

Entanglement and the speed of evolution in mixed states

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Entanglement speeds up evolution of a pure bipartite spin state, in line with the time-energy uncertainty. However, if the state is mixed this is not necessarily the case. We provide a counterexample and point to other factors affecting evolution in mixed states, including classical correlations and entropy.

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I. INTRODUCTION

The speed with which a quantum system evolves from one state to the next is a subject of both theoretical and practical interest. The question sheds light on the nature of time and of quantum evolution in general [1]. Practical applications are evident in the design of a possible quantum computer, in quantum metrology including the establishment of precise frequency standards [2], and in optical and atomic clocks [3]. Some applications will require as fast a period as possible, while others require maximal distinguishability between states. Since practical applications nearly always involve some loss of purity, it is also necessary to consider the evolution of mixed states.

The speed of evolution of a quantum system has been shown to be inversely proportional to energy or energy spread (E or ΔE) [4]. It has also been shown that entanglement speeds up evolution [5]. We find that with mixed states these are not always the case. Mixed states with identical ΔE show completely different time evolution, and there is no clear relationship to the entanglement. In addition to entanglement, factors related to the mixing contribute.

We look at a specific model of a bipartite two-level system and examine factors affecting time evolution, first when the system is in a pure state, and then when it is mixed. With

a pure state the system behaves as expected. However, with mixed states we find surprising behavior. We look at three states that mix a maximally entangled state and separable states in different representative ways [6]. One is a Werner state, which mixes a maximally entangled state with a maximally mixed state: $\rho_{\text{Werner}} = [(1-x)/4]Id + x|\Psi^+\rangle\langle\Psi^+|$ where $|\Psi^+\rangle = (1/\sqrt{2})(|10\rangle + |01\rangle)$. The second, which we name ρ_{Gisin} [7], mixes Ψ^+ with a mixture of product states: $\rho_{\text{Gisin}} = [(1-x)/2](|00\rangle\langle 00| + |11\rangle\langle 11|) + x|\Psi^+\rangle\langle\Psi^+|$. This will enable us to inspect the role of classical correlations of the product states. The third state is a mixture of Ψ^+ with a maximally polarized state with no correlations at all: $\rho_3 = (1-x)|00\rangle\langle 00| + x|\Psi^+\rangle\langle\Psi^+|$. ρ_{Werner} is entangled for $x \geq \frac{1}{3}$, ρ_{Gisin} for $x \geq \frac{1}{2}$, and ρ_3 for all $x > 0$.

Our model has two physically separated spins each in a magnetic field, with the field for each spin orthogonal to the spin axis, so that the evolution operator contains σ_x for each spin. Units are dimensionless with $\omega = 1$. Figure 1 shows the time evolution for these three classes of mixed state for small t . The graphs show the speed of decay from the original state as a function of x . For all three classes of states, taking $\langle E \rangle = \text{tr}(\rho H)$ we find $\Delta E = 2(1+x)$. However we see in Fig. 1 that although the states have the same ΔE , their *time evolution is different*.

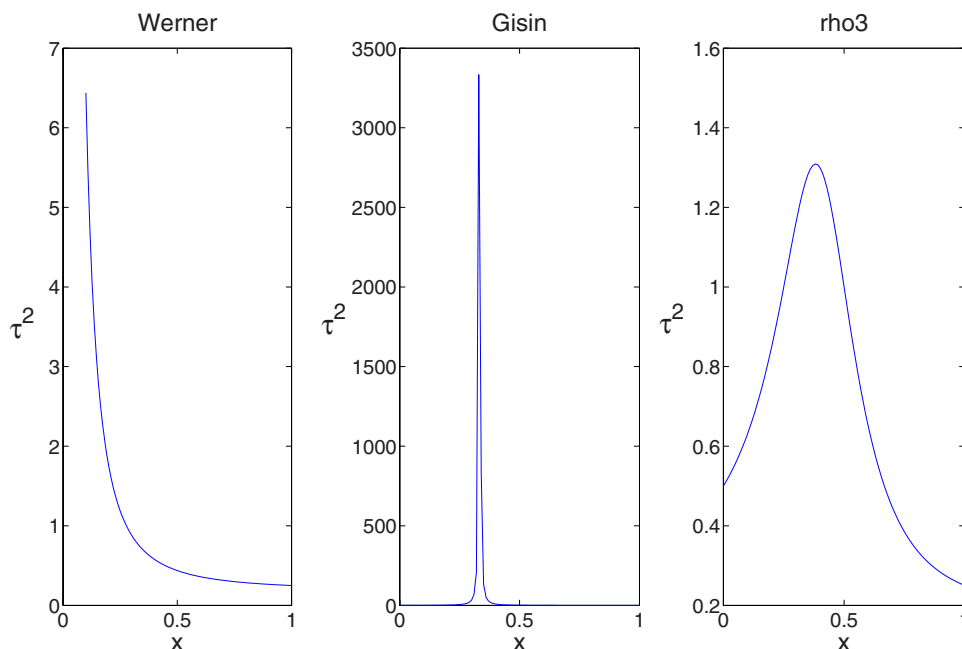


FIG. 1. (Color online) Time evolution for the three matrices (decay coefficient as function of mixing).

As will be discussed in Sec. III, the change in a state is proportional to $\exp(-t^2/\tau^2)$, and Fig. 1 shows τ^2 as a function of x , so that the lower the graph, the faster the rate of decay. With the Werner state entanglement increases with x , and indeed the state is stationary at the identity and the speed of evolution increases with increasing x . However, the center graph of the Gisin state shows increasing speed both for increasing x (increasing entanglement), and also for *decreasing* x . For the Gisin state, decreasing x means increasing classical correlations so that these too may affect the speed. The third matrix gives the most surprising graph. Entanglement, as measured by the concurrence [8], rises linearly with x , but the graph of the speed of decay is clearly not a monotonic function of the entanglement, and for $x < 0.38$ speed increases as entanglement decreases. Thus we see from Fig. 1 that the short-time evolution reveals the essential difference between mixed and pure states: with mixed states the speed is no longer inversely proportional to ΔE , and it is not necessarily a monotonic function of entanglement.

Therefore other factors are at work as well. These are connected to the mixing and include classical correlations and entropy. In this paper we provide an analysis of their contribution to evolution. The paper is organized as follows. First we give details of the model and its evolution when the system is in a pure state. Then we consider mixed states. In that case it becomes necessary to inspect different aspects of evolution separately: evolution for a short time (as discussed above), period of evolution, and maximal distance from the original state. We then separate the effect of classical correlations by looking at the period and distance of states which are mixed but not entangled. We see that classical correlations too can speed up the period of evolution, but at the expense of reducing maximal distance. This can be useful in applications where the change itself is important rather than the distance between states. Finally, we show through numerical means that all correlations speed up evolution while entropy slows it down.

II. DETAILS OF THE MODEL

We take a bipartite two-level system, modeled as two spins in magnetic fields. The spins are physically separated so that a local operation on one will not locally affect the other. The Hamiltonian for the total system is $H = H_a + H_b$, where $H_i = \vec{\sigma}_i \cdot \hat{n}_i$; i refers to the first or second spin and \hat{n}_i refers to the direction of the magnetic field at the location of that spin.

For a pure state it is easily shown that the speed of evolution is a monotonically increasing function of entanglement. Taking a state with arbitrary entanglement, $|\Psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\rangle$, where quantization is along the z axis and entanglement is maximal for $\alpha = \beta = 1/\sqrt{2}$, the state will reach orthogonality when $\langle\Psi|U|\Psi\rangle = 0$, where $U = e^{-i(\vec{\sigma} \cdot \hat{n})_a + (\vec{\sigma} \cdot \hat{n})_b} = [(\cos t)Id - i(\sin t)\vec{\sigma} \cdot \hat{n}]_a [(\cos t)Id - i(\sin t)\vec{\sigma} \cdot \hat{n}]_b$. Equating the real and imaginary parts separately to zero, we find the time to reach an orthogonal state,

$$t_{\perp} = \text{arccot} \sqrt{\langle(\vec{\sigma}_a \cdot \hat{n}_a)(\vec{\sigma}_b \cdot \hat{n}_b)\rangle}. \tag{1}$$

Plugging in, for example, σ_x on both spins, $t_{\perp} = \pi/2$ for a product state, where either α or β is zero. The time decreases

continuously with increasing entanglement, and for a maximally entangled state reaches a minimum at $\pi/4$.

The magnet angles that will give maximal speed are obtained by minimizing the function for time, and by taking into account constraints from the imaginary part of the equation. It turns out that optimal angles for product states must have one of the magnets orthogonal to its spin axis, while for maximally entangled states the optimum depends solely on the relationship between the two magnets. For example, for the anticorrelated singlet optimal magnet angles are antilinear, whereas for a correlated entangled state $(1/\sqrt{2})(|11\rangle + |00\rangle)$ we have $\theta_a = \theta_b$, $\phi = -\phi_b$. Each of the four Bell states has a different set of constraints for optimal angles, so that we have an external physical constraint reflecting the inner structure of the state.

III. MIXED STATES: SHORT-TERM EVOLUTION

Time evolution in this model has (at least) three aspects: evolution for a short time, period, and maximal distance from the original state. With pure states this distinction does not add information, but in the case of mixed states each of the three aspects gives different results.

We begin with evolution for a short time, which we call the kickoff. For pure states

$$\begin{aligned} |\langle\Psi_0|\Psi(t)\rangle|^2 &= |\langle\Psi_0|e^{-i(\hbar H)t}|\Psi_0\rangle|^2 \simeq \left|1 - \frac{it}{\hbar}\langle H\rangle - \frac{t^2}{2\hbar^2}\langle H^2\rangle\right|^2 \\ &= 1 - \frac{t^2}{\hbar^2}(\langle H^2\rangle - \langle H\rangle^2) \cong e^{-(t^2/\hbar^2)\delta E^2} \end{aligned} \tag{2}$$

so that for a short time the decay coefficient is $\delta E^2/\hbar^2$ and the speed of evolution is indeed proportional to the energy spread.

For mixed states we take a direct analog of fidelity in the density matrix formalism:

$$F(\rho(t=0), \rho(t)) = \frac{\text{tr}[\rho(t=0)\rho(t)]}{\text{tr}[\rho(t=0)^2]} = \frac{\text{tr}(\rho_0 U \rho_0 U^\dagger)}{\text{tr}(\rho_0)^2}. \tag{3}$$

The denominator is for normalization. The transparent analogy to fidelity for pure states makes this a convenient measure: for a pure state where $\rho = |\Psi\rangle\langle\Psi|$ this reduces to

$$\frac{\langle\Psi|\Psi\rangle\langle\Psi|U|\Psi\rangle\langle\Psi|U^\dagger|\Psi\rangle}{\langle\Psi|\Psi\rangle}, \tag{4}$$

which is just the square of the fidelity. Expanding F as we did in Eq. (2) by taking $U \simeq 1 - iHt$, we obtain

$$\frac{\text{tr}(\rho_0 U \rho_0 U^\dagger)}{\text{tr}(\rho_0^2)} = 1 - t^2 \frac{\text{tr}(\rho \rho H H) - \text{tr}(\rho H \rho H)}{\text{tr} \rho^2} \simeq e^{-t^2/\tau^2}, \tag{5}$$

so that (taking $\hbar = 1$) $1/\tau^2$ is analogous to δE^2 for pure states. Therefore we can examine the relationship between energy spread and time in the case of density matrices as well. Figure 1 shows τ^2 as a function of x . Though δE^2 is identical for

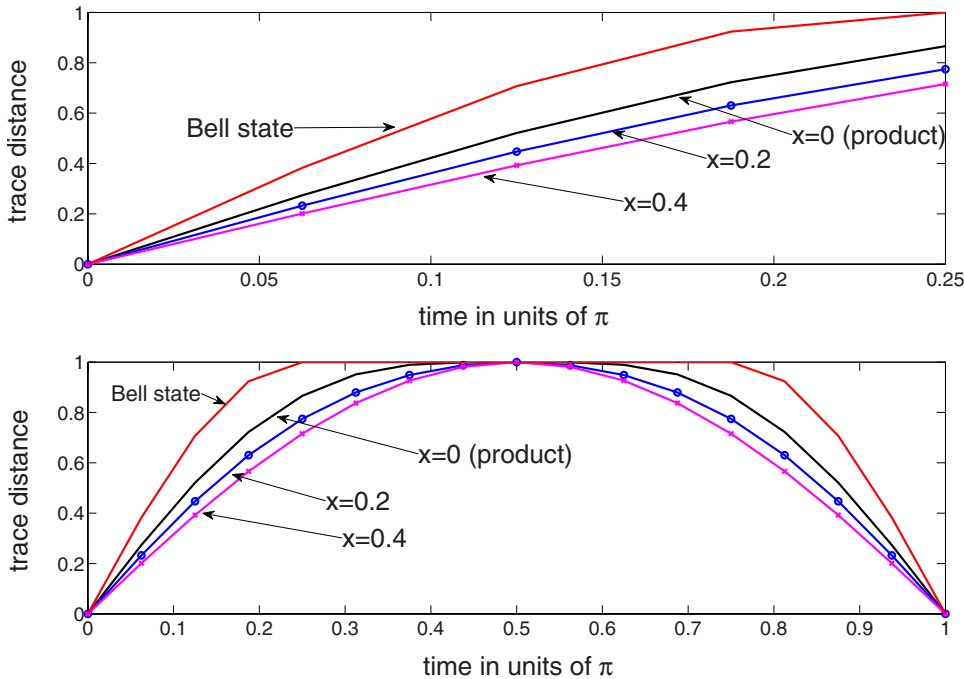


FIG. 2. (Color online) Trace distance for partially entangled mixed state ρ_3 . The concurrence grows linearly with x . Top is for time until $\pi/4$ and bottom shows a full cycle. The lowest curve in both graphs is more entangled than the second lowest.

all three classes of states: ρ_{Wer} , ρ_{Gis} , and ρ_3 , they have clearly different behavior.¹

This behavior is also seen with the more conventional trace distance measure. The trace distance is defined as $D(\rho, \sigma) \equiv \frac{1}{2} \text{tr} |\rho - \sigma|$ [9]. We take $\rho = \rho(t=0)$ and $\sigma = \rho(t) = U\rho U^\dagger$. For pure states this gives the same result as in Sec. II: the more entangled the initial state, the faster it evolves to an orthogonal state. The trace distance for a pure general entangled state $|\psi_{\text{ent}}\rangle = (\cos \gamma)|10\rangle - (\sin \gamma)|01\rangle$ is $D(t) = |\sin(2\gamma) \sin(2t)|$. So at $t = \pi/4$ the distance may equal unity (orthogonal state), but only for maximal entanglement where $\sin \gamma = 1/\sqrt{2}$; for lower entanglement, orthogonality is reached at a later time. For mixed states the graph of the trace distance for small t shows the same surprising results for our three matrices as that shown in Fig. 1. For example, with ρ_3 , for a certain range of values of x , states that are more entangled evolve *more slowly* than states that are less entangled (see Fig. 2).

As before, we see that contrary to expectation, there is no clear indication that the speed increases as a result of increasing entanglement; or rather, it increases with increasing entanglement but also in other cases. Entanglement may be affecting the speed but there must be *another* factor or factors as well.² The question is what these factors are, aside from energy and energy spread. The additional factors must

be related to the mixing, and include classical correlations and entropy.

IV. NONENTANGLED MIXED STATES

The other two aspects of time evolution are helpful in unveiling additional factors, and it is important to distinguish between them. Time evolution in this model is periodic. We therefore look at two different aspects of period. First is the length of a half cycle, that is, the time needed to reach an orthogonal or maximally distant state, which for simplicity we will refer to as “period.” The second aspect of interest is the maximal achievable distance. Both of these are seen with the trace distance measure, and optimizing for each gives different results.

The problem with evolution of mixed states is to separate out the different effects of mixing and of classical and quantum correlations. In order to isolate the effect of classical factors we first examine a mixed state with no quantum correlations at all. In this section we treat a specific toy model, in order to clarify the different effects of period and distance. In the next section we will deal with separable states in general, and attempt to isolate the factors of correlation and entropy.

Quantum correlations were expressed in Sec. II using α and β , which went from 0 (or 1) for a product state to $1/\sqrt{2}$ for maximal entanglement. We now look at a classical analogy in a nonentangled density matrix, $\rho = a|11\rangle\langle 11| + (1-a)|00\rangle\langle 00|$, where for $a = \frac{1}{2}$ we call the state maximally correlated in analogy to the quantum formalism. This is not a standard measure of correlations; that would be better expressed with mutual information, and we do so in Sec. V. The measure in this section attempts to look only at the extent of the departure from a pure product state to a mixed correlated state, and its effect on time evolution.

¹We also graphed τ^2 for the three matrices with angles optimized at each point of x ; the graphs are less dramatic but display the same general shape and slopes.

²Giovanetti *et al.* showed that the evolution is inversely proportional to ΔE , by showing that the time to reach a given fidelity is $T(F) = \beta(F) \pi \hbar / 2 \Delta E$, $\beta(F) = (2/\pi) \arccos(F)$ [10]. We graphed $(1/\beta)(x)$ for our three matrices, taking F as in Eq. (3) and an arbitrary time $t=6$. This graph is *identical* to our graph for $\tau^2(x)$ (Fig. 1). So our decay coefficient is proportional to Giovanetti’s *et al.* time bound, and is a function of the fidelity.

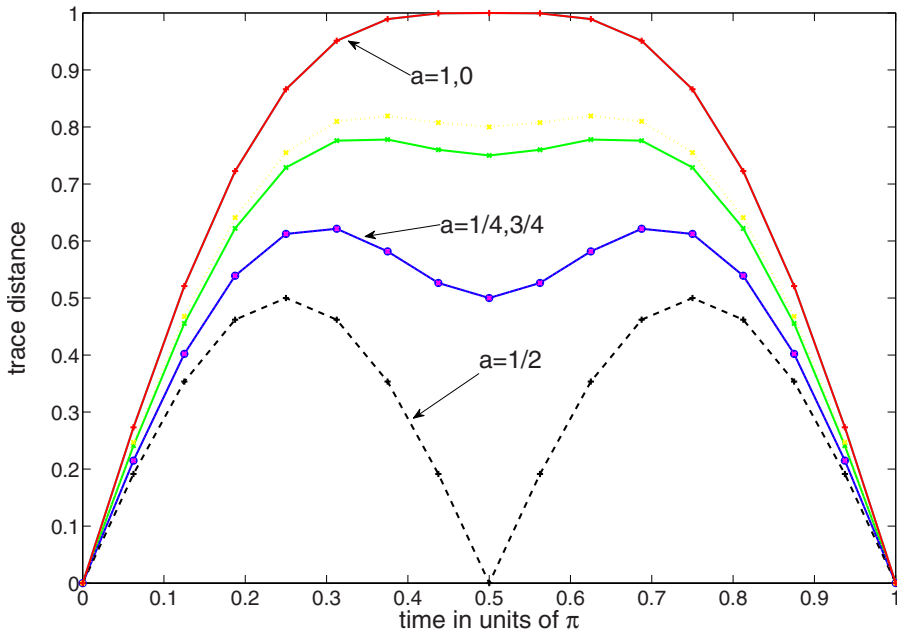


FIG. 3. (Color online) Trace distance of product mixture for various values of a (σ_x on both spins). Top curve (red): $a=0, 1$. Lowest curve (black): $a=1/2$. Second lowest curve (blue, magenta): $a=1/4, 3/4$. Third lowest curve (green): $a=1/8$. Second highest curve (yellow): $a=9/10$.

We look at the time evolution from two different aspects. First we optimize for period, choosing magnet angles that give the fastest possible period regardless of the distance achieved. Afterward we optimize for trace distance rather than period. In the first case the mixing leads to speedup and reduces distance. In the second it causes slowdown but eventually all states reach orthogonality.

Figure 3 shows a graph of the trace distance as a function of time for representative values of a , optimized for period. The fastest period ($\pi/4$) but with the smallest achievable distance ($1/2$) are attained when the state is maximally mixed, $a=1/2$. For a pure product state, $a=1$ and $a=0$, the period is twice as slow ($\pi/2$) but the states reach orthogonality with a distance of 1. The graph shows that as the mixing approaches the product state, the point of maximal distance comes later and the distance grows. Thus we see

that under the appropriate operators increasing mixing reduces the trace distance, but the distance reaches maximum more quickly.

If instead we optimize angles for the largest attainable trace distance (which is at the expense of speed) we see in Fig. 4 that all states achieve optimal distance, but the greater the mixing (the farther from a pure product state), the slower the kickoff and general rate of evolution. This is probably the reason that when optimized for trace distance, entangled mixed states do not necessarily achieve the optimal distance so quickly: the mixing attenuates the result. For practical purposes it is the first case, with optimization for period, that is interesting: in applications where the speed of change is important rather than its extent, increasing mixing of classically correlated states may improve the result. It should be

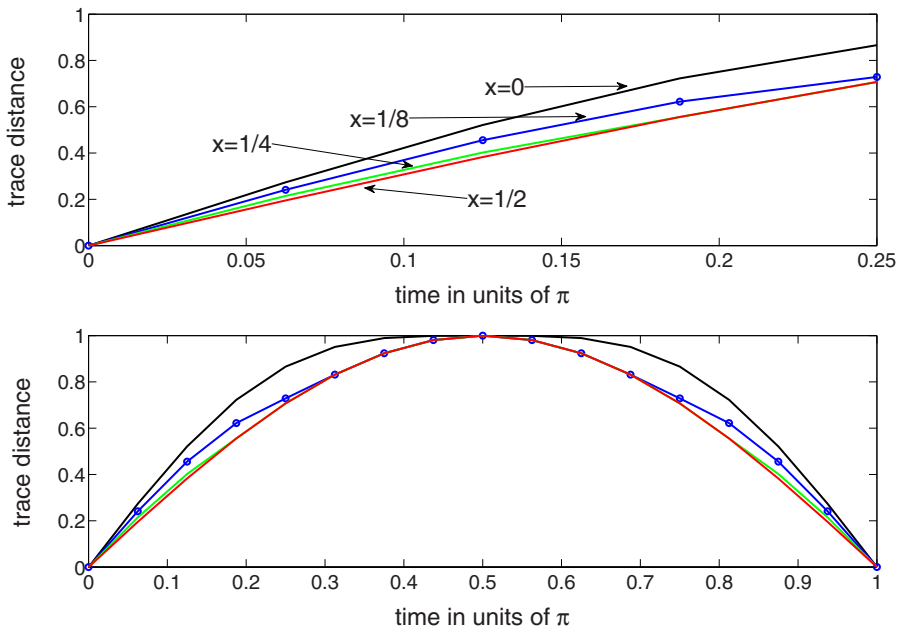


FIG. 4. (Color online) Trace distance for product mixture, optimized for maximum distance. Top is for time until $\pi/4$ and bottom shows a full cycle. In both graphs the highest (black) curve is a pure product state, the lowest is maximally mixed ($a=1/2$), and slope increases as a approaches 0 or 1. Note, however, that all cases reach an orthogonal state at the same time $\pi/2$.

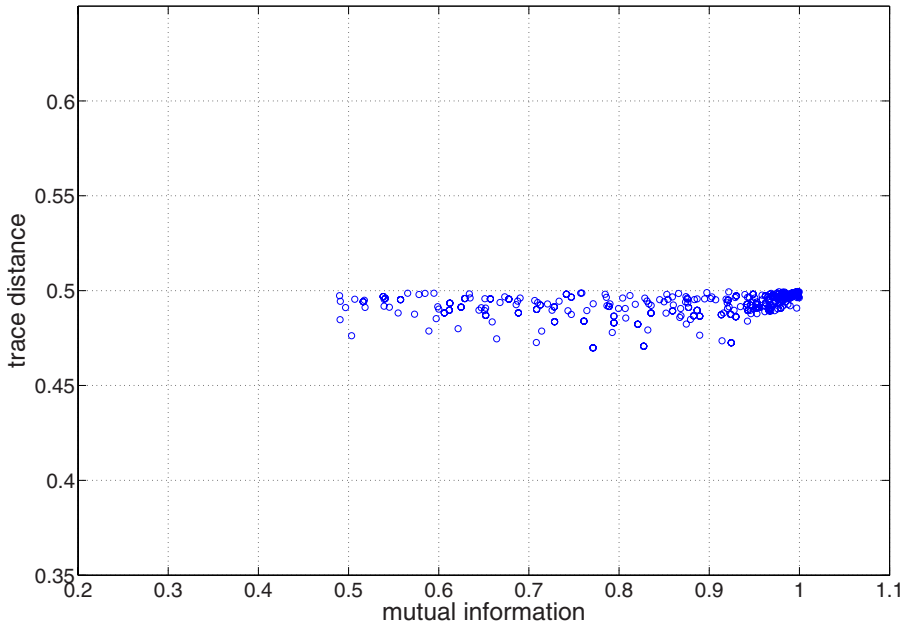


FIG. 5. (Color online) Trace distance as a function of mutual information for separable states. Most, but not all, states that achieve maximal distance have mutual information approaching 1. Mean: 0.8608 (x), 0.4928 (y). Median: 0.9296 (x), 0.4942 (y). Std: 0.1461 (x), 0.005 979 (y).

noted that this applies only to a bipartite state. For larger systems, further investigation is necessary.

V. MUTUAL INFORMATION AND ENTROPY

In the previous section we looked at the effect of mixing as a whole on a toy model, and distinguished between the aspects of period and distance. We now deal with the problem on a more general basis, and attempt to separate out the factors of correlations and of entropy. A measure of the total correlations between two subsystems is mutual information, defined in terms of entropy: $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$. This is actually the relative entropy between ρ_{AB} and $\rho_A \otimes \rho_B$ and so it is a measure of correlations between the subsystems [11]. Groisman *et al.* [12] have shown that mutual information may be seen as describing both classical and quantum correlations. States with maximal classical correlations have $I=1$, and entangled states include classical as well as quantum correlations and have higher I , up to 2 for maximally entangled states. For separable states, a numerical search went through a total of 36×10^6 separable states. Nearly all of these separable states are mixed (pure states constitute only the surface of the Bloch sphere). For these states, we show in Fig. 5 the difference in distance, $D_{\text{dif}} = D(t=\pi/4) - D(t=\pi/2)$, as a function of mutual information, where for each state we optimized angles to give the highest value of D_{dif} . This optimizes for fastest period, that is, reaching the maximal trace distance as quickly as possible. Taking the distance from 0 to $\pi/4$ would be deceptive, because we have seen that one state may have a higher distance at $\pi/4$ than another, but then continue to a maximum at, for example, $\pi/3$, so that it actually reaches its maximum later, while the first state reached its maximum at $\pi/4$. We could have optimized for kickoff or maximal distance rather than period, of course; it was necessary to choose one specific aspect of evolution, and this proved the least ambiguous of the three. In the previous section, when optimizing for

period, we found a specific reference point that proves useful: a distance of $1/2$ for $t=\pi/4$. Indeed, we will see with a numerical search that no unentangled state reaches a greater distance at this point.

We found that the fastest evolving states have a half cycle of $\pi/4$ and so this function D_{dif} shows the maximal distance obtained at the fastest cycle. As with the toy model, here too the maximal trace distance for this period is found to be $1/2$. Therefore if correlations were the sole cause we would expect all optimal states that have a distance of $1/2$ to have mutual information of 1, that is, maximal total correlations for a separable state. In fact, the majority of these states do—but not all of them. Some states that do not have maximal correlations still achieve the optimum trace distance. Therefore trace distance is not a monotonic function of the total correlations, and some other factor is in effect as well.

We therefore used the results of the same search to graph trace distance as a function of von Neumann entropy. The result is similar to the previous section: most states with the optimal trace distance have the minimal entropy $S=1$, as for a maximally correlated state—but again, not all of them. Some states with higher entropy also achieve the maximum trace distance, as shown in Fig. 6. Yet although the same data were used for both—the same states each with its trace distance—the graphs are not exactly inverse images of each other. It appears that in nonentangled mixed states both correlations and entropy affect time evolution, and further work is necessary to clarify the effect.

VI. ENTANGLED MIXED STATES

We now turn to entangled states. It would be desirable to isolate the effect of entanglement on evolution from the other factors in all mixed states. At present we can show this for the three matrices mentioned above, as follows:

When the magnet angles are \hat{z} , $-\hat{z}$, as shown in Fig. 7, all three matrices have the same graph, while the product mix-

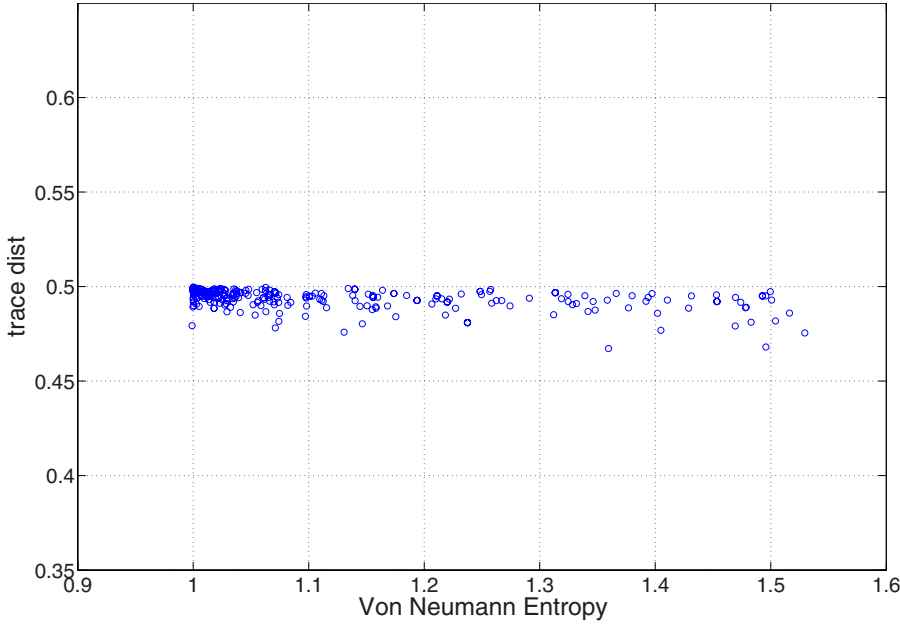


FIG. 6. (Color online) Trace distance as a function of entropy for separable states. Most but not all states that achieve maximal distance have entropy approaching 1. Mean: 1.13 (x), 0.481 (y). Median: 1.057 (x), 0.4951 (y). Std: 0.1572 (x), 0.092 49 (y).

ture does not evolve at all. This is because product states (e.g., $|11\rangle\langle 11|$) are invariant at these angles, so that in this column we are looking only at the effect of the amount of entanglement on the trace distance and have neutralized the effect of mixing product states. In this case all three matrices give exactly the same analytical function for the trace distance $D=|x \sin(2t)|$ and thus the double cycle is preserved for all x , that is, for any degree of entanglement in the mixture. In addition, x attenuates the maximum distance. It is notable that the three matrices have the identical analytical trace distance function, even though the third is always entangled and the others are not. In addition, the general shape is the same whether the matrix is entangled or not (e.g., $x < \frac{1}{3}$ and $x > \frac{1}{3}$ for Werner states). Entanglement clearly speeds up evolution, but the increase is smoothly affected by

something else as well and evidently this is not the mixing with product states which has now been neutralized. It must be noted that entangled states possess classical correlations as well. Therefore evolution in this case may be affected both by quantum and by classical correlations. This would account for the smoothness of the graphs.

However, in addition all three matrices have the same energy spread ΔE so the relation between energy spread and evolution seems here to be in force. This shows that it is the mixing with product states that distorted the relationship, and when they are neutralized it returns.

VII. CONCLUSIONS

We found that the speed of evolution in mixed states differs from that in pure states in that it is influenced not only

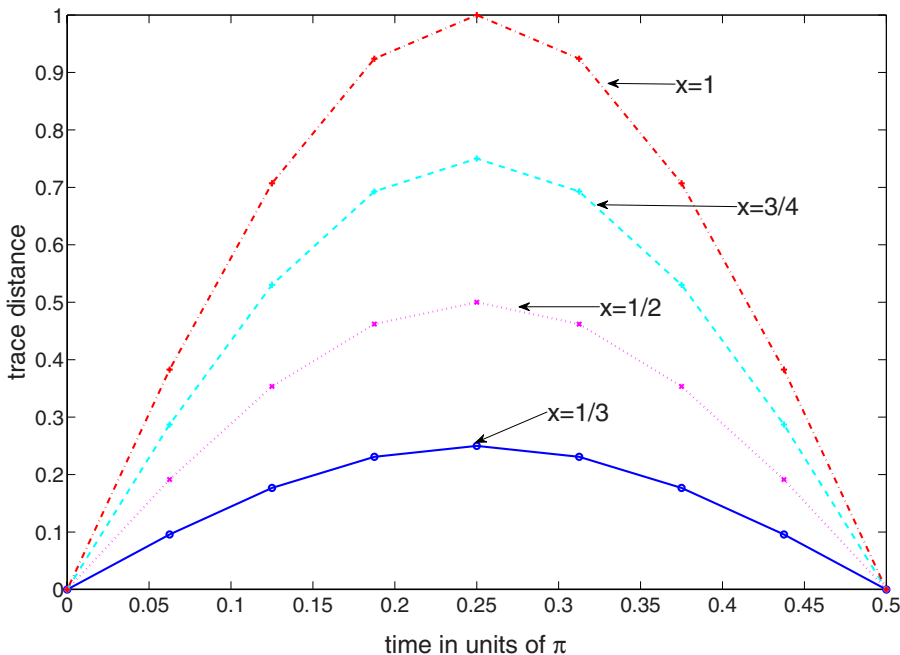


FIG. 7. (Color online) Trace distance as function of time for the three representative matrices. Magnet angles are σ_z and σ_{-z} . With these angles, the graph is the same for all three matrices: ρ_{Werner} , ρ_{Gis} , and ρ_3 . The curves represent mixing values. Lowest (blue), $x = 1/3$, entangled for the Werner state, but not for ρ_{Gis} ; highest (red), $x=1$, maximally entangled for all three matrices and reaches a distance of 1, that is, an orthogonal state. For a product mixture, $x|11\rangle\langle 11| + (1-x)|00\rangle\langle 00|$, the trace distance is zero for all values of x .

by entanglement (which reflects the amount of energy spread) and by external constraints (magnet angle in this model), but also by factors due to the mixing. For the three matrices, the accepted $\Delta E \Delta t$ relationship is preserved only if mixing with product states is neutralized. Mixing introduces classical correlations and entropy, both of which affect evolution. Neither of them alone can account for it, but the relationship between them is not yet clear. This may be because the measure of mutual information includes quantum correlations as well as classical, and the two do not necessarily have the same effect (e.g., maximally mixed classical correlations decrease distance with increased period, quantum correlations do not). Further work might include a measure which excludes quantum correlations and then a functional relationship might be found between the effect of classical correlations and of entropy.

The period can be speeded up by maximally mixed classical correlation, but this reduces the trace distance. When optimized for maximal distance rather than period, such correlations attenuate the possible trace distance and slow the evolution down. In any case it is necessary to clarify which aspect of time evolution—period or maximal distance—is

relevant to the discussion, as optimization for either gives different results.

In sum, time evolution in mixed states appears to be affected by entanglement, classical correlations, entropy, and external constraints. We have attempted to point out the various effects of these factors. These conclusions are a result of numerical methods and of analytic calculations for specific cases. Future work should include an attempt to reach a general analytic expression for the relative contributions of classical and quantum correlations as well as entropy to the speed of evolution.

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