Classification of depolarized maps

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We classify the completely positive maps acting on two d-dimensional systems which commute with all $U \otimes U$ unitaries, where $U \in \mathrm{SU}(d)$. This set of operations map Werner states to Werner states. We find a simple condition for a map to be implementable by stochastic local operations and classical communication (SLOCC). We show that all positive partial transpose (PPT) preserving maps can be implemented by SLOCC. This can be used to prove that the entanglement of Werner states cannot be stochastically increased, even if we allow PPT entanglement for free, which provides a necessary result for proving the result of Masanes [Phys. Rev. Lett. **96**, 150501 (2006)], one of the central results in the theory of entanglement.

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I. INTRODUCTION

Entanglement plays a crucial role in the applications of quantum-information science, such as quantum key distribution [1], superdense coding [2], quantum teleportation [3], and quantum error correction [4], etc. Among several bipartite entangled states, Werner states [5] provide the simplest example of mixed states possessing entanglement and they play an important role in entanglement purification [6], nonlocality [5], entanglement measures [7], etc. Werner states are a useful family of states depending on a single parameter; more theoretical investigations of Werner states can be seen in [8].

It is often of interest in quantum-information processing to determine if a given state can be transformed to some other desired state by local operations [9]. Indeed, convertibility between two (entangled) states using local quantum operations assisted by classical communication (LOCC) is closely related to the problem of quantifying the entanglement associated with each quantum system. Intuitively, one expects that a (single copy) entangled state can be locally and deterministically transformed to a less entangled one but not the other way round. In this paper we consider transformations whose success probability can be smaller than 1, but of course it has to be strictly larger than zero. This set of transformations is called *stochastic local operations and classical communication* (SLOCC), then in general it is possible to transform the state the other way around, i.e., transform a less entangled state to a more entangled state with some success probability. However, in this paper, by studying the depolarized maps that transform Werner states into Werner states, we show that the entanglement of Werner states cannot increase under these maps even though they can be stochastic local operations and classical communication. This is necessary for proving the result in [10], which is one of the central results in the theory of entanglement.

The remainder of the paper is organized as following: in the next two sections, we give a brief introduction on the Werner states and separable maps. We then give our classification of the separable maps that preserve the set of Werner states, which is, by the Jamiolkowsky theorem, equivalent to the classification of a class of symmetric states. In Sec. V we present an application of this classification.

II. WERNER STATES

Consider a tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = C^d$. The swap unitary matrix F is defined as $F|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A$. The symmetric and antisymmetric projectors are, respectively,

$$S = \frac{I+F}{2}, \tag{1}$$

$$A = \frac{I-F}{2}, \tag{2}$$

where I is the identity matrix acting on $\mathcal{H}_A$. Analogously, the symmetric and antisymmetric normalized states are $\hat{S} = S/\mathrm{tr}S$ and $\hat{A} = A/\mathrm{tr}A$.

Werner states, denoted ω_ν, are the ones that commute with all unitaries of the form $U \otimes U$ [5,11], where U acts on C^d, and can be written as

$$\omega_\nu = \nu \hat{A} + (1-\nu)\hat{S}, \tag{3}$$

where $\nu \in [0,1]$. We define the depolarization map as

$$\Lambda[\rho] = \hat{A}\, \mathrm{tr}\rho A + \hat{S}\, \mathrm{tr}\rho S, \tag{4}$$

which is just projection to the Werner states. This map can be physically implemented by LOCC in the following way:

$$\Lambda[\rho] = \int dU\, U \otimes U \rho (U \otimes U)^\dagger, \tag{5}$$

where dU is the Haar measure over SU(d). It is known that ω_ν is separable for $\nu \in [0,1/2]$ and entangled for $\nu \in (1/2,1]$ [5,11].

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III. SEPARABLE MAPS AND SLOCC

In this section, we give a brief introduction of the separable maps and SLOCC, we follow the treatment in [9]. To begin with, a separable complete positive map (CPM), denoted by $\mathcal{E}_s$ takes the following form [12,13]:

$$\mathcal{E}_s:\rho \to \sum_{i=1}^{n} (A_i \otimes B_i)\rho(A_i^\dagger \otimes B_i^\dagger), \tag{6}$$

where ρ acts on $\mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{in}}$, A_i acts on $\mathcal{H}_{\mathcal{A}_{in}}$, and B_i acts on $\mathcal{H}_{\mathcal{B}_{in}}$. If, moreover,

$$\sum_i (A_i \otimes B_i)^\dagger (A_i \otimes B_i) = \mathbb{I}, \tag{7}$$

the map is trace preserving, i.e., if ρ is normalized, so is the output of the map $\mathcal{E}_s(\rho)$. Equivalently, the trace-preserving condition demands that the transformation from ρ to $\mathcal{E}_s(\rho)$ can always be achieved with certainty. It is well known that all LOCC transformations are of the form Eq. (6) but the converse is not true [14].

However, if we allow the map $\rho \to \mathcal{E}_s(\rho)$ to fail with some probability $p<1$, the transformation from ρ to $\mathcal{E}_s(\rho)$ can always be implemented probabilistically via LOCC. In other words, if we do not impose Eq. (7), then Eq. (6) represents, up to some normalization constant, the most general LOCC possible on a bipartite quantum system. These are the SLOCC transformations [15].

Let us also recall the Choi-Jamiołkowski isomorphism [16] between CPMs and quantum states: for every (not necessarily separable) CPM $\mathcal{E}:\mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{in}} \to \mathcal{H}_{\mathcal{A}_{out}} \otimes \mathcal{H}_{\mathcal{B}_{out}}$ there is a unique—again, up to some positive constant α—quantum state $\rho_{\mathcal{E}}$ corresponding to $\mathcal{E}$:

$$\rho_{\mathcal{E}} = \alpha \mathcal{E} \otimes \mathcal{I}(|\Phi^+\rangle_{\mathcal{A}_{in}}\langle\Phi^+| \otimes |\Phi^+\rangle_{\mathcal{B}_{in}}\langle\Phi^+|), \tag{8}$$

where $|\Phi^+\rangle_{\mathcal{A}_{in}} \equiv \Sigma_{i=1}^{d_{\mathcal{A}_{in}}}|i\rangle \otimes |i\rangle$ is the unnormalized maximally entangled state of dimension $d_{\mathcal{A}_{in}}$ (likewise for $|\Phi^+\rangle_{\mathcal{B}_{in}}$). In Eq. (8), it is understood that $\mathcal{E}$ acts on only half of $|\Phi^+\rangle_{\mathcal{A}_{in}}$ and half of $|\Phi^+\rangle_{\mathcal{B}_{in}}$. Clearly, the state $\rho_{\mathcal{E}}$ acts on a Hilbert space of dimension $d_{\mathcal{A}_{in}} \times d_{\mathcal{A}_{out}} \times d_{\mathcal{B}_{in}} \times d_{\mathcal{B}_{out}}$, where $d_{\mathcal{A}_{out}} \times d_{\mathcal{B}_{out}}$ is the dimension of $\mathcal{H}_{\mathcal{A}_{out}} \otimes \mathcal{H}_{\mathcal{B}_{out}}$.

Conversely, given a state $\rho_{\mathcal{E}}$ acting on $\mathcal{H}_{\mathcal{A}_{out}} \otimes \mathcal{H}_{\mathcal{B}_{out}} \otimes \mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{in}}$, the corresponding action of the CPM $\mathcal{E}$ on some ρ acting on $\mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{in}}$ reads

$$\mathcal{E}(\rho) = \frac{1}{\alpha}\mathrm{tr}_{\mathcal{A}_{in}\mathcal{B}_{in}}[\rho_{\mathcal{E}}(\mathbb{I}_{\mathcal{A}_{out}\mathcal{B}_{out}} \otimes \rho^T)], \tag{9}$$

where ρ^T denotes transposition of ρ in some local bases of $\mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{in}}$. For a trace-preserving CPM, it then follows that we must have $tr_{\mathcal{A}_{out}\mathcal{B}_{out}}(\rho_{\mathcal{E}}) = \alpha \mathbb{I}_{\mathcal{A}_{in}\mathcal{B}_{in}}$. A point that should be emphasized now is that $\mathcal{E}$ is a separable map [cf. Eq. (6)] if and only if the corresponding $\rho_{\mathcal{E}}$ given by Eq. (8) is separable across $\mathcal{H}_{\mathcal{A}_{in}} \otimes \mathcal{H}_{\mathcal{A}_{out}}$ and $\mathcal{H}_{\mathcal{B}_{in}} \otimes \mathcal{H}_{\mathcal{B}_{out}}$ [17,18]. Moreover, at the risk of repeating ourselves, the map $\rho \to \mathcal{E}(\rho)$ derived from a separable $\rho_{\mathcal{E}}$ can always be implemented locally, although it may only succeed with some (nonzero) probability. Hence, if we are only interested in transformations that can be performed locally, and not the probability of success in mapping $\rho \to \mathcal{E}(\rho)$, the normalization constant α as well as the normalization of $\rho_{\mathcal{E}}$ become irrelevant.

IV. SYMMETRIC MAPS

Let us consider depolarized maps that transform Werner states into Werner states in $\mathcal{H}_A \otimes \mathcal{H}_B$. By using the Jamiolkowsky theorem [16], each of these maps $\mathcal{E}$ has associated a state $\rho_{\mathcal{E}}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, where also $\mathcal{H}_{A'} = \mathcal{H}_{B'} = C^d$. The action of the map $\mathcal{E}$ associated with $\rho_{\mathcal{E}}$ is the following:

$$\mathcal{E}[\rho] = \mathrm{tr}_{AB}[\rho_{\mathcal{E}}(I_{A'B'} \otimes \rho^T)]. \tag{10}$$

Because of this rule, we refer to $\mathcal{H}_A \otimes \mathcal{H}_B$ as the input space, and to $\mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$ as the output space. One can see that the states $\rho_{\mathcal{E}}$ associated with these maps commute with all unitaries of the form [5,11]

$$U \otimes U \otimes V \otimes V, \tag{11}$$

where U and V act on C^d. From the Jamiolkowsky theorem we know that a map $\mathcal{E}$ is separable if, and only if, its associated state $\rho_{\mathcal{E}}$ is separable across $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ and $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ [17]. In the following we classify all separable states of this kind.

The states that commute with the group (11) are of the form [5,11]

$$\xi = \lambda_1 \hat{A} \otimes \hat{A} + \lambda_2 \hat{A} \otimes \hat{S} + \lambda_3 \hat{S} \otimes \hat{A} + \lambda_4 \hat{S} \otimes \hat{S}, \tag{12}$$

with $\lambda_i \geq 0$ and $\Sigma_i \lambda_i = 1$, where the tensor is across subspace AB and $A'B'$. In the following we specify states with four-dimensional vectors $\vec{\lambda}$. Let us see that the following five states are separable:

$$\vec{\lambda}^{(1)} = (0,0,0,1), \tag{13}$$

$$\vec{\lambda}^{(2)} = \left(0,\frac{1}{2},0,\frac{1}{2}\right), \tag{14}$$

$$\vec{\lambda}^{(3)} = \left(0,0,\frac{1}{2},\frac{1}{2}\right), \tag{15}$$

$$\vec{\lambda}^{(4)} = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right), \tag{16}$$

$$\vec{\lambda}^{(5)} = \left(\frac{1}{2}-\frac{1}{2d},0,0,\frac{1}{2}+\frac{1}{2d}\right). \tag{17}$$

The states $\vec{\lambda}^{(1)},\ldots,\vec{\lambda}^{(4)}$ are separable because they are products of the two separable Werner states ω_0 and $\omega_{1/2}$. The state $\vec{\lambda}^{(5)}$ is separable because it is the output of the LOCC map

$$\Delta[\rho] = \int dU \int dV\, U \otimes U \otimes V \otimes V \rho (U \otimes U \otimes V \otimes V)^\dagger, \tag{18}$$

when the input is the product state $\Sigma_{ks=1}^{d}|ks\rangle_{AB} \otimes |ks\rangle_{A'B'}/d$. The map Ξ associated with the state $\vec{\lambda}^{(5)}$ is precisely the depolarization map (4) and (5).

Let us denote by $\mathcal{P}$ the convex polytope generated by $\vec{\lambda}_1, \ldots, \vec{\lambda}_5$. Clearly, all points in $\mathcal{P}$ correspond to separable states. Let us see that only these states are the separable ones. The set $\mathcal{P}$ can be characterized by a finite number of linear inequalities of the form

$$\vec{\lambda} \cdot \vec{\mu} \geq 0. \tag{19}$$

We can choose the constant in the right-hand side of the inequalities to be zero because all the points in $\mathcal{P}$ are inside the normalization hyperplane. Some of the inequalities $\vec{\mu}$ correspond to the positivity conditions of $\vec{\lambda}$; the rest (the relevant ones) are

$$\vec{\mu}^{(1)} = (1,-1,-1,1), \tag{20}$$

$$\vec{\mu}^{(2)} = (-d-1,d+1,-d+1,d-1), \tag{21}$$

$$\vec{\mu}^{(3)} = (-d-1,-d+1,d+1,d-1). \tag{22}$$

The vectors $\vec{\mu}^{(i)}$ are the extreme points of the dual polytope of $\mathcal{P}$. In general one can find them with standard software, but this case is simple enough to do it by hand.

The Hermitian matrices corresponding to the vectors $\vec{\mu}$ are

$$W_{\vec{\mu}} = \mu_1 A \otimes A + \mu_2 A \otimes S + \mu_3 S \otimes A + \mu_4 S \otimes S. \tag{23}$$

Notice that in order to make

$$\mathrm{tr}\xi W = \vec{\lambda} \cdot \vec{\mu} \tag{24}$$

we must write projectors A and S instead of normalized states $\hat{A}$ and $\hat{S}$. Up to an unimportant proportionality factor, we can write the Hermitian matrices associated with (23) as

$$W^{(1)} = F \otimes F, \tag{25}$$

$$W^{(2)} = dI \otimes F - F \otimes F, \tag{26}$$

$$W^{(3)} = dF \otimes I - F \otimes F. \tag{27}$$

In what follows, we prove that these three operators are entanglement witnesses, that is, their expected values with product states are non-negative:

$$\langle \alpha| \otimes \langle \beta| W^{(i)} |\alpha\rangle \otimes |\beta\rangle \geq 0 \tag{28}$$

for all $|\alpha\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{A'}, |\beta\rangle \in \mathcal{H}_B \otimes \mathcal{H}_{B'}$. For doing this the following identities are useful:

$$\langle \alpha_{AA'} \beta_{BB'} | I \otimes F | \alpha_{AA'} \beta_{BB'} \rangle = \mathrm{tr}(\alpha^\dagger \alpha \beta^\dagger \beta),$$

$$\langle \alpha_{AA'} \beta_{BB'} | F \otimes I | \alpha_{AA'} \beta_{BB'} \rangle = \mathrm{tr}(\alpha \alpha^\dagger \beta \beta^\dagger),$$

$$\langle \alpha_{AA'} \beta_{BB'} | F \otimes F | \alpha_{AA'} \beta_{BB'} \rangle = |\mathrm{tr} \alpha \beta^\dagger|^2,$$

where the matrix α_{ij} is defined as $|\alpha_{AA'}\rangle = \Sigma_{ij} \alpha_{ij} |i\rangle_A \otimes |j\rangle_{A'}$, and analogously for β. From the last equality, one can straightforwardly see that $W^{(1)}$ is a witness. To prove that $W^{(2)}$ is also a witness we define the matrix $\gamma = \alpha \beta^\dagger$ and write the expectation arbitrary product state as

$$\langle \alpha\beta | W^{(2)} | \alpha\beta \rangle = d\, \mathrm{tr} \gamma^\dagger \gamma - |\mathrm{tr} \gamma|^2 \geq 0. \tag{29}$$

The inequality comes from the Cauchy-Schwarz inequality as we can write d as $\mathrm{tr} I^\dagger I$ and $\mathrm{tr}\gamma$ as $\mathrm{tr} I \gamma$. Similarly, one can prove that $W^{(3)}$ is an entanglement witness.

The fact that $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are witnesses implies that all points outside the polytope $\mathcal{P}$ are nonseparable. In conclusion, $\mathcal{P}$ is the set of separable symmetric states.

Remark 1. One can see that the partial transpositions of S and A are

$$S^\Gamma = \frac{I + d\Phi}{2}, \tag{30}$$

$$A^\Gamma = \frac{I - d\Phi}{2}, \tag{31}$$

where Φ stands for the projector onto the maximally entangled state $|\Phi\rangle = \Sigma_{d=1}^{d} |kk\rangle$. If one performs the partial transposition on the generic state (12), and imposes the requirement that the result is positive semidefinite, one obtains the three inequalities associated with $\vec{\mu}^{(1)}$, $\vec{\mu}^{(2)}$, and $\vec{\mu}^{(3)}$, in (20). This means that the set of symmetric positive partial transpose (PPT) states is also $\mathcal{P}$, which means the set of PPT-preserving maps that preserve Werner states coincide with the set of separable maps that preserve the Werner states. So we actually also classified the PPT-preserving maps that preserve Werner states.

V. APPLICATIONS

We can use our classification of depolarized maps to prove that the entanglement of Werner states cannot increase under stochastic local operations and classical communication. It is known that all separable maps can be physically implemented with SLOCC, and, obviously, all SLOCC are separable maps. Then, what we want to prove is that, if

$$\Xi[\nu \hat{A} + (1-\nu)\hat{S}] = \nu' \hat{A} + (1-\nu')\hat{S}, \tag{32}$$

where $\nu \geq 1/2$ and Ξ is a separable map, then $\nu' \leq \nu$.

By using the Jamiolkowsky isomorphism we have that

$$\Xi[\nu \hat{A} + (1-\nu)\hat{S}] = 2\frac{\nu}{d(d-1)}(\lambda_1 \hat{A} + \lambda_2 \hat{S}) + 2\frac{1-\nu}{d(d+1)}(\lambda_3 \hat{A} + \lambda_4 \hat{S}).$$

The new value of the parameter is

$$\nu' = \frac{2\frac{\nu}{d(d-1)}\lambda_1 + 2\frac{1-\nu}{d(d+1)}\lambda_3}{2\frac{\nu}{d(d-1)}(\lambda_1+\lambda_2) + 2\frac{1-\nu}{d(d+1)}(\lambda_3+\lambda_4)}. \tag{33}$$

Now, let us assume $\nu' > \nu$. After some algebra we transform $\nu' > \nu$ into

$$(d-1)(\eta\lambda_4 - \eta^2\lambda_3) + (d+1)(\lambda_2 - \eta\lambda_1) < 0, \quad (34)$$

where $\eta=(1-\nu)/\nu$. Because $0<\eta<1$ then $-\eta\lambda_3 \leqslant -\eta^2\lambda_3$ and $\eta\lambda_2 \leqslant \lambda_3$. Therefore, if the above inequality is true we have

$$(d-1)(\lambda_4 - \lambda_3) + (d+1)(\lambda_2 - \lambda_1) < 0, \quad (35)$$

which is precisely $\vec{\lambda}\cdot\vec{\mu}^{(2)}<0$. This is in contradiction with the fact that Ξ is separable; therefore $\nu' \leqslant \nu$ must hold.

VI. CONCLUSIONS

In summary, we have explicitly characterized the complete positive maps acting on two d-dimensional systems which commute with all $U\otimes U$ unitaries. A simple condition is also given on those maps which can be implemented by SLOCC. We achieved this by using the Jamiolkowsky theorem: instead of characterizing the completely positive maps directly, we equivalently characterized the states of four d-dimensional systems which commute with all $U\otimes U\otimes V \otimes V$ unitaries, where $U,V \in \mathrm{SU}(d)$. This enabled us to give conditions on the separable and PPT-preserving maps that preserve the Werner states. With these conditions, we showed that the entanglement of Werner states cannot be increased under SLOCC.

The fact that the entanglement of Werner states cannot be increased under SLOCC has been the key to proving that each bipartite entangled state can increase the teleportation power of another state [10]. This establishes that all entangled states are useful for quantum-information processing.

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[1] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[2] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] D. Gottesman, e-print arXiv:quant-ph/0004072.
[5] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[6] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[7] P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **86**, 2681 (2001).
[8] J. Lee and M. S. Kim, Phys. Rev. Lett. **84**, 4236 (2000); T. Hiroshima and S. Ishizaka, Phys. Rev. A **62**, 044302 (2000); A. O. Pittenger and M. H. Rubin, Opt. Commun. **179**, 447 (2000); S. Bose and V. Vedral, Phys. Rev. A **61**, 040101(R) (2000); S. Ishizaka and T. Hiroshima, *ibid.* **62**, 022310 (2000); A. Felicetti, S. Mancini, and P. Tombesi, *ibid.* **65**, 062107 (2002).
[9] Yeong-Cherng Liang, Lluis Masanes, and Andrew C. Doherty, Phys. Rev. A **77**, 012332 (2008).
[10] L. Masanes, Phys. Rev. Lett. **96**, 150501 (2006).
[11] K. G. H. Vollbrecht and R. F. Werner, e-print arXiv:quant-ph/0010095.
[12] E. M. Rains, e-print arXiv:quant-ph/9707002.
[13] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[14] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **59**, 1070 (1999).
[15] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[16] A. Jamiokowski, Rep. Math. Phys. **3**, 275 (1972).
[17] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
[18] Vlad Gheorghiu and Robert B. Griffiths, Phys. Rev. A **76**, 032310 (2007).