only nontrivially on the same qubits as  $U_k$  and are therefore quasilocal. Note that  $1 + U_k$  is a projector onto the subspace of eigenvectors of  $U_k$  with eigenvalue  $+1$  and  $Z_k$  changes any eigenstate  $|\Psi_{i_1, \dots, i_k=0, \dots, i_n}\rangle$  to  $|\Psi_{i_1, \dots, i_k=1, \dots, i_n}\rangle$ , and vice versa. Thus,  $c_k$  is an operator which has only one  $2^{n-1}$ -fold-degenerate eigenvalue, namely, 0, and the corresponding eigenspace is  $\text{span}\{|\Psi_{i_1, \dots, i_k=0, \dots, i_n}\rangle$  with  $i_j \in \{0, 1\}$ , for  $j \neq k\}$ . Thus, we have that  $c_k |\phi\rangle = 0$  if and only if  $U_k |\phi\rangle = |\phi\rangle$ , which implies that the only state which is an eigenstate to all operators  $c_k$  is the graph state. Using the definition of  $c_k$  it is easy to show that  $c_k = U \sigma_{i_k}^{(k)} U^\dagger$ , where  $U$  transforms the computational basis into the graph basis. Due to the results presented in Sec. III, this shows that the graph state is the unique stationary state of the process described by  $\mathcal{L}_{\{c_k\}}$ . Since any stabilizer state is up to some local unitary (actually local Clifford) operation  $V$  equivalent to a graph state, the operators  $c(S)_k = V c_k V^\dagger$  are quasilocal and  $\mathcal{L}_{\{c(S)\}}$  has as unique stationary state the stabilizer state. Furthermore, the relaxation of this process is independent of  $n$  and there exists no purely imaginary eigenvalue of the Liouvillian.

Note that we can write  $c_k = Z_k P_k$ , where  $P_k = \frac{1}{2}(1 - U_k)$  is the projector onto the eigenspace to eigenvalue  $-1$  of  $U_k$ . Thus, the evolution corresponding to these operators can be implemented using a feedback mechanism.

Another way to show that the stabilizer states can be prepared using quasilocal jump operators would be to use the relation between the unitary that generates the state and the operators  $\{c_k\}$ , as described in Sec. IV.

## C. Preparation of MP and PEP states

As mentioned above, one has to choose non-Hermitian matrices to guarantee that the desired state is the unique stationary state. Since otherwise, at least the completely mixed state is a stationary state too. The aim of this section is to demonstrate the general construction of the jump operators. Therefore, we consider the so-called product entangled pair state [23]. In one dimension these states are called matrix product states (MPSs). Let us consider  $n$   $d$ -dimensional systems. Any state describing these systems can be written as

$$|\Psi\rangle = \sum_{i_1, \dots, i_n=1}^d \text{tr}(A_1^{[1]} A_2^{[2]} \dots A_n^{[n]}) |i_1, \dots, i_n\rangle, \quad (12)$$

where  $A_i^{[k]}$  are  $D \times D$  matrices, with  $D$  being the bond dimension [24]. Using this way of presenting the states and especially the generalization to 2D has been proven to be very powerful to determine, for instance, the ground states of some Hamiltonians [23].

Similar to the procedure above, we design now the interaction between the reservoir and the system, such that the desired MPS or PEPS is the unique stationary state. In [25] it has been shown that for many MPSs and PEPSs one can construct a frustration-free Hamiltonian which has this state as the unique ground state. These Hamiltonians are of the form

$$H = \sum_k h_k, \quad (13)$$

where the Hermitian operators  $h_k$  are quasilocal. The ground state  $|\Psi\rangle$  of these Hamiltonians is uniquely defined by the equations

$$h_k|\Psi\rangle = \lambda_{\min}|\Psi\rangle \quad \forall k, \quad (14)$$

where  $\lambda_{\min}$  denotes the minimal eigenvalue of  $h_k$ . That is, the ground state of the Hamiltonian corresponds to the ground state of the quasilocal Hamiltonians.

In order to obtain these states now as stationary states of a dissipative process we construct the operators  $c_k$  which have a unique common eigenstate, which is the desired MPS or PEPS. In order to do so we consider the ground states of the operators  $h_k$ ,  $D(h_k)$  (note that this must be more than one dimensional) and again construct the non-Hermitian operators  $c_k$  whose eigenvectors span  $D(h_k)$ . Then, the only common eigenstate of the operators  $c_k$  is  $|\Psi\rangle$ . To ensure that there exists no mixed stationary state, one might need to consider more than one operator per site.

Let us illustrate the general idea by considering as an example the ground state of the AKLT model [12]. Historically, this state occurred first in the context of condensed matter physics. There it was shown to be the unique ground state of a Heisenberg-like Hamiltonian (see below) [12]. Recently this state has attracted interest in quantum-information theory, due to its useful properties for quantum communication [26]. The Hamiltonian has the following form:

$$H = \sum_k h_k \quad \text{with } h_k = \vec{S}_k \cdot \vec{S}_{k+1} + \frac{1}{3}(\vec{S}_k \cdot \vec{S}_{k+1})^2, \quad (15)$$

where the operators  $S^\alpha$  with  $\alpha \in \{x, y, z\}$  denote the spin-1 operators. The quasilocal Hamiltonians  $h_k$  act nontrivially on systems  $k$  and  $k+1$ . As mentioned before, the ground state of the Hamiltonian, which we denote by  $|\Psi\rangle$ , is the ground state of all quasilocal Hamiltonians  $h_k$ .

Since the Hamiltonian is translationally invariant we omit the index  $k$  as well as the identity operator whenever it does not lead to any confusion. The operators  $h_k$  have the eigenvalues  $4/3$ , which is fivefold degenerate, and  $-2/3$ , which is fourfold degenerate. We denote by  $|\Phi_k\rangle$  ( $|\Psi_k\rangle$ ) an orthonormal basis of the subspace corresponding to the eigenvalue  $4/3$  ( $-2/3$ ), respectively. The operators  $c_k$  (acting on system  $k$  and  $k+1$ ) can be defined as

$$c_k = |\Psi_1\rangle\langle\Phi_1| + |\Psi_2\rangle\langle\Phi_2| + |\Psi_3\rangle\langle\Phi_3| + |\Psi_4\rangle\langle\Phi_4| + |\Psi_5\rangle\langle\Phi_5|. \quad (16)$$

It is easy to see that the only states which are eigenstates of  $c_k$  are states in the subspace spanned by  $(\{|\Psi_k\rangle\}_{k=1}^4)$ , i.e., the ground state subspace of  $h_k$ . Thus, the set of operators

$\{c_k\}$  is sufficient to guarantee that there is no other pure stationary state.

In order to show that there is no mixed state, we note that the operators  $c_k$  can be written as  $c_k = U_k P_k$ , where  $U_k$  is a unitary matrix and  $P_k = \frac{1}{2}(2/3\mathbb{1} + h_k)$  is the projector onto the eigenstates of  $h_k$  with eigenvalue  $4/3$ . The choice of the unitary is by no means unique and we will use this freedom to show that also in this case there is no mixed stationary state. Let us write  $c_k^\alpha = U_k^\alpha P_k$ , then  $c_k^\alpha|\Phi\rangle = 0 \forall k, \alpha$  if and only if  $|\Phi\rangle = |\Psi\rangle$ . For any state  $|\chi\rangle$  different from  $|\Psi\rangle$ , there exists a  $k$  such that  $P_k|\chi\rangle \neq 0$ . Without loss of generality we assume  $k=n$ . We construct now a completely positive map,  $\mathcal{E}_k(\rho) = \sum_\alpha (c_k^\alpha)^\dagger \rho c_k^\alpha$  such that  $\mathcal{E}(|\Psi\rangle\langle\Psi|) \equiv \mathcal{E}_n \circ \mathcal{E}_{n-1} \circ \dots \circ \mathcal{E}_1(|\Psi\rangle\langle\Psi|) \propto \mathbb{1}_{2,3,\dots,n} \otimes P_n$  and therefore  $\langle\chi|\mathcal{E}(|\Psi\rangle\langle\Psi|)|\chi\rangle \neq 0$ . Since all the terms occurring in  $\mathcal{E}(|\Psi\rangle\langle\Psi|)$  are positive semidefinite, this implies that there exists a monomial  $P = c_n^{\alpha_n} \dots c_1^{\alpha_1}$  such that  $\langle\chi|P^\dagger|\Psi\rangle\langle\Psi|P|\chi\rangle = |\langle\Psi|P|\chi\rangle|^2 \neq 0$ . Thus, the state is unique. We choose sufficiently many  $U^\alpha$  such that  $\sum_\alpha (U_k^\alpha)^\dagger \rho U_k^\alpha = \frac{1}{9}\mathbb{1}_k \otimes \text{tr}_{k,k+1}(\rho)$  for any state  $\rho$ .<sup>3</sup> Using that  $\text{tr}_k(P_k) \propto \mathbb{1}$ , we find

$$\begin{aligned} \mathcal{E}_2(\mathcal{E}_1(|\Psi\rangle\langle\Psi|)) &\propto \mathcal{E}_2(P_1 \otimes \text{tr}_{1,2}(|\Psi\rangle\langle\Psi|)) \propto \mathbb{1}_1 \otimes P_2 \\ &\otimes \text{tr}_{1,2,3}(|\Psi\rangle\langle\Psi|). \end{aligned} \quad (17)$$

Continuing in this way we end up with  $\mathbb{1}_{2,3,\dots,n} \otimes P_n$ , which shows that the ground state of the AKLT model is unique.

#### D. Driven $\eta$ condensate

The second example for a dissipatively driven state given in [11] is the  $\eta$  condensate of paired fermions. First described by Yang [28], the  $\eta$  condensate is an exact excited eigenstate of the Fermi-Hubbard (FH) model for fermions with two internal states  $\uparrow, \downarrow$  on a bipartite lattice in  $d$  dimensions with  $M^d$  sites. Its properties derive from the fact that the FH Hamiltonian

$$H_{\text{FH}} = -J \sum_{\langle l,l'\rangle, \sigma=\uparrow,\downarrow} f_{\sigma,l}^\dagger f_{\sigma,l'} + U \sum_l f_{\uparrow,l}^\dagger f_{\downarrow,l}^\dagger f_{\downarrow,l} f_{\uparrow,l} \quad (18)$$

and the  $\eta$  creation operator

$$\eta^\dagger \equiv \frac{1}{\sqrt{M^d}} \sum_l S(l) f_{\uparrow,l}^\dagger f_{\downarrow,l}^\dagger \quad (19)$$

obey the commutation relation  $[\eta^\dagger, H_{\text{FH}}] = U\eta^\dagger$ , where  $S(l)$  in (19) alternates between  $\pm 1$  in a checkerboard pattern. The  $f_{\sigma,l}^\dagger$  operators create a fermion with spin  $\sigma = \uparrow, \downarrow$  on site  $l$  and obey canonical anticommutation relations. In (18),  $J$  denotes the tunneling rate and  $U$  the on-site interspecies interaction.

From this observation, it follows that the state  $|\eta, N\rangle \equiv (\eta^\dagger)^N |0\rangle$  is an exact excited eigenstate of  $H_{\text{FH}}$ , with energy  $NU$ , irrespective of the boundary conditions. In the following, we assume periodic boundary conditions. In position space the  $\eta$  condensate can be understood as a superposition state of all possible vectors  $|D^{(d,k)}\rangle$ , which denote position basis states in  $\mathcal{H}_N$  that have only double occupation, at posi-

<sup>3</sup>This can always be achieved with finitely many  $U_k^\alpha$  since the dimension of the Hilbert space is finite; see, for instance, [27].

tions  $d_k$ ,  $k \in (1, \dots, N)$ :  $|\eta, N\rangle = \sum_{\{d_k\}} S(\{d_k\}) |D^{(d_k)}\rangle$ , where  $S(\{d_k\}) = \pm 1$  if the number of even  $d_k$  is itself even (odd) on the even-indexed sites. A pair of fermions on the same site is called a *doublon* in the following.

The  $\eta$  condensate, while never a ground state of the Hubbard model, is interesting from the perspective of many-body physics, as it exhibits perfect superfluidity in any spatial dimension, without any approximations.  $\eta$  pairing ( $\langle \eta^\dagger \eta \rangle \neq 0$ ) in the ground state has also been considered for doped negative- $U$  Hubbard models [29] and extended Hubbard models (see, e.g., [30]).

We define the jump operators

$$\begin{aligned} c_l^{(1)} &= (\eta_l^\dagger - \eta_{l+1}^\dagger)(\eta_l + \eta_{l+1}), \\ c_l^{(2)} &= n_{\uparrow,l} f_{\downarrow,l}^\dagger f_{\downarrow,l+1} + n_{\uparrow,l+1} f_{\downarrow,l+1}^\dagger f_{\downarrow,l}, \\ c_l^{(3)} &= (f_{\uparrow,l}^\dagger + f_{\uparrow,l+1}^\dagger)(f_{\uparrow,l} + f_{\uparrow,l+1})(1 - n_{\downarrow,l})(1 - n_{\downarrow,l+1}), \\ c_l^{(4)} &= (f_{\downarrow,l}^\dagger f_{\downarrow,l+1} + f_{\downarrow,l+1}^\dagger f_{\downarrow,l}) n_{\uparrow,l} n_{\uparrow,l+1} \end{aligned} \quad (20)$$

on the Hilbert space  $\mathcal{H}_N$ , in which all states have  $N$  spin-up and  $N$  spin-down fermions. Here,  $\eta_l^\dagger \equiv f_{\uparrow,l}^\dagger f_{\downarrow,l}^\dagger$  and  $n_{\sigma,l} \equiv f_{\sigma,l}^\dagger f_{\sigma,l}$ .

A dark state  $|\Psi\rangle \in \mathcal{H}_N$  is defined through the condition  $c_l^{(k)}|\Psi\rangle = 0 \quad \forall k, l$ . In the orthonormal position basis,  $|\Psi\rangle$  is defined through the coefficients  $A_{i_1, j_1, \dots, i_M, j_M}$ , where  $i_l = 1$  ( $=0$ ) if there is (no) spin-up fermion on  $l$ , and  $j_l$  is analogously defined for spin-down fermions. Besides normalization, all  $A$ 's must obey  $\sum_{l=1}^M i_l = \sum_{l=1}^M j_l = N$ . In this notation,  $c_l^{(1)}$  imposes that the coefficients for any dark state must obey  $A_{1,1,0,0} = -A_{0,0,1,1}$  and  $A_{1,1,1,0} = A_{1,1,0,1} = A_{1,0,1,1} = A_{0,1,1,1} = 0 \quad \forall i_k, j_k, k \neq l, l+1$  (the  $i_k$ 's and  $j_k$ 's in  $A$  are suppressed for brevity). The first of the two conditions is the essential property of the  $\eta$  condensate:  $|D^{(d_k)}\rangle$  differing in the position of one doublon by one site carry opposite signs.  $c_l^{(2)}$  imposes the constraint  $A_{1,0,0,1} = A_{0,1,1,0} = 0$ , which signifies that fermions of opposite spin on adjacent sites are associated into doublons (on the site of the spin-up fermion).  $c_l^{(3)}$  then imposes  $A_{1,0,0,0} = -A_{0,0,1,0}$  and  $A_{1,0,1,0} = 0$ . These jump operators induce a diffusion process, which delocalizes the spin-up fermions, provided they do not encounter spin-down fermions or doublons. Furthermore, two spin-up fermions may not sit on adjacent sites in these dark states.  $c_l^{(4)}$  finally imposes  $A_{1,1,1,0} = -A_{1,0,1,1}$ , i.e., spin-down fermions must also be delocalized over the lattice, but only atop the spin-up fermions.

Using the conditions imposed by  $c^{(1)}$ ,  $c^{(2)}$ , and  $c^{(3)}$  and the constraint on the number of spin-up and spin-down fermions, it is straightforward to show that the  $\eta$  condensate is the only dark state. Note that  $|\eta, N\rangle$  is trivially also a dark state to  $c^{(4)}$ . We assume that there are nonzero coefficients in  $|\Psi\rangle$  for configurations with unpaired spin-up and spin-down fermions. The conditions for the dark state then immediately yield a contradiction. This is the case, as the condition  $A_{1,0,0,0} = -A_{0,0,1,0}$  means that any unpaired spin-up fermion may be shifted in position if no spin-up fermion, spin-down fermion, or doublon is on the adjacent site. By assumption and the condition, all the coefficients to these configurations must also be nonzero. By shifting sufficiently often, one of these

obstacles will eventually be encountered—at which point the other conditions will yield a contradiction.

Thus, any dark state may only have nonzero coefficients for the doublon basis states  $|D^{(d_k)}\rangle$ . The first condition from the  $c^{(1)}$  then immediately yields that all these coefficients must be equal to  $S(\{d_k\})$ . This is the case, because starting from any doublon configuration with nonzero amplitude, repeated application of this condition yields that any other possible doublon configuration on the lattice must also be nonzero, with its sign also obeying the rules for the  $\eta$  condensate.

The proof of uniqueness again requires that for any  $|\Psi\rangle \in \mathcal{H}_N$  a monomial  $P(\{c_l^{(k)}\})$ ,  $k=1, \dots, 4$ , can be constructed such that  $\langle \eta, N | P | \Psi \rangle \neq 0$ . The proof proceeds differently for two different cases: (a) for  $|\Psi\rangle \notin \text{span}\{|D^{(d_k)}\rangle\}$ , where  $|D^{(d_k)}\rangle$  denotes a basis state carrying only doublons on sites  $d_k$ ,  $k \in (1, \dots, N)$ ; (b) for  $|\Psi\rangle \in \text{span}\{|D^{(d_k)}\rangle\}$ .

(a) The proof for the uniqueness of the dark state indicates that we can proceed analogously to the case of the BEC, but now in position space instead of momentum space: Pick one position basis state  $|\phi^{(0)}\rangle$  with unpaired fermions occurring in  $|\Psi\rangle$  with nonzero amplitude. It is clear from the above proof that we can always find a sequence of applications of  $c_l^{(3)}$ 's and  $c_l^{(4)}$ 's such that a spin-up and a spin-down fermion are on adjacent sites (we observe that  $c_l^{(4)}$  allows a doublon to swap positions with a spin-up fermion on an adjacent site). Application of  $c^{(2)}$  then associates these into a doublon. It is straightforward to see that this sequence can be constructed such that it maps  $|\phi^{(0)}\rangle$  into a particular  $|D^{(d_k)}\rangle$ , and any state orthogonal to  $|\phi^{(0)}\rangle$  to zero.

(b) We can proceed directly, analogous to the proof of uniqueness for the ground state of the AKLT model, by generalizing the jump operators  $c_l^{(1)}$  to  $c_l^{(1)\alpha} = U_l^\alpha P_l$ , with  $P_l$  the projector on the symmetric state of one doublon on sites  $l$  and  $l+1$ . The unitaries  $U_l^\alpha$  now must conserve particle number. We define completely positive maps  $\mathcal{E}_l$  again as in Sec. V C. Their application in sequence on the  $\eta$  condensate yields a density matrix with nonvanishing overlap to  $|\Psi\rangle$ , which proves the existence of a monomial  $P$  leading to finite overlap, as for the ground state of the AKLT model.

## VI. CONCLUSION AND OUTLOOK

In this paper we have shown how to use dissipative processes for state preparation. That dissipation, in conjunction with measurements, can be used to prepare pure states is known (see, e.g., [31,32]), where usually single- or few-particle states are considered. Here, we demonstrated how dissipation can be employed to generate multipartite states. For an arbitrary state composed out of  $n$   $d$ -level systems we constructed a dissipative process (with constant relaxation time), consisting of  $n$  jump operators, which has this state as the unique stationary state. For certain states, such as the cluster states, we showed that this process can be implemented using only quasilocal operations. Apart from that we demonstrated how a quasilocal dissipative process, which is suitable for state preparation, can be constructed for a given state, which has a quasilocal description. We illustrated this method by considering the ground state of the AKLT model.



Furthermore, we showed that the quasilocal dissipative processes, which we considered in [11], have as unique stationary states the BEC state and the  $\eta$  condensate, respectively.

The processes discussed here might also be used to gain some insight in to the computational complexity of certain problems, like for instance the so-called satisfiability problem [33]. There, the aim is to find out whether there exists a common solution to a set of Boolean equations. To be more precise, the problem is to determine an  $n$ -bit string for which  $m = \text{poly}(n)$  given clauses, involving only three (two) variables for the 3-SAT (2-SAT) problem respectively, hold true. It is straightforward to construct a dissipative process such that any computational state, corresponding to the classical solution of the problem, is a stationary state of the process. Using the formalism we developed here, we are going to investigate the difference between the  $NP$ -complete 3-SAT problem and the 2-SAT problem, which can be solved (classically) in polynomial time. It might well be that looking at this problem from this completely different point of view leads to a new insight. Furthermore, it would be interesting

to establish a relationship between the relaxation time and the nonlocality of the jump operators, which cause the system to evolve to the desired state. Apart from that, the investigation of dissipative processes, which lead to higher-dimensional dark state subspaces, might lead to interesting applications for quantum computation and storage of quantum information.

In [34] the authors investigated independently similar aspects of dissipative processes. It was shown there that these processes can also be employed for universal efficient quantum computation.

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