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Dynamical symmetry of Dirac hydrogen atom with spin symmetry and its connection with Ginocchio's oscillator

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The Dirac hydrogen atom with spin symmetry is shown and has a SO(4) symmetry. The generators are derived, and the corresponding Casimir operator leads to the energy spectrum naturally. This type of hydrogen atom is connected to a four-dimensional Dirac system with equal scalar and vector harmonic oscillator potential by the Kustaanheimo-Stiefel transformation with a constraint.

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INTRODUCTION

Hydrogen atom and harmonic oscillator are two simple, solvable, and real models. A common feature of them is that their orbits of motion are closed in classical mechanics [1]. This indicates there are more constants of motion in these systems than the orbit angular momentum. They have been shown as the Rung-Lenz vector [2,3] in the hydrogen atom and the second order tensors [4] in the harmonic oscillator. These conserved quantities as well as angular momentum generate the SO(4) and SU(3) Lie groups, respectively. They are not geometrical but the symmetries in the phase space are called dynamical symmetries. These symmetries lead to an algebraic approach to determine the energy levels. Generally, the N-dimensional (ND) hydrogen atom has the SO(N+1) and the isotropic oscillator has the SU(N) symmetry. The SO(4) generators in the three-dimensional (3D) hydrogen atom can be written as two sets of decoupled SU(2) operators, whose Casimir operators are equal [5]. On the other hand, one can introduce a constraint condition on the four-dimensional (4D) oscillator and separate it into two 2D oscillators with the same energy. These two systems have the same algebraic structure and can be connected by the famous Kustaanheimo-Stiefel (KS) transformation [6]. The same transformation has been applied to the path integral treatment of the hydrogen atom [7] and the radial coherent state for the Coulomb problem [8].

In the relativistic quantum mechanics, the motion of the spin-1/2 particle satisfies the Dirac equation. Neither the Dirac hydrogen atom [9] nor the Dirac oscillator [10] has a dynamical symmetry. The main reason of the breaking of dynamical symmetries is the spin-orbit coupling. There was not any transformation reported to connect these two Dirac systems.

In recent years, the Dirac equation with scalar and vector potentials of equal magnitude (SVPEM) has been studied widely [11–18]. When the potentials are spherical, the Dirac equation is said to have the spin or pseudospin symmetry corresponding to the same or opposite sign. Then, the total angular momentum can be divided into conserved orbital and spin parts [see Eq. (2) for the spin symmetry case], which form the SU(2) algebra separately. These symmetries, which have been observed in the hadron and nuclear spectroscopies for a long time [19,20], are derived from the investigation of the dynamics between a quark and an antiquark [21–23]. The very recent studies [17,18] have revealed that the motion of a spin-1/2 particle with SVPEM satisfies the same differential equation and has the same energy spectrum as a scalar particle. For the spin or pseudospin symmetry systems, Alberto *et al.* [17] have indicated that the spin-orbit and Darwin terms of either the upper component or the lower component of the Dirac spinor vanish, which made it equivalent, as far as energy is concerned, to a spin-0 state. These results suggest that one can image the spin-1/2 particle with SVPEM as a relativistic scalar particle with an additional spin but without the spin-orbit coupling. From this point of view, we speculate the dynamical symmetries in a nonrelativistic hydrogen atom and harmonic oscillator would be held in the Dirac systems with spin or pseudospin symmetry.

Actually, Ginocchio [24,25] has found the U(3) and pseudo-U(3) symmetry in the Dirac equation with spin or pseudospin symmetry when the potential takes the harmonic oscillator form. The aim of this work is to answer the following questions: Does the Dirac equation with spin or pseudospin symmetry have a SO(4) dynamical symmetry, when the potential takes the Coulomb form? And what is the relation of it and Ginocchio's oscillator? For the sake of brevity, we only give the details of the spin symmetry case. The results of the pseudospin symmetry case will be easily obtained by making some corrections.

SO(4) SYMMETRY

The Dirac Hamiltonian with spin symmetry, in the relativistic units, $\hbar = c = 1$, is given by

$$H = \vec{\alpha} \cdot \vec{p} + \beta M + (1 + \beta)\frac{V(r)}{2}, \quad (1)$$

where $\vec{\alpha}$ and β are the Dirac matrices, and M is the mass. It commutes with the deformed orbital and spin angular momentum [22]

$$\vec{L} = \begin{bmatrix} \vec{l} & 0 \\ 0 & U_p \vec{l} U_p^\dagger \end{bmatrix}, \quad \vec{S} = \begin{bmatrix} \vec{s} & 0 \\ 0 & U_p \vec{s} U_p^\dagger \end{bmatrix}, \quad (2)$$

where $\vec{l} = \vec{r} \times \vec{p}$, $\vec{s} = \frac{\vec{\sigma}}{2}$ are the usual spin generators, $\vec{\sigma}$ are the Pauli matrices, and $U_p = U_p^\dagger = \frac{\vec{\sigma} \cdot \vec{p}}{p}$ is the helicity unitary opera-

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tor [13]. The components of them form the SU(2) Lie algebra separately, and the sum of them equals the total angular momentum on account of $U_p\vec{l}U_p^\dagger + U_p\vec{s}U_p^\dagger = \vec{l}+\vec{s}$. Ginocchio has shown $\vec{L}$ in Eq. (2) to be three of the generators of the U(3) symmetry group. The other conserved quantities are supposed to take the form as

$$Q = \begin{bmatrix} Q_{11} & Q_{12}\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}Q_{21} & \vec{\sigma}\cdot\vec{p}Q_{22}\vec{\sigma}\cdot\vec{p} \end{bmatrix}.$$

If it commutes with the spin symmetry Hamiltonian (1), the matrix elements should satisfy the equations

$$Q_{12} = Q_{21},$$

$$[Q_{11}, V(r)] + [Q_{12}, p^2] = 0,$$

$$[Q_{12}, V(r)] + [Q_{22}, p^2] = 0,$$

$$Q_{11} = Q_{12}[2M + V(r)] + Q_{22}p^2. \quad (3)$$

In a nonrelativistic hydrogen atom, the constants of motion are the orbital angular momentum $\vec{l}$ and the Rung-Lenz vector [2,3,5], $\vec{R} = \frac{\vec{f}}{2Mk} - \frac{\vec{r}}{r}$, where $\vec{f} = \vec{p}\times\vec{l} - \vec{l}\times\vec{p}$, and k is the parameter in the Coulomb potential $V(r)^h = -\frac{k}{r}$. One can obtain the following relations easily:

$$[\vec{f}, p^2] = 0, \quad \left[-\frac{\vec{r}}{r}, V^h(r)\right] = 0,$$

$$\frac{1}{2Mk}[\vec{f}, V^h(r)] + \frac{1}{2M}\left[-\frac{\vec{r}}{r}, p^2\right] = 0.$$

Inserting them into Eq. (3), we can find the solutions of the Coulomb potential $V(r) = V^h(r) = -\frac{k}{r}$,

$$\vec{Q} = \begin{bmatrix} 2M\vec{R} + \frac{k\vec{r}}{r^2} & \left(-\frac{\vec{r}}{r}\right)\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\left(-\frac{\vec{r}}{r}\right) & \vec{\sigma}\cdot\vec{p}\left(\frac{1}{k}\frac{\vec{f}}{p^2}\right)\vec{\sigma}\cdot\vec{p} \end{bmatrix}.$$

The commutation relations of conserved $\vec{L}$ and $\vec{Q}$ are

$$[L_i, L_j] = i\epsilon_{ijk}L_k,$$

$$[L_i, Q_j] = i\epsilon_{ijk}Q_k,$$

$$[Q_i, Q_j] = i\frac{-4}{k^2}(H^2 - M^2)\epsilon_{ijk}L_k, \quad (4)$$

where $i, j, k = 1, 2, 3$; and,

$$\vec{Q}\cdot\vec{L} = 0,$$

$$\vec{Q}^2 = \frac{4}{k^2}(H^2 - M^2)(L^2 + 1) + (H + M)^2. \quad (5)$$

For a fixed energy level, H is a constant. We can define the normalized generators $\vec{A} = [\frac{-4}{k^2}(H^2 - M^2)]^{-1/2}\vec{Q}$. Evidently, the two later relations in Eq. (4) are transformed into $[L_i, A_j] = i\epsilon_{ijk}A_k$ and $[A_i, A_j] = i\epsilon_{ijk}L_k$. These results show that the Dirac hydrogen atom with spin symmetry has a SO(4) symmetry.

By the standard process of the nonrelativistic hydrogen atom [5], the relations of the generators lead to the energy spectrum. Set $\vec{I} = (\vec{L}+\vec{A})/2$ and $\vec{K} = (\vec{L}-\vec{A})/2$, which satisfy the commutation relations, $[I_i, I_j] = i\epsilon_{ijk}I_k$, $[K_i, K_j] = i\epsilon_{ijk}K_k, [I_i, K_j] = 0$. $\vec{I}$ and $\vec{K}$ constitute two commutative SU(2) algebra. Considering Eq. (5), it is immediate to obtain the Casimir operators

$$\vec{I}^2 = \vec{K}^2 = \frac{1}{4}\left[-\frac{k^2}{4}\frac{H+M}{H-M} - 1\right] = j(j+1),$$

where $j = 0, 1/2, 1, 3/2, \ldots$. Hence the eigenvalues of the Hamiltonian are given by

$$E^{\pm} = \frac{\pm 4n^2 - k^2}{4n^2 + k^2}M, \quad n = 2j + 1 = 1, 2, 3, \ldots \quad (6)$$

which come up to [11,12].

When $M \to \infty$, $H \to M$, the nonrelativistic limit of the energy levels is $E^{+} \to M - \frac{k^2}{2n^2}M$, the second term of which agrees with the nonrelativistic results [5]. We can also get the nonrelativistic limits of the conserved quantities

$$H - M \to \begin{bmatrix} \frac{p^2}{2M} - \frac{k}{r} & 0 \\ 0 & \frac{p^2}{2M} \end{bmatrix}, \quad \frac{\vec{Q}}{2M} \to \begin{bmatrix} \vec{R} & 0 \\ 0 & \frac{1}{2Mk}\vec{f} \end{bmatrix}.$$

The upper-left elements of the above matrices are nothing but the nonrelativistic hydrogen atom Hamiltonian and the Rung-Lenz vector, and the lower-right ones are their limits when $k \to 0$.

KS TRANSFORMATION

The treatment of dynamical symmetry in the above section can be generalized to ND Dirac systems with SVPEM. This will be discussed in a forthcoming paper. Here we give the spectrum of the 4D Dirac system with equal scalar and vector harmonic oscillator potentials (4D Ginocchio's oscillator), and show its relation with the Dirac hydrogen atom with spin symmetry. Denoting the 4D spatial coordinates and momentums as u_μ and P_μ ($\mu = 1, 2, 3, 4$), the Dirac Hamiltonian with SVPEM is given by

$$\mathcal{H} = \alpha^\mu P_\mu + \beta m + (1+\beta)\frac{\mathcal{V}(u)}{2}.$$

Here, $\alpha^{1,2,3} = \vec{\alpha}$ and β are the same as the 3D case, and $\alpha^4 = -\sigma_2 \otimes I$. Then, the Dirac equation can be written as

$$\begin{bmatrix} m + \mathcal{V}(u) - \varepsilon & B \\ B^\dagger & -m - \varepsilon \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \quad (7)$$

where $B = \sigma^\mu P_\mu$ with $\sigma^{1,2,3} = \vec{\sigma}$ and $\sigma^4 = i$. It equals the second order differential equations

$$[P^2+(\varepsilon+m)\mathcal{V}(u)-(\varepsilon^2-m^2)]\varphi_1=0,$$

$$(\varepsilon+m)\varphi_2+B^\dagger\varphi_1=0.$$

Considering ε is a constant for a given energy level, the first equation takes the same form as the Schrödinger equation with the mass $\tilde{m}=(\varepsilon+m)/2$ and eigenvalue $\tilde{\varepsilon}=\varepsilon-m$. When $\mathcal{V}(u)=\frac{1}{2}m\omega^2u^2=\frac{1}{2}\tilde{m}\Omega^2u^2$, the energy spectrum of the 4D Schrödinger equation is $\tilde{\varepsilon}=(N+2)\Omega$, where $N=n_1+n_2+n_3+n_4$ is the total number operator [8] of the 4D nonrelativistic harmonic oscillator. Hence the eigenvalues in Eq. (7) satisfy the equation

$$(\varepsilon+m)^2(\varepsilon-m)^2-2m\omega^2(\varepsilon+m)(N+2)^2=0. \tag{8}$$

The so-called KS transformation, between the spatial coordinates $\vec{r}=(x_1,x_2,x_3)$ and $u_\mu=(u_1,u_2,u_3,u_4)$, is given by [6]

$$x_1=2(u_1u_3-u_2u_4),$$

$$x_2=2(u_1u_4+u_2u_3),$$

$$x_3=u_1^2+u_2^2-u_3^2-u_4^2,$$

which connects the 3D hydrogen atom and 4D harmonic oscillator with the constraint condition

$$u_4P_1-u_1P_4+u_2P_3-u_3P_2=0. \tag{9}$$

Under this transformation and constraint, it is easy to obtain $B=2\Gamma\vec{\sigma}\cdot\vec{p}$, where $\Gamma=u_1+iu_2\sigma_3-iu_3\sigma_2+iu_4\sigma_1$, $\Gamma\Gamma^\dagger=u^2=r$. Then, the Dirac equation (7) can be written as

$$\begin{bmatrix}2\Gamma & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{m-\varepsilon}{4r}+\frac{1}{8}m\omega^2 & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -m-\varepsilon\end{bmatrix}\begin{bmatrix}2\Gamma^\dagger & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=0. \tag{10}$$

Setting

$$M+E=m+\varepsilon,\quad M-E=\frac{1}{8}m\omega^2,\quad k=\frac{1}{4}(\varepsilon-m), \tag{11}$$

Eq. (10) is equivalent to nothing but the 3D Dirac equation of the hydrogen atom with spin symmetry

$$\begin{bmatrix}M-\frac{k}{r}-E & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -M-E\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}=0, \tag{12}$$

where $\psi_1=2\Gamma^\dagger\varphi_1$ and $\psi_2=\varphi_2$. The constraint in Eq. (9) requires the number operators in the 4D Schrödinger harmonic oscillator satisfy [8] $n_1+n_2=n_3+n_4$. Therefore the quantum number in Eq. (8) is confined to be even, $N+2=2n=2,4,6,\ldots$. Substituting the relations from Eq. (11) into Eq. (8), one can obtain the energy spectrum in Eq. (6).

CONCLUSION

In summary, we have shown that the 3D Dirac hydrogen atom with spin symmetry has the SO(4) symmetry. The nature of this symmetry is not geometrical but dynamical. The relation of its Hamiltonian and the Casimir operators of the symmetry group leads to an algebraic solution of the relativistic energy spectrum. The nonrelativistic limits of the conserved quantities coincide with their nonrelativistic counterparts accurately. As a result of the symmetries of the Coulomb and harmonic oscillator problem, we prove the Dirac hydrogen atom with spin symmetry can be connected to a 4D Ginocchio's oscillator by the KS transformation with a constraint condition. Dynamical symmetry and the KS transformation are two important concepts in nonrelativistic quantum mechanics. The results in this paper and [25] show they also exist in Dirac systems with SVPEM. The spin symmetry and pseudospin symmetry exist frequently in antinucleon and nucleon spectra [11]. We can foretell there are more properties of the Schrödinger equations present in these systems.

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