$\mathcal{PT}$-symmetric brachistochrone problem, Lorentz boosts, and nonunitary operator equivalence classes

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The $\mathcal{PT}$-symmetric (PTS) quantum brachistochrone problem is re-analyzed as a quantum system consisting of a non-Hermitian PTS component and a purely Hermitian component simultaneously. Interpreting this specific setup as a subsystem of a larger Hermitian system, we find nonunitary operator equivalence classes (conjugacy classes) as natural ingredients which contain at least one Dirac-Hermitian representative. With the help of a geometric analysis the compatibility of the vanishing passage time solution of a PTS brachistochrone with the Anandan-Aharonov lower bound for passage times of Hermitian brachistochrones is demonstrated.

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I. INTRODUCTION

Non-Hermitian $\mathcal{PT}$-symmetric quantum mechanical (PTQM) models [1,2] with exact $\mathcal{PT}$ symmetry (PTS) and diagonalizable spectral decomposition are known to be equivalent to Hermitian quantum mechanical models [3]. Under the corresponding equivalence transformations non-Hermitian Hamiltonians with differential expressions of local type are in general mapped into strongly nonlocal Hermitian Hamiltonians [4,5], whereas non-Hermitian matrix Hamiltonians are by conjugation mapped into Hermitian matrix Hamiltonians. These equivalence relations led to the natural conclusion [5] that PTQM models with exact PTS are a kind of economical writing of possibly complicated Hermitian QM models—and known properties of Hermitian QM will straightforwardly extend to PTQM in its exact symmetry sector.

In the recent consideration [6] a PTS quantum brachistochrone model has been proposed which indicates on a violation of the strict one-to-one equivalence PTQM$\Leftrightarrow$standard QM. The model is of 2×2 matrix type, mathematically easily tractable and therefore it may serve as a toy model for physical concepts. Here we will show that the violation of the one-to-one equivalence follows from the fact that the $\mathcal{PT}$-symmetric brachistochrone model is built from Hermitian operators and $\mathcal{PT}$-symmetric (and therefore non-Hermitian) operators simultaneously. The model is not reducible to a setup with purely Hermitian operators. Rather the Hermiticity of one component of the model will be connected with the non-Hermiticity of another component, and vice versa. The apparent physical inconsistency can be resolved by considering the model as an effective subsystem of a larger Hermitian system going in this way beyond the one-to-one equivalence assumed, e.g., in [7].

Moreover, we will find a hyperbolic structure underlying the $\mathcal{PT}$-symmetric model connected with the complex orthogonal group $O(2,\mathbb{C})$ and indicating on certain structural analogies of the $\mathcal{PT}$-symmetric brachistochrone with Lorentz boosted spinor systems. In this way it will appear natural to reconsider $\mathcal{PT}$-symmetric models connected by "boosts" as model families and corresponding operators and observables as elements of $O(2,\mathbb{C})$ conjugacy classes. In rough analogy to special relativity we may introduce different reference frames. It turns out that for $\mathcal{PT}$-symmetric models of the type of [6] the conjugacy classes contain at least one Dirac-Hermitian operator. We find that the probabilistic content of models belonging to the same conjugacy class allows for a natural interpretation as frame independence.

The basic subject of the present work is the PTQM brachistochrone problem of [6]. In Sec. II, we analyze the equivalence relations between the representations of the $\mathcal{PT}$-symmetric system in terms of non-Hermitian and Hermitian Hamiltonian. Re-parametrizing the mapping operator between $\mathcal{PT}$-symmetric and Hermitian Hamiltonian we show that it can be re-interpreted as a boost operator of a two-component spinor setup. Using the close structural analogy to representations of relativistic systems in different reference frames and the representation invariance of the probabilistic content of the model we introduce operator equivalence classes. In Sec. III the passage time and probability content of the brachistochrone are analyzed in detail. The underlying geometrical structures of the brachistochrone problem are discussed in Sec. IV in terms of Möbius transformations and deformations of the Fubini-Study metric. They are visualized as mapping between Bloch sphere setups and provide an explanation of the vanishing passage time effect as geometric mapping artifact. The results are summarized in the Conclusions (Sec. V).

II. OPERATOR EQUIVALENCE CLASSES

A. The $\mathcal{PT}$-symmetric brachistochrone

Let us briefly recall the quantum brachistochrone problem as formulated in [6]. Given an initial state $|\psi_i\rangle$ and a final state $|\psi_f\rangle$ of a quantum system the problem consists of obtaining a PTS Hamiltonian H, $[\mathcal{PT},H]=0$ which minimizes the time t needed for the evolution $U(t)=e^{-itH}:|\psi_i\rangle\mapsto|\psi_f\rangle=U(t)|\psi_i\rangle$. In [6] the Hamiltonian H, the parity operator $\mathcal{P}$, and the initial and final states $|\psi_i\rangle$ and $|\psi_f\rangle$ were assumed as

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$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad r,s,\theta \in \mathbb{R}, \quad \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$|\psi_i\rangle = (1,0)^T, \quad |\psi_f\rangle = (0,1)^T. \tag{1}$$

The time inversion operator $\mathcal{T}$ is antilinear and acts in the present model as complex conjugation. The Hamiltonian H has eigenvalues $E_{\pm}=r\cos(\theta) \pm \sqrt{s^2-r^2\sin^2(\theta)}$ so that exact $\mathcal{PT}$ symmetry with Im $E_{\pm}=0$ and diagonalizability of H hold for $s^2 > r^2 \sin^2(\theta)$. Parameter configurations with $\theta=0$ correspond to a purely Hermitian (real symmetric) Hamiltonian, whereas configurations with $s^2=r^2\sin^2(\theta)$ are related to the boundary between exact and spontaneously broken PTS. These latter configurations are characterized by coalescing eigenvalues $E_+=E_-=E_0 := r\cos(\theta)$ and eigenvectors, lost diagonalizability of

$$H \sim \begin{pmatrix} E_0 & 1 \\ 0 & E_0 \end{pmatrix}$$

and correspond to exceptional points [8]. For fixed

$$\omega := E_+ - E_- \tag{2}$$

a Hamiltonian H was found in [6] which led to a vanishing evolution time $t=0$.

B. Non-Hermitian Hamiltonian

As plausibly argued in [7], a vanishing passage time is impossible for a PTQM model which by an equivalence transformation can be one-to-one mapped into a purely Hermitian 2×2 matrix model. The apparent contradiction between the results of [6] and those of [7] can be resolved by noticing that the states $|\psi_i\rangle$ and $|\psi_f\rangle$ can be interpreted as eigenstates of a spin-$\frac{1}{2}$ operator

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z^\dagger$$

which is not $\mathcal{PT}$ symmetric in the representation (1) for $\mathcal{P}$. This means that the starting assumptions of [6] (only H is $\mathcal{PT}$ symmetric) and [7] (all the system is $\mathcal{PT}$ symmetric) are different and therefore the conclusions are different.

Moreover, the approach of [6] implicitly indicates that physical effects beyond the 2×2 Hermitian matrix model can be obtained from systems which comprise Hermitian and $\mathcal{PT}$-symmetric (non-Hermitian) subsystems simultaneously. For this purpose it suffices to interpret the $\mathcal{PT}$-symmetric (non-Hermitian) components as dimensionally reduced (down-projected) components of a larger Hermitian system.

Specifically, for the model [6] the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H in Eq. (1) induces a nonunitary evolution. This nonunitary evolution described by the Schrödinger equation

$$i\partial_t \psi = H\psi \tag{3}$$

can be regarded as effective evolution in the dimensionally reduced (down-projected) subsystem induced by the unitary evolution of the larger closed system. Explicitly, the relation between the down-projected and closed systems is easily demonstrated with the help of a time-independent Hermitian block matrix Hamiltonian $\hat{H}=\hat{H}^\dagger$ of the large system and its Schrödinger equation

$$i\partial_t \hat{\psi} = \hat{H}\hat{\psi} \tag{4}$$

which takes the form

$$i\partial_t \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \tag{5}$$

Here χ denotes the wave function components in the subsystem living in the Hilbert space components complementary to ψ. For time-independent Hamiltonians H and $\hat{H}$ the compatibility of Eqs. (3) and (4) is ensured by a constraint on H and the matrix blocks $A=A^\dagger$, B, $D=D^\dagger$ in the form of an algebraic matrix Riccati equation

$$H^2 - (A + BDB^{-1})H - BB^\dagger + BDB^{-1}A = 0. \tag{6}$$

In general, this constraint is not invariant under Hermitian conjugation so that accordingly H is, in general, non-Hermitian. The effect of the nonunitary evolution of the down projection is easily understood by noticing that vectors $\hat{\psi}$, $\hat{\varphi}$ orthogonal in a large (Hilbert) space $\hat{\mathcal{H}}=\mathcal{H}_1 \oplus \mathcal{H}_2 \ni \hat{\psi}, \hat{\varphi}$ remain orthogonal under unitary evolution in this space. Their down-projected components are, in general, nonorthogonal in the lower-dimensional subspaces

$$\langle \hat{\psi}|\hat{\varphi}\rangle_{\hat{\mathcal{H}}} = (\psi_1,\varphi_1)_{\mathcal{H}_1} + (\psi_2,\varphi_2)_{\mathcal{H}_2} = 0,$$

$$(\psi_1,\varphi_1)_{\mathcal{H}_1} = -(\psi_2,\varphi_2)_{\mathcal{H}_2} \neq 0 \tag{7}$$

and evolve in these subspaces nonunitarily.

Further on, we restrict our attention to the effective down-projected system with non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H whose eigenvalues are purely real (sector of exact $\mathcal{PT}$ symmetry). In contrast to a Hermitian Hamiltonian the eigenvectors of H are in general nonorthogonal in Hilbert space and therefore H is not a von Neumann observable. In this regard it should be noted that in modern quantum theory the concepts of "observable" and "measurement" are understood in a wider sense than in the early times of Bohr, von Neumann, Dirac, etc. In particular, one does not associate Hermiticity with a necessary attribute of an observable [9] anymore. Nonorthogonal vector sets appear naturally after measurements of observables. They are used in constructing nonorthogonal decompositions of the identity operator, so called positive operator valued measures (POVMs), and provide a consistent probabilistic interpretation of the measurement process [9–13]. The corresponding approach is one of the cornerstones of quantum information and computation theory [11–13]. von Neumann observables and their orthogonal projector decompositions of the identity operator are connected with repeatable measurements and provide sharp observables [13], whereas "generalized observables" with nonorthogonal identity decompositions are unsharp (smeared) observables and correspond to nonrepeatable, purely probabilistic measurements [13].

C. Lorentz boost analogy

Let us recall a few basic facts on the $\mathcal{PT}$-symmetric Hamiltonian H in Eq. (1). This Hamiltonian is self-adjoint with regard to the indefinite $\mathcal{PT}$ inner product $\mathcal{PT}|E_k\rangle\cdot|E_l\rangle$ (see [14]) and it is therefore self-adjoint in the Krein space [15] $(\mathcal{K}_\mathcal{P},[.,.]_\mathcal{P})$, $\mathcal{K}_\mathcal{P}\cong\mathbb{C}^2$ with the indefinite metric defined by the parity inversion $\mathcal{P}$ as $[.,.]_\mathcal{P}:=\langle\cdot|\mathcal{P}|\cdot\rangle$. Moreover, there exists a Hermitian operator $\eta=\eta^\dagger>0$ so that

$$\eta H = H^\dagger \eta \tag{8}$$

and hence that H is self-adjoint (quasi-Hermitian in the sense of [16]) in the Hilbert space $(\mathcal{H}_\eta,\langle.,.\rangle_\eta)$, $\mathcal{H}_\eta\cong\mathbb{C}^2$ endowed with η as positive definite metric $\langle.,.\rangle_\eta:=\langle\cdot|\eta|\cdot\rangle$ [17]. Identifying the $\mathcal{CPT}$ inner product (see, e.g., [2]) with this η-defined inner product one finds $\mathcal{CPT}|E_k\rangle\cdot|E_l\rangle=\langle E_k|(\mathcal{CP})^T|E_l\rangle=\langle E_k|\eta|E_l\rangle$ and hence $\eta^T=\mathcal{CP}$. Together with the relation $\eta^{-1}=\mathcal{CP}$ obtained in [18] this implies $\eta^T=\eta^{-1}$. For general $N\times N$ matrix models this means that η is an element of the complex orthogonal group $O(N,\mathbb{C})\ni\eta$ [and $\eta\in SO(N,\mathbb{C})$ in the case of $\det(\eta)=1$] additionally to the Hermiticity $\eta=\eta^\dagger$.

In contrast to Hermitian Hamiltonians, the spectrum of $\mathcal{PT}$-symmetric Hamiltonians may consist of real eigenvalues as well as of complex conjugate eigenvalue pairs. Concerning the brachistochrone problem we restrict our attention to $\mathcal{PT}$-symmetric Hamiltonians with a purely real spectrum. Such Hamiltonians H are known to be in a one-to-one equivalence relation to Hermitian Hamiltonians h [3],

$$\rho: H \mapsto h = \rho H \rho^{-1}, \quad h = h^\dagger, \tag{9}$$

where due to Eq. (8) it holds that $\eta=\rho^\dagger\rho$. Obviously, up to a unitary transformation of h one may set $\rho=\rho^\dagger$ so that

$$\eta = \rho^2 \tag{10}$$

and ρ itself is a complex orthogonal rotation as well: $\rho \in O(N,\mathbb{C})$.

Let us now explicitly apply these considerations to the brachistochrone system of [6]. For this purpose we represent the Hamiltonian (1) as

$$H = a_0 I_2 + s\begin{pmatrix} i\sin(\alpha) & 1 \\ 1 & -i\sin(\alpha)\end{pmatrix},$$

$$\sin(\alpha) := \frac{r}{s}\sin(\theta), \quad a_0 := r\cos(\theta). \tag{11}$$

Its bi-orthogonal non-normalized eigenvectors have the form [8]

$$|E_\pm\rangle = c_\pm \chi_\pm, \quad |\tilde{E}_\pm\rangle = d_\pm^* \chi_\pm^*, \quad c_\pm, d_\pm \in \mathbb{C}^*,$$

$$\chi_\pm := \left(1, -i\sin(\alpha) \pm \sqrt{1-\sin^2(\alpha)}\right)^T \tag{12}$$

and it holds that $\langle\tilde{E}_\mp|E_\pm\rangle=0\ \forall\alpha$, and $\langle\tilde{E}_\pm|E_\pm\rangle=c_\pm d_\pm \chi_\pm^T\chi_\pm\neq 0\ \forall\alpha\neq(N+1/2)\pi$, $N\in\mathbb{Z}$. The values $\alpha=(N+1/2)\pi$ correspond to exceptional points (EPs) of the spectrum [8] and the eigenvectors due to $\chi(\alpha=\pm\pi/2)=(1,\mp i)^T$ become isotropic (self-orthogonal) $\chi_\pm^T\chi_\pm=0$ at these points. In [8] several arguments have been listed which indicate on a strong similarity of these isotropic eigenvectors and the isotropic lightlike vectors well known from special relativity. Here, we take this analogy literally and conjecture the ansatz $\sin(\alpha)=v/c$ so that $\chi_\pm$ contains terms which disappear in the light-cone limit $|v|\to c$ in the typical relativistic way,

$$\chi_\pm = \left(1, -i\frac{v}{c} \pm \sqrt{1-\frac{v^2}{c^2}}\right)^T. \tag{13}$$

On its turn this suggests the usual reparametrization

$$\sin(\alpha) = v/c =: \tanh(\beta) \tag{14}$$

with [see Eqs. (2) and (11)]

$$\cosh(\beta) = \frac{2s}{\omega}. \tag{15}$$

From $\mathcal{P}$ in Eq. (1) and the operator $\mathcal{C}$ (see, e.g., [2]) which encodes the dynamical mapping between the Krein-space $\mathcal{PT}$ inner product and the Hilbert space $\mathcal{CPT}$ inner product we find the explicit representation of the metric

$$\eta = \mathcal{PC} = \frac{1}{\cos(\alpha)}\begin{pmatrix} 1 & -i\sin(\alpha) \\ i\sin(\alpha) & 1\end{pmatrix} = \begin{pmatrix} \cosh(\beta) & -i\sinh(\beta) \\ i\sinh(\beta) & \cosh(\beta)\end{pmatrix} = e^{\beta\sigma_y},$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \det(\eta)=1, \quad \eta\in SO(2,\mathbb{C}) \tag{16}$$

and with Eq. (10) the transformation $\rho\in SO(2,\mathbb{C}): H\mapsto h$,

$$\rho = e^{\beta\sigma_y/2} = \begin{pmatrix} \cosh(\beta/2) & -i\sinh(\beta/2) \\ i\sinh(\beta/2) & \cosh(\beta/2)\end{pmatrix}. \tag{17}$$

Due to $\rho^\dagger\sigma_z\rho=\sigma_z$ the transformation is pseudounitary $\rho\in SU(1,1)$ as well. In terms of the β parametrization the Hamiltonian H can be represented via Eqs. (11), (14), and (15), i.e., via $(r,s,\theta)\mapsto(r,\omega,\beta)$, as

$$H(\beta) = a_0 I_2 + \frac{\omega}{2}\begin{pmatrix} i\sinh(\beta) & \cosh(\beta) \\ \cosh(\beta) & -i\sinh(\beta)\end{pmatrix},$$

$$a_0 = r\cos(\theta) = \sqrt{r^2 - \frac{\omega^2}{4}\sinh^2(\beta)}. \tag{18}$$

Its Hermitian equivalent takes the form

$$h = \rho H \rho^{-1} = a_0 I_2 + \frac{\omega}{2}\sigma_x. \tag{19}$$

The transformation invariant energy offset $a_0 I_2$ produces a general phase factor of the wave function and, for tuned $r=\frac{\omega}{2}\cosh(\beta)$, takes the β-independent value $a_0=\omega/2$. The dynamically relevant nontrivial matrix terms of the Hamiltonians H and h show a certain structural similarity with the chiral components of the Dirac equation in its Weyl representation [19],

$$D_W\Psi \equiv \begin{pmatrix} -m & p_0+\boldsymbol{\sigma}\cdot\mathbf{p} \\ p_0-\boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \varphi_R(\mathbf{p}) \\ \varphi_L(\mathbf{p}) \end{pmatrix} = 0, \quad (20)$$

$$\varphi_{R,L}(\mathbf{p}) := e^{\pm\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2}\varphi_{R,L}(0). \quad (21)$$

Here $\varphi_{R,L}(\mathbf{p})$ denote the chiral right and left two-component spinors of a spin-$\frac{1}{2}$ particle with energy p_0, rest mass m, and momentum $\mathbf{p}$ directed along the unit vector $\mathbf{n}$,

$$p_0 = m\cosh(\phi), \quad \mathbf{p} = \mathbf{n}m\sinh(\phi), \quad (22)$$

$\varphi_R(0)=\varphi_L(0)$ are the corresponding rest frame chiral spinors, and $e^{\pm\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2}=\cosh(\phi/2)I_2\pm\boldsymbol{\sigma}\cdot\mathbf{n}\sinh(\phi/2)$ are the pure boosts relating the spinors in the two frames.

With the help of the rotation

$$V=\frac{1}{\sqrt{2}}(I_2-i\sigma_y)\in \mathrm{SU}(2) \quad (23)$$

we find from Eq. (19)

$$\mathfrak{h} := V^{-1}[h-a_0I_2]V = m\sigma_z, \quad m := \frac{\omega}{2}, \quad (24)$$

and from Eq. (18)

$$\mathfrak{H}(\beta) := V^{-1}[H(\beta)-a_0I_2]V$$
$$= \rho^{-1}\hat{h}\rho = \sigma_z(p_0+\sigma_yp_y) = \sigma_z(p_0+\boldsymbol{\sigma}\cdot\mathbf{p}) \quad (25)$$

with

$$p_0 := m\cosh(\beta), \quad p_y := m\sinh(\beta), \quad (26)$$

$p_x=p_z=0$ and $\mathfrak{H}(-\beta)=\sigma_z(p_0-\boldsymbol{\sigma}\cdot\mathbf{p})$ so that

$$\Sigma_zD_W\psi = \begin{bmatrix} -\sigma_zm & \mathfrak{H}(\beta) \\ \mathfrak{H}(-\beta) & -\sigma_zm \end{bmatrix}\begin{bmatrix} \varphi_R(p_y) \\ \varphi_L(p_y) \end{bmatrix} = 0,$$

$$\Sigma_z := \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = I_2\otimes\sigma_z \quad (27)$$

with $\varphi_{R,L}(p_y)=e^{\pm\beta\sigma_y/2}\varphi_{R,L}(0)$. This means that h and φ can be related via the chiral components $\varphi_R(0)=\varphi_L(0)=V^{-1}\varphi$ to a massive spin-$\frac{1}{2}$ particle (with rest mass $m=\omega/2$) in its rest frame (co-moving frame). In contrast, $H(\beta)$ and ψ can be associated to the same particle observed from a Lorentz boosted frame (laboratory or observer frame) [19]. Energy and momentum are, as usual, related by the mass shell condition $p_0^2-p_y^2=m^2$, which guaranties the compatibility of the system (27). The transformation $\rho(\beta)=e^{\beta\sigma_y/2}$ is then the usual Lorentz boost acting in the two-component spinor representation [20].

D. Operator equivalence classes and their statistical content

The structural analogy of the $\mathcal{PT}$-symmetric matrix system and the chiral components of relativistic particle systems leads to the following natural assumption. Similar as physical observables in relativistic systems can be measured in different reference frames, we can associate PTQM systems represented in terms of Hamiltonians H and h with different physical reference frames. In analogy to relativistic systems where the observables described in different reference frames are related by Lorentz transformations and can be associated to orbit sections of unitary representations of the Lorentz group one can assume for PTQM systems that the corresponding observables in different reference frames are related by equivalence transformations of the complex orthogonal group $\mathrm{SO}(N,\mathbb{C})$. In accordance with Sec. II B, the fact that for the PTQM system the corresponding transformations are not unitary but of $\mathrm{SO}(N,\mathbb{C})$ type can be attributed to the projection of the larger unitary system to its lower dimensional subsystem [21].

For the brachistochrone model [6] we have one frame $\mathcal{F}_1$ associated with the non-Hermitian Hamiltonian H and the Hermitian spin operator S_z. The equivalence transformation ρ maps H into the Hermitian Hamiltonian $h=\rho H\rho^{-1}$ which can be associated with a second reference frame $\mathcal{F}_2$. Simultaneously with H the spin operator S_z maps into

$$S_z \mapsto s_z = \rho S_z\rho^{-1} \quad (28)$$

which due to the nonunitarity of $\rho(\beta\neq 0)$, i.e., $\rho^\dagger=\rho\neq\rho^{-1}$, is non-Hermitian $s_z\neq s_z^\dagger=\rho^{-1}S_z\rho$. Hence in both frames $\mathcal{F}_1$ and $\mathcal{F}_2$ the system is described in terms of Dirac-Hermitian as well as non-Dirac-Hermitian operators. Therefore it cannot be regarded as fundamental in the sense that there exists a frame where all operators are Dirac-Hermitian simultaneously and where they can be treated as von Neumann observables.

In conventional (von Neumann) quantum mechanics, the expectation values of the physical observables as well as the probabilities to observe them are invariant under unitary transformations of the operators and state vectors so that the physics is not affected by the concrete representation. Below we show that the probabilities and expectation values of an observable are independent of the representation used to calculate them. In particular, this means that the properties of a given observable can be calculated in a representation (or "reference frame") where the observable is associated with a Hermitian operator. Therefore the usual quantum mechanical orthogonal measurements may be applied to this observable in the specific frame. Using this property we show that both probabilities and average values can also be calculated in a representation where the observable is described by a non-Hermitian operator ("non-Hermitian" or "observer reference frame"). This leads to a generalization of the notions of both the "statistical operator" describing the state of a quantum object and the projection operators on eigenstates of the observable.

Let a quantum system be in pure state $|\phi\rangle$, $\langle\phi|\phi\rangle=1$, and we wish to measure an observable (energy) described by a Dirac-Hermitian operator $h=h^\dagger$. Then according to the axioms of conventional (von Neumann) quantum mechanics we can detect only eigenvalues E_i of h,

$$h|e_i\rangle = E_i|e_i\rangle, \quad \langle e_i|e_j\rangle = \delta_{ij} \quad (29)$$

(for simplicity we assume h acting in a finite-dimensional Hilbert space) with the probabilities

$$p_i = |\langle e_i|\phi\rangle|^2 = \mathrm{Tr}(P_i\varrho), \quad P_i = |e_i\rangle\langle e_i|, \quad \varrho = |\phi\rangle\langle\phi|. \tag{30}$$

If the value E_k appeared as a measurement result then after the measurement the state of the system is described by the state vector (up to an unessential normalization factor) $|e_k\rangle$. If now we change the representation ("reference frame") using a nonunitary nonsingular similarity transformation [see Eq. (28)]

$$|\phi\rangle = \rho|\psi\rangle = \rho^{-1}|\tilde{\psi}\rangle, \quad |e_i\rangle = \rho|E_i\rangle = \rho^{-1}|\tilde{E}_i\rangle,$$

$$h = \rho H \rho^{-1} = \rho^{-1}H^\dagger \rho \tag{31}$$

with $|E_i\rangle$ and $|\tilde{E}_i\rangle = \rho^2|E_i\rangle$ being eigenvectors of H and $H^\dagger$, respectively,

$$H|E_i\rangle = E_i|E_i\rangle, \quad H|\tilde{E}_i\rangle = E_i|\tilde{E}_i\rangle \tag{32}$$

then

$$p_i = \langle\tilde{\psi}|E_i\rangle\langle\tilde{E}_i|\psi\rangle = \mathrm{Tr}(\Pi_i \Upsilon) = \langle\psi|\tilde{E}_i\rangle\langle E_i|\tilde{\psi}\rangle = \mathrm{Tr}(\Upsilon^\dagger \Pi_i^\dagger), \tag{33}$$

where

$$\Pi_i = |E_i\rangle\langle\tilde{E}_i|, \quad \Upsilon = |\psi\rangle\langle\tilde{\psi}|. \tag{34}$$

The expectation value of the energy can be expressed in terms of $|E_i\rangle$, $|\tilde{E}_i\rangle$ and $|\psi\rangle$, $|\tilde{\psi}\rangle$ as well,

$$\begin{aligned}\langle E\rangle &= \langle\phi|h|\phi\rangle = \mathrm{Tr}(h\varrho)\\ &= \sum_{i=1}^{N} E_i\langle\tilde{\psi}|E_i\rangle\langle\tilde{E}_i|\psi\rangle\\ &= \langle\tilde{\psi}|H\psi\rangle = \mathrm{Tr}(H\Upsilon)\\ &= \langle\psi|H^\dagger\tilde{\psi}\rangle = \mathrm{Tr}(\Upsilon^\dagger H^\dagger).\end{aligned} \tag{35}$$

Thus the operators Υ, $\Upsilon^\dagger$ play the role of statistical operators for a pure state associated with the vector $|\psi\rangle$, whereas Π_k, $\Pi_k^\dagger$ describe the observable corresponding to the Hamiltonian (energy) of the system in these "non-Hermitian frames." The state with definite value of the energy is described either by the quasiprojectors $\Pi_i = |E_i\rangle\langle\tilde{E}_i|$ or by their adjoints $\Pi_i^\dagger$.

If the outcome E_k is detected as a measurement result then the state of the system after the measurement (up to a normalization factor) in the Hermitian frame is described by the vector $P_k|\psi\rangle \sim |e_k\rangle$ and in non-Hermitian frames by either $|E_k\rangle = \rho^{-1}|e_k\rangle$ or $|\tilde{E}_k\rangle = \rho|e_k\rangle$ with statistical operators $\Upsilon_k = |E_k\rangle\langle\tilde{E}_k|$ and $\Upsilon_k^\dagger$.

Unitary equivalent classes, where the same physical observables are represented by different operators related with unitary transformations, are, evidently, subclasses of these more general equivalence transformations.

Our final comment here is that the probabilities (33) are intimately related with experiments on unambiguous state discrimination which, in turn, are based on generalized observables, POVM, and Naimark's dilation (extension) theorem. Moreover, quasiprojectors (34) appear in a natural way when an observable related with a specific symmetry operator in an extended space is measured [22].

III. SPIN FLIPS UNDER A NON-HERMITIAN EVOLUTION

Let us illustrate this scheme by re-analyzing the $\mathcal{PT}$-symmetric quantum brachistochrone problem of [6] as pseudounitary evolution (spin-flip) problem of a Hermitian spin-$\frac{1}{2}$ observable. According to the equivalence relations found in Sec. II D there are two equivalent ways to calculate the spin-flip probabilities. One may either consider the pair S_z, H with $S_z := \sigma_z = S_z^\dagger$, $H \neq H^\dagger$ and find the pseudounitary evolution operator [23] $U(t) = e^{-itH}$ acting on the spin eigenstates $|\uparrow\rangle$, $|\downarrow\rangle$ of S_z. Or, alternatively, one may consider the equivalent pair s_z, h consisting of a non-Hermitian operator $s_z \neq s_z^\dagger$ whose eigenstates undergo a unitary evolution $u(t) = e^{-ith}$ governed by $h = h^\dagger$. We choose the first way of calculation (following [6]) and obtain $U(t)$ via exponentiation of the $\mathcal{PT}$-symmetric 2×2-matrix Hamiltonian (1) as

$$\begin{aligned}U(t) &= e^{-iHt} = \sum_k e^{-iE_k t}|\psi_k(0)\rangle\langle\psi_k(0)|\\ &= \frac{e^{-irt\cos\theta}}{\cos\alpha}\begin{pmatrix}\cos\left(\frac{\omega t}{2} - \alpha\right) & -i\sin\left(\frac{\omega t}{2}\right)\\ -i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2} + \alpha\right)\end{pmatrix} \neq U^\dagger(t),\end{aligned} \tag{36}$$

where $\sin\alpha := \frac{r}{s}\sin\theta$ and $\omega := 2s|\cos\alpha| = \Delta E$ is the difference of eigenvalues of H. Applying $U(t)$ to the initial spin-up state, $|\uparrow\rangle$, we reproduce the previously reported result of [6]

$$|\psi(t)\rangle = \frac{e^{-irt\cos\theta}}{\cos\alpha}\begin{pmatrix}\cos\left(\frac{\omega t}{2} - \alpha\right)\\ -i\sin\frac{\omega t}{2}\end{pmatrix}. \tag{37}$$

The probabilities to find the spin either up or down at any time moment $t > 0$ for a system being in the state (37) are calculated using the usual quantum mechanical prescriptions in the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ (measurement of $S = \sigma_z$)

$$p_\uparrow(t) = \frac{\langle\psi(t)|\uparrow\rangle\langle\uparrow|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle},$$

$$p_\downarrow(t) = \frac{\langle\psi(t)|\downarrow\rangle\langle\downarrow|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} \tag{38}$$

and give in the present case

$$p_\uparrow = \frac{\cos^2(\omega t/2 - \alpha)}{\cos^2(\omega t/2 - \alpha) + \sin^2(\omega t/2)},$$

$$p_\downarrow = \frac{\sin^2(\omega t/2)}{\cos^2(\omega t/2 - \alpha) + \sin^2(\omega t/2)}. \tag{39}$$

From here we find the time intervals

$$\Delta t_{\uparrow\to\downarrow} = \frac{\pi + 2\alpha}{\Delta E}, \quad \Delta t_{\downarrow\to\uparrow} = \frac{\pi - 2\alpha}{\Delta E} \tag{40}$$

necessary for the first spin flips from up to down and back, respectively. For all values $\alpha \in [-\pi/2, 0)$ the evolution time lies below the Anandan-Aharonov lower bound $\Delta t_{\uparrow\to\downarrow} \leqslant \frac{\pi}{\Delta E} =: \Delta_{AA}$ for a spin-flip evolution in a Hermitian system [24]. In the special case $\alpha \to -\pi/2$ with ΔE fixed the zero-passage time result $\Delta t_{\uparrow\to\downarrow}(\alpha \to -\pi/2) \to 0$ from [6] is reproduced. In [8] this regime has been related to an exceptional point of the spectrum of H where its two eigenvectors coalesce so that the Hilbert space distance between them vanishes. Subsequently we show that in the equivalent system with Hermitian $h = h^\dagger$ the originally orthogonal $|\psi_i\rangle$, $|\psi_f\rangle \in \mathcal{H}$ in the Hilbert space $\mathcal{H}$ are mapped into nearly coalescing $|\varphi_i\rangle, |\varphi_f\rangle \in \tilde{\mathcal{H}}$ so that $\Delta t_{\uparrow\to\downarrow}(\alpha \to -\pi/2) \to 0$ is not connected with a violation of the Anandan-Aharonov lower bound $\Delta t_{\uparrow\to\downarrow} \geqslant \frac{\pi}{\Delta E}$. Rather it can be attributed to changes in the Hilbert space metric induced by the mapping $\rho: \mathcal{H} \to \tilde{\mathcal{H}}$.

Before we turn to the corresponding geometrical considerations three comments are in order:

1. The total time for a spin flip followed by a flip back $\Delta t_{\uparrow\to\downarrow\to\uparrow}$ remains invariant $\Delta t_{\uparrow\to\downarrow\to\uparrow} = \frac{2\pi}{\Delta E}$ independently of the non-Hermiticity parameter α—a result obtained recently also in [25].

2. With regard to the concept of conjugacy orbits (operator conjugacy classes) the general solution technique for a PTS brachistochrone problem will comprise the following steps. (i) Given a family of diagonalizable PTS Hamiltonians H one finds the transformations ρ which render them Hermitian in the Hilbert space $\tilde{\mathcal{H}}$. (ii) Initial and final state $|\psi_{i,f}\rangle \in \mathcal{H}$ are to be mapped into $|\varphi_{i,f}\rangle \in \tilde{\mathcal{H}}$. (iii) With these vectors as initial and final states the brachistochrone problem is solved along standard techniques for Hermitian Hamiltonians [26] singling out a specific Hermitian Hamiltonian h_b. (iv) Its non-Hermitian representative H_b from the same conjugacy orbit is the solution of the PTS brachistochrone problem.

3. Concerning the implementation of the brachistochrone [6] and the state preparation–state discrimination we note the following aspects. The basic background of the interpretation is connected with the initial and final states of the large Hermitian system. The preparation of the initial state and the probing of the final state of the brachistochrone subsystem can then be accomplished by simple projection measurements. In the case that the down projection from the large system to the subsystem is performed in such a way that a (H, S_z) type subsystem is obtained then the initial and final states of the subsystem can be associated with the usual S_z spin-projection measurements in this subsystem. If instead the down projection is chosen to result in an (h, s_z) type subsystem then the nonorthogonal initial $|\varphi_I\rangle$ and final $|\varphi_F\rangle$ subsystem states cannot be reinterpreted as eigenstates of a single effective von Neumann observable in this subsystem. Instead, preparation and probing (discrimination) of $|\varphi_I\rangle$ and $|\varphi_F\rangle$ should remain directly related with the initial and final states of the large Hermitian system. (A detailed discussion of these points will be presented in Ref. [22].) Nevertheless, effectively $|\varphi_I\rangle$ and $|\varphi_F\rangle$ can be associated with POVM components living in the subsystem Hilbert space. In summary, the physical mutual consistency of the two representations (H, S_z) and (h, s_z) is guaranteed by well defined relations between different measurement procedures which can be performed on the same large Hermitian system.

IV. GEOMETRY OF THE $\mathcal{PT}$-SYMMETRIC BRACHISTOCHRONE

The origin of the zero-passage time solution of the $\mathcal{PT}$-symmetric brachistochrone problem is easily understood by studying the geometric properties of the η-related mapping ρ and its action on the projective Hilbert (state) space of the model $\mathbb{CP}^1 \cong (\mathbb{C}^2 - \{0\})/\mathbb{C}^* \cong \hat{\mathbb{C}}$. Here, $\mathbb{C}^* := \mathbb{C} - \{0\}$, and $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane. We briefly discuss these properties globally in terms of (linear fractional) Möbius transformations of the extended complex plane $\hat{\mathbb{C}} \ni z$, in terms of the deformation mapping of the $\mathbb{CP}^1$-related Bloch sphere, as well as locally in terms of the Fubini-Study metric.

An arbitrary state vector $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2$ can be represented as [27]

$$|\psi\rangle = \cos(\theta)|0\rangle + e^{i\varphi}\sin(\theta)|1\rangle = \begin{pmatrix} \cos(\theta) \\ e^{i\varphi}\sin(\theta) \end{pmatrix} \cong \begin{pmatrix} 1 \\ z \end{pmatrix}\mathbb{C} \tag{41}$$

with $z = e^{i\varphi}\tan(\theta) \in \hat{\mathbb{C}}$ as a coordinate of the extended complex plane $\hat{\mathbb{C}}$. A linear transformation

$$S: |\psi\rangle \mapsto |\psi'\rangle = S|\psi\rangle, \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{C}) \tag{42}$$

acts then as linear fractional (Möbius) transformation $M(2) \cong PSL(2, \mathbb{C})$ [28] [automorphism $\mathrm{Aut}(\hat{\mathbb{C}})$] on $z \in \hat{\mathbb{C}}$

$$S: z \mapsto z' = f(z) := \frac{Dz + C}{Bz + A}. \tag{43}$$

Apart from the their decomposition properties (translation, rotation, dilation, inversion) Möbius transformations are classified by their type and fixed points $z = f(z)$. For $S \in SL(2, \mathbb{C})$ the type is given by $T(S) := [\mathrm{tr}(S)]^2$ as $T = 4$—parabolic, $T \in [0, 4)$—elliptic, $T \in (4, \infty)$—hyperbolic and $\mathbb{C} \ni T \notin [0, 4]$—loxodromic [29]. For the similarity transformation ρ in Eq. (17) it holds that $T(\rho) = 4\cosh^2(\beta/2)$ so that for $\beta \neq 0$ it is hyperbolic and in the trivial case $\beta = 0$, $\rho = I_2$—parabolic. All nontrivial transformations $\rho(\beta)$ have the same pair of fixed points $z_e \equiv z_\pm = \pm i$ independently of the value of $\beta \neq 0$. Comparison with Eq. (13) shows that the fixed point states $|I_\pm\rangle := (1, \pm i)^T \mathbb{C}^*$ correspond to the eigenvectors at the exceptional points of $H(\alpha = \mp \frac{\pi}{2} + 2N\pi)$, $N \in \mathbb{Z}$. A point $z = z_\varepsilon + \Delta$, $|\Delta| \ll 1$ close to a fixed point maps as

$$z \mapsto z' = f(z_\varepsilon + \Delta) \approx f(z_\varepsilon) + f'(z_\varepsilon)\Delta =: z_\varepsilon + \Delta' \tag{44}$$

so that from $f'(z_\varepsilon) = \exp(-\varepsilon\beta)$ one finds a distance dilation $f' > 1$ for $\varepsilon\beta < 0$ and a contraction $f' < 1$ for $\varepsilon\beta > 0$. Hence for $\beta>0$ ($\beta<0$) the fixed point z_+ (z_-) acts as an attractor and z_- (z_+) as a repellor (see, e.g., [30]).

Closely related to the distances on $\hat{\mathbb{C}} \ni z$ is the Fubini-Study metric (see, e.g., [31]) on $\mathcal{P}(\mathcal{H})=\mathbb{CP}^1$. In terms of the affine coordinate z this metric reads

$$ds^2 = \frac{2dzdz^*}{(1+|z|^2)^2} := g(z,z^*)dzdz^*. \tag{45}$$

Under the mapping $\rho:\mathcal{P}(\mathcal{H})\to\mathcal{P}(\tilde{\mathcal{H}})$ it transforms into $d\tilde{s}^2 = g(z',z'^*)dz'dz'^*$ with z' given by Eq. (43) and $S=\rho$. In terms of the original affine coordinate z it takes the form

$$d\tilde{s}^2 = \frac{2dzdz^*}{[\cosh(\beta)(1+|z|^2)+i\sinh(\beta)(z^*-z)]^2} \tag{46}$$

and coincides with the η—deformed Fubini-Study metric of the Hilbert space $(\mathcal{H}_\eta,\langle\cdot|\cdot\rangle_\eta)$ [see Eq. (A5) in the Appendix]. For the fixed point vicinities with $z\approx z_\varepsilon=\varepsilon i$ the metric (46) reproduces (already in zeroth-order approximation) the typical contraction or dilation (attractor or repellor) behavior $d\tilde{s}^2\approx 2e^{-\varepsilon\beta}dzdz^*$ found via Eq. (44).

A next piece of information can be gained by considering the mapping ρ globally as automorphism of the Bloch sphere. The Bloch sphere representation of a quantum state ψ is given by the correspondence $\psi\in\mathbb{C}^2\to\mathbb{CP}^1\cong S^2\subset\mathbb{R}^3$ which for a state parametrization (41) has the form [32]

$$x=\sin(2\theta)\cos(\varphi),$$

$$y=\sin(2\theta)\sin(\varphi),$$

$$z=\cos(2\theta). \tag{47}$$

We use this representation together with the projective mapping of an arbitrary non-normalized state vector $\psi\in\mathbb{C}^2$

$$\psi=\begin{pmatrix}a\\b\end{pmatrix}\cong\begin{pmatrix}\cos(\theta)\\e^{i\varphi}\sin(\theta)\end{pmatrix}\mathbb{C}^*, \tag{48}$$

and the easily derived relations

$$\varphi=\arg(b)-\arg(a),$$

$$\cos(2\theta)=\frac{|a|^2-|b|^2}{|a|^2+|b|^2},$$

$$\sin(2\theta)=\frac{2|a||b|}{|a|^2+|b|^2} \tag{49}$$

to analyze the ρ-induced transformations graphically. The corresponding plots in Fig. 1 demonstrate the global deformations induced by ρ. Clearly visible is the relative position of the states in the Hilbert spaces. In the space $(\mathcal{H},\langle\cdot|\cdot\rangle)$ the eigenstates (eigenvectors) $|E_\pm\rangle$ of the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H are nonorthogonal (nonantipodal), whereas the initial and final eigenstates $|\psi_i\rangle,|\psi_f\rangle$ of the Hermitian spin operator σ_z are orthogonal (antipodal). The mapping $\rho:\mathcal{H}\to\tilde{\mathcal{H}}$ acts in such a way that it transforms $|E_\pm\rangle$ into the states $|e_\pm\rangle$ which are orthogonal in $(\tilde{\mathcal{H}},\langle\cdot|\cdot\rangle)$. Simultaneously, it transforms $|\psi_i\rangle,|\psi_f\rangle$ into the nonorthogonal $|\varphi_i\rangle=\rho|\psi_i\rangle$ and $|\varphi_f\rangle=\rho|\psi_f\rangle$ dilating or contracting in this way the distance $\mathrm{dist}_{\mathcal{H}}(|\psi_i\rangle,|\psi_f\rangle)=\pi$ into $\mathrm{dist}_{\tilde{\mathcal{H}}}(|\varphi_i\rangle,|\varphi_f\rangle)\gtrless\pi$. The antipodal fixed point states $|I_\pm\rangle$ remain invariant under ρ. The states $|E_\pm\rangle$ are located on a big circle passing through the fixed points $|I_\pm\rangle$, and $|\psi_i\rangle,|\psi_f\rangle$ on another $\pi/2$-rotated big circle through $|I_\pm\rangle$. Under the transformation ρ all but the fixed point states are moved along these big circles away from the repellor fixed point and toward the attractor fixed point.

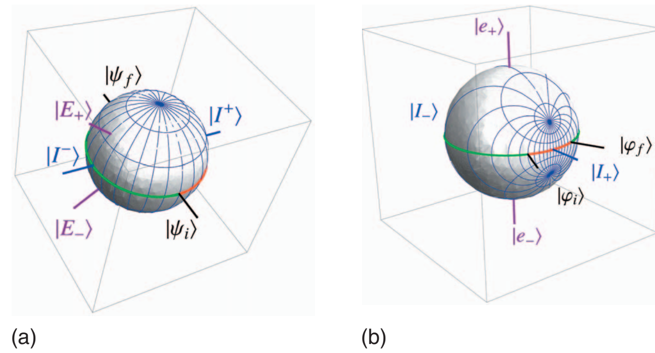

FIG. 1. (Color) The transformation ρ maps the initial and final states $|\psi_{i,f}\rangle\in\mathcal{H}$ as well as the energy eigenstates $|E_\pm\rangle\in\mathcal{H}$ into $|\varphi_{i,f}\rangle,|e_\pm\rangle\in\tilde{\mathcal{H}}$, respectively, and leaves the EP-related fixed point states $|I_\pm\rangle$ invariant. The contraction or dilation properties of the evolution paths (highlighted red or green curves) are defined by their location relative to the originally nonorthogonal energy eigenstates $|E_\pm\rangle$.

In $\tilde{\mathcal{H}}$ the evolution between the states $|\varphi_i\rangle=\rho|\psi_i\rangle$ and $|\varphi_f\rangle=\rho|\psi_f\rangle$ is governed by the unitary transformation $u(t)=e^{-ith}$ with Hermitian Hamiltonian h. This unitary transformation corresponds to the usual rigid rotation of the Bloch sphere [26] (elliptic type Möbius transformation) with the two mapped energy eigenstates $|e_\pm\rangle=\rho|E_\pm\rangle$ as antipodal transformation fixed points. In [8] it has been shown that the vanishing-passage-time solution of the brachistochrone problem of [6] corresponds to an EP limit with coalescing energy eigenstates $|E_+\rangle\to|E_-\rangle$. The mapping ρ "orthogonalizes" them into $|e_\pm\rangle$ but simultaneously transforms the orthogonal $|\psi_{i,f}\rangle$ into coalescing $|\varphi_i\rangle\to|\varphi_f\rangle$ and induces a corresponding vanishing distance $\mathrm{dist}_{\tilde{\mathcal{H}}}(|\varphi_i\rangle,|\varphi_f\rangle)\to 0$. The evolution type is not affected by this equivalence, i.e., the transformation $U(t):\mathcal{H}\to\mathcal{H}$ remains pseudounitary with regard to $(\mathcal{H}_\eta,\langle\cdot|\cdot\rangle_\eta)$ and $u(t):\tilde{\mathcal{H}}\to\tilde{\mathcal{H}}$ unitary with regard to $\tilde{\mathcal{H}}$. For $u(t):|\varphi_i\rangle\mapsto|\varphi_f\rangle$ the Anandan-Aharonov lower bound [24] on the passage time remains valid.

Finally, we note that the hyperbolic type Möbius transformation on the Bloch sphere with its two transformation fixed points and the distance contraction and dilation mechanism is a generic projective transformation which in relativistic physical systems induces the well known aberration effect [33] of shifting the positions of far stars toward the direction of motion of the relativistically moving observer.

V. CONCLUDING REMARKS

In the present paper we have interpreted the $\mathcal{PT}$-symmetric brachistochrone setup of [6] as a quantum system consisting of a non-Hermitian $\mathcal{PT}$-symmetric component and a Hermitian component simultaneously. This interpretation allowed us to formulate a general recipe for the construction of partially $\mathcal{PT}$-symmetric quantum systems which are not 1:1 equivalent to purely Hermitian systems. Using a strong structural analogy with the reference frames for inertial observers in special relativity we associated $\mathcal{PT}$-symmetric models in different representations with corresponding measurement frames. We showed that operators which are Dirac Hermitian are connected with non-Dirac-Hermitian operators in another frame. The probabilistic content of the models is frame independent. With the help of a geometric analysis of the equivalence mapping between mutually $\mathcal{PT}$ symmetric and Hermitian operators the compatibility of the vanishing passage-time solution with the Anandan-Aharonov lower bound [24] for Hermitian system has been demonstrated.

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APPENDIX: η-DEFORMED FUBINI-STUDY METRIC

The Fubini-Study metric [31] on a standard QM-related projective Hilbert space $\mathbb{P}(\tilde{\mathcal{H}})=\mathbb{CP}^N$ is given in terms of state vectors $|\varphi\rangle \in \tilde{\mathcal{H}}=\mathbb{C}^{N+1}$ as

$$d\tilde{s}^2 = 2\frac{\langle\varphi|\varphi\rangle\langle d\varphi|d\varphi\rangle - \langle d\varphi|\varphi\rangle\langle\varphi|d\varphi\rangle}{\langle\varphi|\varphi\rangle^2}. \tag{A1}$$

When the states $|\varphi\rangle$ are the result of a linear invertible mapping $\rho:|\psi\rangle\mapsto|\varphi\rangle=\rho|\psi\rangle$ with $\rho^\dagger\rho=\eta$ then for $|\psi\rangle\in\mathcal{H}=\mathbb{C}^{N+1}$ the metric (A1) becomes "η-deformed" [see also [7], but note an erroneous numerator in Eq. (8) of that paper]

$$d\tilde{s}^2 = 2\frac{\langle\psi|\eta|\psi\rangle\langle d\psi|\eta|d\psi\rangle - \langle d\psi|\eta|\psi\rangle\langle\psi|\eta|d\psi\rangle}{\langle\psi|\eta|\psi\rangle^2}. \tag{A2}$$

For the affine chart $U_0 \ni |\psi\rangle=(1,z_1,\ldots,z_N)^T=:(1,z)^T$ of the projective space $\mathbb{CP}^N\supset U_0$ one sets for convenience

$$\eta=\begin{pmatrix} a & c^\dagger \\ c & D\end{pmatrix}, \quad a\in\mathbb{R}, \quad c\in\mathbb{C}^N, \quad D\in\mathbb{C}^{N\times N} \tag{A3}$$

and finds (due to $\eta=\eta^\dagger$, $D=D^\dagger$)

$$d\tilde{s}^2 = 2\frac{dz^\dagger[qD-(c+Dz)\otimes(c+Dz)^\dagger]dz}{q^2},$$

$$q := a + c^\dagger z + z^\dagger c + z^\dagger D z \in \mathbb{R}. \tag{A4}$$

In the case of $\det(\eta)=1$ and $|\psi\rangle\in\mathbb{CP}^1$ this reduces via $D\in\mathbb{R}$ to

$$d\tilde{s}^2 = \frac{2dz^* dz}{[a+c^*z+cz^*+D|z|^2]^2}. \tag{A5}$$

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