Quantum criticality as a resource for quantum estimation

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We address quantum critical systems as a resource in quantum estimation and derive the ultimate quantum limits to the precision of any estimator of the coupling parameters. In particular, if L denotes the size of a system and λ is the relevant coupling parameters driving a quantum phase transition, we show that a precision improvement of order $1/L$ may be achieved in the estimation of λ at the critical point compared to the noncritical case. We show that analog results hold for temperature estimation in classical phase transitions. Results are illustrated by means of a specific example involving a fermion tight-binding model with pair creation (BCS model).

I. INTRODUCTION

It is often the case that a quantity of interest is not directly accessible, either in principle or due to experimental impediments. In these situations one should resort to indirect measurements, inferring the value of the quantity of interest by inspecting a given probe. This is basically a parameter estimation problem whose solution may be found using tools from classical estimation theory [1] or, when quantum systems are involved, from its quantum counterpart [2]. Indeed, quantum estimation theory has been successfully applied to find optimal measurements and, in turn, to evaluate the corresponding lower bounds on precision, for the estimation of parameters imposed by both unitary and nonunitary transformations. These include single-mode phase-shift [3–5], displacement [6], squeezing [7,8], as well as depolarizing [9] or amplitude-damping [10] channels in a finite-dimensional system, lossy channel in infinite-dimensional ones [11,12], and the position of a single photon [13]. Here we focus on the estimation of the parameters driving the dynamics of an interacting many-body quantum system, i.e., the coupling constants defining the system's Hamiltonian and temperature. It is a generic fact that when those are changed the system is driven into different phases. When this happens at zero temperature one says that a *quantum phase transition* (QPT) has occurred [14]. Different phases of a system are, by definition, characterized by radically different physical properties, e.g., the expectation value of some distinguished observable (order parameter). This in turn implies that the corresponding quantum states have to be statistically distinguishable more effectively than states belonging to the same phase. It is a non trivial result that the degree of statistical distinguishability is quantified by the Hilbert-space (pure states) distance [15] or by the density operator distance (mixed states) [16]. This amount to saying that the Hilbert-space geometry is an *information-space geometry* [17–19]. One is then naturally led to consider the distance functions between infinitesimally close quantum states obtained by an infinitesimal change of the parameters defining the system's Hamiltonian. Boundaries between different phases are then tentatively identified with the set of points where this small change of parameters gives rise to a major change in the distance, i.e., major enhancement of the statistical distinguishability. This is precisely the strategy advocated in [20–25] in the so-called metric (or fidelity) approach to critical phenomena.

The purpose of this paper is to establish the following result: quantum criticality represents a resource for the quantum estimation of Hamiltonian parameters. Indeed, by exploiting the geometrical theory of quantum estimation, we will show that the accuracy of the estimation of coupling constants and field strengths at the critical points is greatly enhanced with respect to the noncritical ones. We will also discuss ultimate limits imposed by quantum mechanics to this scheme of parameter estimation. These results are, under several points of view, an analog of those showing that entanglement is a useful metrological resource [26–28]. In particular, if L denotes the size of a system and λ is the relevant coupling parameters driving a quantum phase transition, we will show that a relative improvement of order L may be achieved in the estimation of λ at the critical point.

The paper is structured as follows. In Sec. II we provide a brief introduction to the geometrical theory of quantum estimation, whereas in Sec. III we review the metric approach to quantum criticality. In Sec. IV we show how quantum critical systems represent a resource for quantum estimation and derive general results about ultimate quantum limits to precision. In Sec. V we illustrate properties of the optimal measurements, also for systems at finite temperature, and the connection with the optimal measurement for state discrimination. In Sec. VI we illustrate our general results by means of a specific example involving a fermion tight-binding model with pair creation. Section VII closes the paper with some concluding remarks.

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II. GEOMETRY OF QUANTUM ESTIMATION THEORY

The solution of a parameter estimation problem amounts to finding an estimator, i.e, a mapping $\hat{\lambda} = \hat{\lambda}(x_1, x_2, \ldots)$ from the set χ of measurement outcomes into the space of parameters. Optimal estimators in classical estimation theory are those saturating the Cramer-Rao inequality,

$$V_\lambda[\hat{\lambda}] \geq F^{-1}(\lambda),$$

which poses a lower bound on the mean square error $V_\lambda[\hat{\lambda}]_{jk} = E_\lambda[(\hat{\lambda} - \lambda)_j (\hat{\lambda} - \lambda)_k]$ in terms of the Fisher information

$$F(\lambda) = \int_\chi d\hat{\lambda}(x) p(\hat{\lambda}|\lambda)[\partial_\lambda \ln p(\hat{\lambda}|\lambda)]^2.$$

Of course for unbiased estimators, as those we will deal with, the mean square error is equal to the covariance matrix $V_\lambda[\hat{\lambda}]_{jk} = E_\lambda[\hat{\lambda}_j \hat{\lambda}_k] - E_\lambda[\hat{\lambda}_j] E_\lambda[\hat{\lambda}_k]$.

When quantum systems are involved any estimation problem may be stated by considering a family of quantum states $\varrho(\lambda)$ which are defined on a given Hilbert space $\mathcal{H}$ and labeled by a parameter λ living on a d-dimensional manifold $\mathcal{M}$, with the mapping $\lambda \mapsto \varrho(\lambda)$ providing a coordinate system. This is sometimes referred to as a quantum statistical model. In turn, a quantum estimator $\hat{\lambda}$ for λ is a self-adjoint operator, which describes a quantum measurement followed by any classical data processing performed on the outcomes. As in the classical case, the goal of a quantum inference process is to find the optimal estimator. The ultimate precision attainable by quantum measurements in inferring the value of a parameter or a set of parameters is expressed by the quantum Cramer-Rao (QCR) theorem [16,29] which sets a lower bound for the mean square error of the quantum Fisher information. The QCR bound is independent on the measurement and very much based on the geometrical structure of the set of the involved quantum states. The *symmetric logarithmic derivative* $\mathcal{L}(\lambda)$ (SLD) is implicitly defined as the (set) Hermitian operator(s) satisfying the equation(s)

$$\partial_\mu \varrho(\lambda) = \frac{1}{2}\{\varrho(\lambda)\mathcal{L}_\mu(\lambda) + \mathcal{L}_\mu(\lambda)\varrho(\lambda)\},$$

where $\partial_\mu := \partial/\partial_\mu$, $(\mu = 1, \ldots, d)$. From the above equation, when $\varrho_j + \varrho_k > 0$, one obtains

$$\langle\varphi_j|\mathcal{L}_\mu(\lambda)|\varphi_k\rangle = 2\langle\varphi_j|\partial_\mu\varrho(\lambda)|\varphi_k\rangle/(\varrho_j + \varrho_k),$$

where we have used the spectral resolution $\varrho(\lambda) = \Sigma_k \varrho_k |\varphi_k\rangle\langle\varphi_k|$. The quantum Fisher information $\mathcal{H}(\lambda)$ (QFI) is a matrix defined as follows:

$$\mathcal{H}_{\mu\nu}(\lambda) = \text{Tr}\left[\varrho(\lambda)\frac{1}{2}\{\mathcal{L}_\mu(\lambda)\mathcal{L}_\nu(\lambda) + \mathcal{L}_\nu(\lambda)\mathcal{L}_\mu(\lambda)\}\right].$$

The QFI is symmetric, real, and positive semidefinite, i.e., represents a metric for the manifold underlying the quantum statistical model [17,18]. The QCR theorem states that the mean square error of any quantum estimator is bounded by the inverse of the quantum Fisher information. In formula

$$V_\lambda[\hat{\lambda}] \geq \mathcal{H}^{-1}(\lambda). \tag{1}$$

QCR is an ultimate bound: it does depend on the geometrical structure of the quantum statistical model and does not depend on the measurement. Notice that due to noncommutativity of quantum mechanics the bound may be not attainable, as it is the case for several multiparameter models. When one has at disposal M identical copies of the state ϱ_λ the QCR bound reads $V_\lambda[\hat{\lambda}] \geq 1/M\mathcal{H}(\lambda)$, which is easily derived upon exploiting the additivity of the quantum Fisher information. A relevant remark [16,29] is that the SLD itself represents an optimal measurement and the corresponding Fisher information is equal to the QFI.

III. GEOMETRY OF QUANTUM PHASE TRANSITIONS

Now we turn to illustrating the achievements of the metric approach to QPTs that are relevant to the problem investigated in this paper. The crucial point is that this approach, being based on the state-space geometry is universally applicable to any statistical system and does not require any preliminary understanding of the structure of the different phases, e.g., symmetry breaking patterns and order parameters. In principle, not even the system's Hamiltonian has to be known as long as the relevant states are (see, e.g., the analysis of matrix-product state QPTs [24]). Of course the main conceptual as well as technical obstacle one has to overcome in order to get this simple strategy at work is provided by the fact that the relevant phenomena are intrinsically thermodynamical limit ones. More technically, one considers the set of Gibbs thermal states $\varrho_\beta(\lambda) := Z^{-1}e^{-\beta H(\lambda)}$ $(Z := \text{Tr}[e^{-\beta H(\lambda)}])$ associated with a parametric family of Hamiltonians $\{H(\lambda)\}_{\lambda \in \mathcal{M}}$. Physically the λ's are to be thought of as coupling constant strengths and external fields defining the many-body Hamiltonian $H(\lambda)$. The systems one is interested in are characterized by the fact that, in the thermodynamical limit, they feature a zero temperature, i.e., quantum, phase transitions (QPT) for critical values λ_c [14]. We now consider the Bures metric $ds_B^2 = \Sigma_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu$ over the manifold of density matrices where [30]

$$g_{\mu\nu} = \frac{1}{2}\sum_{jk}\frac{\langle\varphi_j|\partial_\mu\varrho|\varphi_k\rangle\langle\varphi_k|\partial_\nu\varrho|\varphi_j\rangle}{\varrho_k + \varrho_j}. \tag{2}$$

The key result of the metric (or fidelity) approach to QPTs is that the set of critical parameters can be identified and analyzed in terms of the scaling and finite-size scaling behavior of the metric (2). More precisely the metric has the following properties: (i) In the thermodynamical limit and in the neighborhood of the critical values λ_c the zero-temperature metric has the scaling behavior $ds_B^2 \sim L^d|\lambda - \lambda_c|^{-\nu\Delta_g}$, where L is the system size, d is the spatial dimensionality, ν is the correlation length exponent $(\xi \sim |\lambda - \lambda_c|^{-\nu})$, and $\Delta_g = 2\zeta + d - 2\Delta_V$. Here ζ is the dynamical exponent and Δ_V is the scaling dimension of the operator coupled to λ. (ii) At the critical points, or more generally in the critical region defined by $L \ll \xi$, the finite-size scaling is $ds_B^2 \sim L^{d+\Delta_g}$. The main point is that for a wide class of QPTs Δ_g can be greater than zero, whereas at the regular points the scaling is always extensive,

i.e., $ds_B^2 \sim L^d$. To be precise superextensive behavior requires that the perturbation be sufficiently relevant. The superextensive behavior gives rise, for $L \to \infty$, to a peak (drop) of the metric (fidelity) that allows one to identify the boundaries between the different phases. Moreover, it can be proven [31] that, for local Hamiltonians, superextensive behavior of any of the metric elements is a sufficient condition for gaplessness, i.e., criticality. On the other hand, criticality is *not* a sufficient condition for such a superextensive scaling. There are indeed QPTs driven by local operators not sufficiently relevant (renormalization group sense) where no peak in the metric is observed at the critical point [31]. (iii) When the temperature is turned on one can still see the signatures of quantum criticality. This is done by studying the scaling, as a function of the temperature, of the elements of the metric (2) or fidelity [32]. In particular, when the temperature is small but bigger than the system's energy gap one has $ds_B^2 \sim T^{-\beta}$ ($\beta > 0$). When one sits, in the parameter space, at the critical point this result can be extended all the way down to zero temperature giving rise to a divergent behavior that matches $L|\lambda - \lambda_c|^{-\nu\Delta_g}$. Remarkably, crossovers between semiclassical and quantum critical regions in the (T, λ) plane can be identified by studying the largest eigenvalue of the metric or its curvature [33].

IV. CRITICALITY AS A RESOURCE

To the aims of this paper it is crucial to notice that the QFI is proportional to the Bures metric (2). Indeed, by evaluating the trace defining the QFI in the eigenbasis of $\varrho(\lambda)$ one readily finds [16] $g_{\mu\nu} = \frac{1}{4}\mathcal{H}_{\mu\nu}$. This simple remark, along with the results of the metric approach to criticality summarized in the previous section, immediately lead to the main conclusion of this paper: the estimation of a physical quantity driving a quantum phase transition is dramatically enhanced at the quantum critical point. It is important to notice that the Hamiltonian dependence on the quantity to be estimated can be even an indirect one. More precisely, suppose that H depends on a set of coupling constants λ and those in turn depend on the unknown (to be estimated) quantities λ'; i.e., $\lambda = f(\lambda')$, through a known function f; we assume also that f is smooth and that its derivatives are bounded and the system's size independent. From the tensor nature of the Bures metric, i.e., $g'_{\mu\nu} = g_{\alpha\beta}(\partial\lambda_\alpha/\partial\lambda'_\mu)(\partial\lambda_\beta/\partial\lambda'_\nu)$, one has that the QFI associated to the λ' has the same dependence on the system's size of the QFI associated to the λ's and the same divergencies in the thermodynamical limit. From the QCR bound (1) then it follows that if one is able to engineer a system such that the coupling constants λ defining its Hamiltonian (featuring QPTs) depends on the unknown quantities λ' in the way outlined above then the λ''s can be estimated with a greater efficiency in the points corresponding to the QPTs. For example, if $\lambda = \mu_0 - \lambda'$, then λ' can be effectively estimated around $\lambda' = \mu_0 - \lambda_c$ (μ_0 is assumed to be *exactly* known).

In order to quantitatively assess the improvement in the estimation accuracy, let us focus on the single parameter case $\lambda \in \mathbf{R}$. In this case the QCR reads $V_\lambda[\hat{\lambda}] \geq (4g_{11})^{-1}$. From the results summarized in the previous section one easily sees

that (i) in the neighborhood of the critical point and at zero temperature the optimal covariance $V_\lambda[\hat{\lambda}]$ scales like $|\lambda - \lambda_c|^{\nu\Delta_g}$. This is a remarkable fact for at least two reasons. On the one hand, it means that the covariance itself scales as the parameter and thus divergence of the QFI are allowed [12]. On the other hand, for those QPTs such that $\Delta_g > 0$, the covariance can be then pushed all the way down to zero by getting closer and closer to the quantum critical point; (ii) the covariance of $\hat{\lambda}$ may achieve the limit $L^{-\alpha}$ where $\alpha = d$ in all regular, i.e., noncritical points, while at the critical ones one can achieve $\alpha > d$. For example, in the class of one-dimensional systems studied in [22–24] one has that the estimation accuracy goes from order L^{-1} in the regular points to L^{-2} at the critical points; (iii) at the critical point in the parameter space and for finite temperature T the covariance of the optimal estimator scales like T^β. The exponent β is related to the Δ_g above and for a class of QPTs is greater than zero [33]. In this case by approaching zero temperature accuracy grows unboundedly, i.e., QFI diverges [12]. In contrast, at the regular points accuracy remains finite even for $T \to 0$ (and the finite system's size). In the quasifree fermionic case mentioned above one has $\beta = 1$.

A. Finite-size corrections

In view of practical applications involving realistic samples, we now consider corrections arising from the finite system size. As we have seen, when $L \to \infty$, the maximum of the QFI is located at the critical point λ_c. Instead, for finite L, the location of the maximum is shifted by an amount which goes to zero as $L \to \infty$. To obtain an estimate of the shift one must include an off-scaling contribution. The previous formulas for the scaling of ds_B^2, $\mathcal{H}$ in the off- and quasicritical regions can be combined in a single equation valid in a broader regime as follows:

$$\frac{\mathcal{H}}{L^d} = L^{\Delta_g}\phi(z) + L^{\Delta_g}D(\lambda) + L^{\Delta_g - \epsilon}C(\lambda) + \text{ (smaller terms)}.$$

$$(3)$$

Here z is the scaling variable $z = (L/\xi)^{1/\nu}$, $\epsilon > 0$, and ϕ is a scaling function satisfying $\phi(z) \sim z^{-\nu\Delta_g}$ when $z \to \infty$ (the off-critical region), whereas we must have $\phi(0) \neq 0$ to comply with the behavior in the quasicritical region ($z \to 0$). The existence of such a scaling function is a consequence of the scaling hypothesis. Instead the functions $C(\lambda)$, $D(\lambda)$ are analytic around λ_c and are responsible for corrections to scaling. The maximum of $\mathcal{H}(\lambda)$ defines a pseudocritical point λ_L^* whose location, for large L is given by

$$\lambda_L^* - \lambda_c \approx -\frac{\phi'}{\phi''}L^{-1/\nu} - \frac{D'}{\phi''}L^{-2/\nu} - \frac{C'}{\phi''}L^{-2/\nu - \epsilon}.$$
$$(4)$$

One can say that the shift exponent of the QFI is $1/\nu$, $2/\nu$, or larger depending on the form of the functions above. A similar shift of the location of the quantum critical point may be observed when the temperature is turned on.

V. OPTIMAL MEASUREMENTS

Our results establish the quantum limits to the degree of accuracy in estimating the coupling constant of a locally interacting many-body system. It is crucial to stress again that the QCR bound is, in this case, attainable with the corresponding SLD representing an observable to be measured in order to achieve the optimal estimation. Of course the SLD may be, in general, a very complex observable which itself depends on the ground or thermal state of the system. For a family of pure states $\varrho(\lambda) = |\psi(\lambda)\rangle\langle\psi(\lambda)|$, the SLD is easily derived as

$$\mathcal{L}_\mu(\lambda) = 2\partial_\mu\varrho(\lambda) = 2(|\psi\rangle\langle\partial_\mu\psi| + |\partial_\mu\psi\rangle\langle\psi|).$$

Using first order perturbation theory, one finds $\langle n|\partial_\mu\psi\rangle = (E_n - E_0)^{-1}\langle n|\partial_\mu H|0\rangle$ where now the $|n\rangle$'s (E_n's) are the eigenvectors (eigenvalues) of $H(\lambda)$ being $|0\rangle$ the ground state. Therefore, we may write the SLD as

$$\mathcal{L}_\mu(\lambda) = 2[(P_0\partial_\mu H G(E_0) + \text{H.c.}],$$

where $P_0 = |0\rangle\langle 0|$, $G(E) = P_1[H(\lambda) - E]^{-1}P_1$, being $P_1 = I - P_0$. In the one parameter case the nonvanishing eigenvalues of the SLD are given by $\pm 2\sqrt{\langle\partial\psi|\partial\psi\rangle - |\langle\psi|\partial\psi\rangle|^2} = \pm 2ds_B/d\lambda$ from which the unknown parameter λ can be estimated. The corresponding QFI is given by [25]

$$\mathcal{H}_{\mu\nu}(\lambda) = 4\sum_{n>0}\frac{\langle 0|\partial_\mu H|n\rangle\langle n|\partial_\nu H|0\rangle}{(E_n - E_0)^2}. \quad (5)$$

From this expression one sees that the origin of the divergence of QFI at the critical point is the vanishing (in the thermodynamical limit) of one of the $E_n - E_0$ factors.

A. Finite temperature

Now we would like to show that also classical, i.e., temperature driven, phase transitions provide, in principle, a valuable resource for estimation theory. Consider the metric induced on the (inverse) temperature axis by the mapping $\beta \rightarrow \varrho(\beta)$. In [25,33] it has been shown that $ds^2 \sim d\beta^2(\langle H^2\rangle_\beta - \langle H\rangle_\beta^2) = d\beta^2\beta^{-2}c_V(\beta)$, where $c_V(\beta)$ denotes the specific heat at the inverse temperature β. In [25] it was noticed that this relation suggests a neat and deep interplay between Hilbert space geometry and thermodynamics. Here we stress that also quantum estimation gets involved. Indeed, following the same lines used in the QPTs case, one easily realizes that when the specific heat shows an anomalous increase then the same happens to the QFI associated to the parameter β. From this fact stems the fact that at the classical phase transitions with diverging specific heat one can estimate temperature with arbitrarily high accuracy. Conversely, if the specific heat is bounded from above the estimation accuracy of the temperature is bounded from below. Again, in analogy with the QPT case discussed above, if β can be made dependent, in some known fashion, on some other parameter λ', then this latter can be estimated with better accuracy at the phase transition. In other words, we have a quantitative statement of the intuitive expectation about the fact that *thermometers* are more precise at the points where changes of state of matter occur.

B. Quantum discrimination

In our analysis the problem of interest is that of estimating the value of a parameter or a set of parameters. On the other hand, when one deals with *discrimination* of parameters rather than estimation, the relevant metric ds_{QCB}^2 can be derived from the so-called quantum Chernoff bound [34], which arises in the problem of discriminating two quantum states in the setting of asymptotically many copies. In view of the inequality $\frac{1}{2}ds_B^2 \leq ds_{QCB}^2 \leq ds_B^2$ all the results obtained here can be generalized to ds_{QCB}^2. This means that quantum criticality is a resource for quantum discrimination as well.

C. Applications: Two-stage adaptive measurements

A question may arise on how our results may be exploited in practice, being the form of the SLD, which maximizes the Fisher information, typically dependent on the true, unknown, state of the quantum system. The apparent loophole in the argument is closed by noticing that one can still achieve the same rate of distinguishability by a two-stage adaptive measurement procedure [35]. Roughly speaking the estimation scheme goes as follows: one starts by performing a generic, perhaps suboptimal, measurement on a vanishing fraction of the copies of the system and obtains a preliminary estimation. Then one measures the remaining copies, taking the the preliminary estimation as the true value. This guarantees that the Fisher information obtained by the second series of measurements approaches the QFI as the number of measurements goes to infinity, and that the resulting estimator saturates the QCR bound. Notice that the achievability of the QCR bound is ensured when a single parameter has to be estimated [36], though the actual implementation of the optimal measurement, which is in general, not unique [37], may be more or less challenging depending on the specific features of the system under investigation. We also notice that any maximum-likelihood (ML) [38] estimator, defined as the estimator $\hat{\lambda} = \hat{\lambda}(x_1, x_2, \ldots)$ maximizing the likelihood function $\mathcal{L}(\lambda) = \Pi_k^M p(x_k|\lambda)$, M being the number of measurements and $p(x|\lambda)$ the conditional probability density of the measured quantity, is *consistent*, i.e., it converges in probability to the true value and is asymptotically ($M \gg 1$) *efficient*, i.e., it saturates the Cramer-Rao bound in the limit of many measurements, thus achieving the ultimate bound in precision.

VI. QUANTUM ESTIMATION IN THE BCS MODEL

In order to appreciate the critical enhancement of precision in a specific physical situation we consider a fermionic tight-binding model with pair creation, i.e., the BCS-type model defined by the Hamiltonian

$$H_J = -J\sum_{i=1}^{L}(c_i^\dagger c_{i+1} + \gamma c_i^\dagger c_{i+1}^\dagger + \text{H.c.}) - 2h\sum_{i=1}^{L}c_i^\dagger c_i, \quad (6)$$

where L denotes the system size and periodic boundary conditions have been used. Upon considering the thermal or the ground state of the above Hamiltonian we have a uniparametric statistical model where J is the parameter to be esti-

mated, h is the external (tunable) field, and the anisotropy γ is a fixed quantity. For $\gamma \neq 0$ this model undergoes a quantum phase transition of Ising type at $h=J$ (because of symmetries there is an analogous critical point at $h=-J$). Precisely around the Ising transition point we will investigate the enhancement in precision offered by criticality.

Moving to Fourier space the Hamiltonian may be rewritten as

$$H_J = \sum_{k \in BZ} \epsilon_k (n_k + n_{-k} - 1) + \sum_{k \in BZ} (i\Delta_k c_k^\dagger c_{-k}^\dagger + \text{H.c.})$$
$$= \sum_{k>0} [-\epsilon_k \tau_k^z + \Delta_k \tau_k^y], \qquad (7)$$

where $\epsilon_k = -J\cos(k) - h$, $\Delta_k = -J\gamma\sin(k)$, the Brillouin zone BZ ranges from $-\pi$ to π and the momenta are of the form $k = 2\pi n/L$, with integer n. The expression (7) for H_J follows from the observation that in the subspace spanned by $\{|0\rangle, c_k^\dagger c_{-k}^\dagger |0\rangle\}$ the operators $n_k + n_{-k} - 1 = -\tau_k^z$ and $(ic_k^\dagger c_{-k}^\dagger + \text{H.c.}) = \tau_k^y$, represent a set of Pauli operators τ_k^j (and they are zero in the complementary space).

In the above quasispin formulation the Hamiltonian in Eq. (7) has a block form, and so is the thermal state $\varrho_J = Z^{-1}\exp\{-\beta H_J\}$. Being each block essentially two-dimensional, the SLD of the model can be obtained by computing the SLD in each block, with the aid of formula (18) of [39]. One then arrives at

$$\mathcal{L}(J) = \sum_{k>0} b_k^z \tau_k^z + b_k^y \tau_k^y,$$

where at $T=0$,

$$b_k^y = \frac{h}{J}\frac{\Delta_k \epsilon_k}{\Lambda_k^3}, \quad b_k^z = \frac{h}{J}\frac{\Delta_k^2}{\Lambda_k^3}, \qquad (8)$$

where $\Lambda_k = \sqrt{\epsilon_k^2 + \Delta_k^2}$.

Going back to real space we have

$$\mathcal{L}(J) = \frac{1}{2}L b^z(0) - \sum_{l,j} c_l^\dagger b^z(l-j)c_j$$
$$+ \frac{1}{2}\left[i\sum_{l,j} c_l^\dagger b^y(j-l)c_j^\dagger + \text{H.c.}\right], \qquad (9)$$

where

$$b^z(d) \equiv \frac{1}{L}\sum_{k \in BZ} e^{-ikd}b_k^z,$$

and analogously for $b^y(d)$. For example, for $L=4$ the general formula reduces to

$$\mathcal{L}(J) = \frac{h\gamma}{(h^2+\gamma^2 J^2)^{3/2}}\left\{J\gamma - \frac{J\gamma N}{2} + \frac{J\gamma}{2}[c_1^\dagger c_3 + c_2^\dagger c_4 + \text{H.c.}]\right.$$
$$\left. + h[c_1^\dagger c_2^\dagger - c_1^\dagger c_4^\dagger + \text{H.c.}]\right\}, \qquad (10)$$

which represents a collective measurement on the system. $N = \Sigma_j c_j^\dagger c_j$ is the total number operator.

In general, the coefficients $b^z(d)$ and $b^y(d)$ decay exponentially at normal points of the phase diagram and thus the

SLD is a local operator. On the other hand, the decay is only algebraic in the critical region $|h-J|L \lesssim 1$ and this corresponds to a SLD given by a collective measurement. Indeed, at the Ising transition one can show that $b^y(d) \sim d^{-1}$. For $b^z(d)$ the integral of the Fourier coefficient does not converge for large L, so that one has to keep the sum with L finite. The sum is well approximated by $b^z(d) \simeq 2.9|\frac{d}{L} - \frac{1}{2}|(-1)^d$. Notice that for $\gamma=0$ the SLD at zero temperature is identically zero since for any finite L changing J only results in a level crossing.

When the temperature is turned on, one can draw similar conclusions. The local character of the optimal measurement is enhanced at regular points $\forall L$, whereas in the critical regime nonlocal measurements are needed. More specifically, the coefficients $b^{z,y}(d)$ decay exponentially at regular points and algebraically in the region $|h-J| \lesssim T$.

The ultimate precision is determined by the QFI $\mathcal{H}_J$, which, for such a quasifree Fermi system reduces to (see [25])

$$\mathcal{H}(J) = \sum_{k>0}(\partial_J \vartheta_k)^2 = \sum_{k>0}\frac{h^2\gamma^2\sin(k)^2}{\Lambda_k^4}, \qquad (11)$$

where $\vartheta_k = \arctan(\epsilon_k/\Delta_k)$. Upon introducing the scaling variable $z = L(h-J)$ one is interested in the behavior around the Ising transition, i.e., for $z \ll 1$ where one can observe superextensive behavior. Indeed, upon expanding for small z and using a Euler-Maclaurin formula, one finally gets

$$\mathcal{H}(J) = \frac{L^2}{24J^2\gamma^2} - \frac{L}{2\pi^2 J^2\gamma^2} + \frac{z}{L}\left[\frac{(\gamma^2-1)L^2}{12J^3\gamma^4} + O(L)\right]$$
$$- \frac{z^2}{L^2}\left[\frac{L^4}{720J^4\gamma^4} + O(L^2)\right] + O(z^3). \qquad (12)$$

From this expression one can explicitly see the improvement of precision due to the underlying quantum critical behavior of the system. Besides, one can locate the pseudocritical point, defined as a value of the field leading to the maximum of $\mathcal{H}(J)$. Differentiating the above formula with respect to h one obtains

$$h_L^* = J + 30J(\gamma^2-1)L^{-2} + O(L^{-3}),$$

and use this information to achieve the quantum Cramer-Rao bound also for finite-size systems.

With the notations of Sec. IV A, we note we can write the scaling form of $\mathcal{H}(J)/L$ as in Eq. (3) with $\Delta_g = \nu = \epsilon = 1$ and

$$\phi(z) = \frac{1}{24J^2\gamma^2} - \frac{z^2}{720J^4\gamma^4} + O(z^3), \qquad (13)$$

$$D(h) = \frac{(\gamma^2-1)}{12J^3\gamma^4}(h-J) + O((h-J)^2), \qquad (14)$$

$$C(h) = \frac{1}{2\pi^2 J^2\gamma^2}O(h-J). \qquad (15)$$

Having $\phi'(0)=0$ the shift exponent turns out to be $2/\nu=2$.

At finite temperature and in the thermodynamical limit the QFI has been calculated previously for a quasifree Fermi system [33]. In the present case we obtain

$$\frac{\mathcal{H}(J)}{L} = \frac{\beta^2}{8\pi} \int_0^\pi \frac{dk}{\cosh^2(\beta\Lambda_k/2)} \frac{[J + h\cos(k)]^2}{\Lambda_k^2} \quad (16)$$

$$+ \frac{1}{2\pi} \int_0^\pi \frac{\cosh(\beta\Lambda_k) - 1}{\cosh(\beta\Lambda_k)} \frac{h^2\gamma^2 \sin(k)^2}{\Lambda_k^4} dk. \quad (17)$$

In this case we verified numerically that the maximum of $\mathcal{H}(J)$ always occurs at $h=J$ where it has a cusp. Then, since the first integral vanishes when the temperature goes to zero we evaluate the second term at the critical point. For $h=J$ the dispersion Λ_k is linear around $k=\pi$, which gives the dominant contribution to the integral. One then obtains

$$\mathcal{H}(J) = \frac{2\mathcal{C}}{\pi^2} \frac{L}{T|J\gamma|} + O(T^0),$$

$\mathcal{C}=0.915$ being the Catalan constant. From the above formula one can again appreciate the enhancement of the bound to precision occurring when T goes to zero in the critical regime.

VII. CONCLUSIONS AND OUTLOOKS

In conclusion, upon bringing together results from the geometric theory of quantum estimation and the geometric theory of quantum phase transition we have quantitatively shown that phase transitions represent a resource for the estimation of Hamiltonian parameters as well as of temperature. To this aim we used the quantum Cramer-Rao bound and the equivalence of the notion of the quantum Fisher metric and that of the quantum (ground or thermal) state metric. We have also found an explicit form of the observable achieving the ultimate precision. A specific example involving a fermionic tight-binding model with pair creation has been presented in order to illustrate the critical enhancement and the properties of the optimal measurement. The improvement in estimation tasks brought about by quantum criticality is reminiscent of the one associated to quantum entanglement in computational as well as metrological tasks. The analysis reported in this paper makes an important point of principle and establishes the ultimate quantum limits to the precision with which one can estimate coupling constants characterizing a quantum Hamiltonian. Since the Hamiltonian completely characterizes the quantum dynamics one can say that our results shed light on the ultimate limits imposed by quantum theory to the observer capability of knowing a quantum dynamics. Remarkably the boundaries between different phases of quantum matter are where these limits gets looser and a deeper knowledge can be achieved.

[1] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).

[2] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, NY, 1976).

[3] A. S. Holevo, Rep. Math. Phys. **16**, 385 (1979).

[4] M. D'Ariano *et al.*, Phys. Lett. A **248**, 103 (1998).

[5] A. Monras, Phys. Rev. A **73**, 033821 (2006).

[6] C. W. Helstrom, Found. Phys. **4**, 453 (1974).

[7] G. J. Milburn, W. Y. Chen, and K. R. Jones, Phys. Rev. A **50**, 801 (1994).

[8] G. Chiribella, G. M. D'Ariano, and M. F. Sacchi, Phys. Rev. A **73**, 062103 (2006).

[9] A. Fujiwara, Phys. Rev. A **63**, 042304 (2001).

[10] J. Zhenfeng *et al.*, e-print arXiv:quant-ph/0610060.

[11] V. D'Auria *et al.*, J. Phys. B **39**, 1187 (2006).

[12] A. Monras and M. G. A. Paris, Phys. Rev. Lett. **98**, 160401 (2007).

[13] B. R. Frieden, Opt. Commun. **271**, 71 (2007).

[14] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).

[15] W. K. Wootters, Phys. Rev. D **23**, 357 (1981).

[16] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).

[17] D. C. Brody and L. P. Hughston, Proc. R. Soc. London, Ser. A **454**, 2445 (1998); **455**, 1683 (1999).

[18] M. G. A. Paris, e-print arXiv:0804.2981 Int. J. Quantum Inf. (to be published).

[19] S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, Phys. Rev. A **77**, 012317 (2008).

[20] P. Zanardi and N. Paunkovic, Phys. Rev. E **74**, 031123 (2006).

[21] H. Q. Zhou and J. P. Barjaktarevic, J. Phys. A **41**, 412001 (2008).

[22] P. Zanardi *et al.*, J. Stat. Mech.: Theory Exp. 2007, L02002.

[23] M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B **75**, 014439 (2007).

[24] M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B **76**, 104420 (2007).

[25] P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007).

[26] G. M. D'Ariano, P. LoPresti, and M. G. A. Paris, Phys. Rev. Lett. **87**, 270404 (2001).

[27] A. Acin, Phys. Rev. Lett. **87**, 177901 (2001).

[28] V. Giovannetti, S. Lloyd, and L. Maccone, Nature (London) **412**, 417 (2001); Science **306**, 1330 (2004); Phys. Rev. Lett. **96**, 010401 (2006).

[29] C. W. Helstrom, Phys. Lett. **25A**, 101 (1967); IEEE Trans. Inf. Theory **14**, 234 (1968).

[30] H.-J. Sommers and K. Zyczkowski, J. Phys. A **36**, 10083 (2003).

[31] L. Campos Venuti and P. Zanardi, Phys. Rev. Lett. **99**, 095701 (2007).

[32] P. Zanardi, H. T. Quan, X. Wang, and C. P. Sun, Phys. Rev. A **75**, 032109 (2007).

[33] P. Zanardi, L. Campos Venuti, and P. Giorda, Phys. Rev. A **76**, 062318 (2007).

[34] V. Kargin, Ann. Stat. **33**, 959 (2005); K. M. R. Audenaert, J. Calsamiglia, R. Munoz-Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete, Phys. Rev. Lett. **98**, 160501 (2007).

[35] O. E. Barndorff-Nielsen and R. D. Gill, J. Phys. A **33**, 4481

(2000); A. Luati, Ann. Stat. **32**, 1770 (2004).

[36] A. Fujiwara, J. Phys. A **36**, 12489 (2006).

[37] M. Sarovar and G. J. Milburn, J. Phys. A **39**, 8487 (2006).

[38] Z. Hradil and J. Rehacek, Lect. Notes Phys. **649**, 59 (2004).

[39] C. Invernizzi, M. Korbman, L. Campos Venuti, and M. G. A. Paris, e-print arXiv:quant-ph/0807.3213v1, Phys. Rev. A (to be published).