# Dynamical constants for electromagnetic fields with elliptic-cylindrical symmetry

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Taking into account the characteristics of a free scalar field in elliptic coordinates, a new dynamical variable is found for the free electromagnetic field. The conservation law associated to this variable cannot be obtained by direct application of standard Noether theorem since the symmetry generator is of second order. Consequences on the expected mechanical behavior of an atomic system interacting with electromagnetic waves exhibiting such a symmetry are also discussed.

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## I. INTRODUCTION

Symmetries in physical systems manifest as dynamical constants. In this way, via Noether theorem, the homogeneity and isotropy of free space is directly related to the conservation of linear momentum $\vec{P}$ and angular momentum $\vec{J}$, respectively, and time homogeneity is related to energy $\mathcal{E}$ conservation. For a system of particles interacting through intermediate fields, global conservation laws are usually translated, via the equations of motion, into local conservation laws describing the interchange of dynamical variables between particles and fields. Under proper circumstances, the effects of macroscopic materials on the fields are synthesized into boundary conditions for the fields. These boundary conditions define a set of field configurations (modes) that satisfy them. The mean value of the different dynamical variables can be evaluated for each mode inside the region where boundary conditions are imposed. If this value of the dynamical variable can change in time just through its flow on the boundaries, a conservation law for the dynamical variable associated to the field holds. For instance, the geometry of waveguides define the electromagnetic (EM) modes and the parameters which characterize them are linked to dynamical properties of the field. For rectangular symmetry, the Cartesian wave vector $\vec{k}$ is related to the momentum $\vec{P}$ of the EM field, the frequency to the energy $\mathcal{E}$ per photon, and the polarization to the spin angular momentum along $\vec{k}$, $S_z$. For cylindrical symmetry the energy $\mathcal{E}$ per photon, linear momentum $\hat{P}_z$, orbital momentum along the symmetry axis $L_z$, and helicity $S_z$, define the parameters that characterize the electromagnetic Bessel modes: frequency $\omega$, wave vector component along the symmetry axis $k_z$, azimuthal integer $m$, and polarization $\sigma$. As shown 20 years ago, Bessel modes [1], as well as other electromagnetic configurations coinciding with modes inside waveguides of a given geometry, can be generated approximately in free space by interferometric means. This has lead to the possibility of creating, in the quantum realm, photons with a variety of quantum numbers which, in fact, can be entangled [2].

The purpose of this work is to study the dynamical properties of electromagnetic waves in elliptic-cylindrical coordinates. The corresponding modes are known as Mathieu fields. The cylindrical symmetry leads, under ideal conditions, to propagation invariance along the symmetry axis. Four of the five dynamical variables behind the parameters that characterize Mathieu modes can be directly identified. The symmetry generator behind the fifth parameter is trivially obtained from the wave equations but its relation to the electromagnetic dynamical variable is not, since the symmetry generator is of second order. We shall give an explicit expression for this dynamical variable.

The zeroth order Mathieu beams were first generated in free space by an annular slit illuminated with a strip pattern produced by a cylindrical lens [3]; higher order Mathieu beams have already been generated by holographic means [4]. Given the increasing use of light to control the motion of atomic systems and microparticles we shall also make an analysis of the behavior of an atom in a Mathieu field. Emphasis will be given on the dependence of mechanical effects on the values of that parameter and the associated dynamical variable.

## II. MASSLESS SCALAR FIELD IN ELLIPTIC COORDINATES

Elliptic-cylindrical coordinates are defined by the transformations [5]

$$x + \imath y = f\cosh(u + \imath v), \quad u \in [0,\infty), \quad v \in [0,2\pi),$$

$$z = z, \tag{1}$$

where the real valued constant $f$ is half the interfocal distance and the coordinates $u$ and $v$ are the so-called radial- and angular-like variables, while $z$ is the axial variable. Unitary vectors related to a given coordinate $x$ will be written $\hat{e}_x$. The relevant scaling factor for this coordinate system is

$$h = h_u = h_v = f\sqrt{(\cosh 2u - \cos 2v)/2}. \tag{2}$$

Considering the solution of the wave equation under the assumption of propagation invariance along the $z$ direction,

$$\nabla^2\Psi = \partial^2_{ct}\Psi, \quad \Psi(\vec{r},t) = \psi(\vec{r}_\perp)e^{\imath(k_z z - \omega t)}, \tag{3}$$

the Helmholtz equation is obtained—partial derivatives with respect to a given variable $x$ are compactly denoted by $\partial_x$, also the notation $\partial_{ct} = \frac{1}{c}\partial_t$ will be used. In elliptic coordinates, the Helmholtz equation takes the form

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$$\left(\frac{1}{h^2}(\partial_u^2+\partial_v^2)+k_\perp^2\right)\psi(u,v)=0,\quad k_\perp^2=\frac{\omega^2}{c^2}-k_z^2. \tag{4}$$

This equation is separable, $\psi(u,v)=U(u)V(v)$, and yields the set of differential equations

$$(\partial_u^2-b+2q\cosh 2u)U(u)=0, \tag{5}$$

$$(\partial_v^2+b-2q\cos 2v)V(v)=0, \tag{6}$$

known as the modified and ordinary Mathieu equations, in that order, which will be called radial and angular Mathieu differential equations from now on. The real constant $q$ is related to both half the interfocal distance $f$ and the magnitude of the perpendicular component of the wave vector $k_\perp$ by

$$q=(fk_\perp/2)^2. \tag{7}$$

From a field theoretical point of view, $q$ is directly related to the perpendicular momentum carried by the wave $\psi$. For a given value of $q$, the possible values of $b$ compatible with the boundary condition $V(v+\pi)=V(v)$ or $V(v+2\pi)=V(v)$ are called the characteristic values [6,7]. They are usually ordered in ascending values and renamed $a_n$ ($b_n$) for even, $p=e$, (odd, $p=o$) solutions. The parity of the order $n$ determines if the function is $\pi$-or $2\pi$-periodic for even or odd order $n$, respectively. The mathematical set $\{\psi(u,v)^{(p,n,q)}\}$ is complete and orthogonal [8].

From Mathieu radial (5) and angular (6) equations, it is straightforward to construct an operator that shares eigenfunctions with the squared transverse momentum,

$$\mathbb{B}\psi(u,v)=b\psi(u,v),$$

$$\mathbb{B}=-\frac{f^2}{2h^2}(\cos 2v\,\partial_u^2+\cosh 2u\,\partial_v^2). \tag{8}$$

A physical interpretation of the dimensionless eigenvalue $b$ can be found writing the operator $\mathbb{B}$ in Cartesian coordinates

$$\mathbb{B}=-\left(x^2-\frac{f^2}{2}\right)\partial_y^2-\left(y^2+\frac{f^2}{2}\right)\partial_x^2+2xy\partial_x\partial_y+x\partial_x+y\partial_y,$$

$$\mathbb{B}=\frac{1}{2}(l_{z+}l_{z-}+l_{z-}l_{z+})-\frac{f^2}{2}\nabla_\perp^2\doteq l_{[z+}\,l_{z-]}-\frac{f^2}{2}\nabla_\perp^2, \tag{9}$$

where the operator $l_{z\pm}=-\imath[(\vec{r}\pm f\hat{e}_x)\times\vec{\nabla}]_z$, in the context of the quantum mechanics of a particle, is proportional to the $z$ component of the angular momentum with respect to either focii of the elliptic-cylindrical coordinate system.

Previous studies of the Helmholtz equation, Eq. (4), already report this identification [9,10]. That is the case of the analysis of the spectra of a quantum elliptic billiard for which $\mathbb{B}$ commutes with the free particle Hamiltonian [9,11]. In fact, $\beta=l_{z+}l_{z-}+2mf^2H$ is a constant of motion of the classical analog of this system with $m$ the mass of the particle and $H$ the classical Hamiltonian within the billiard. It has been found [9,12] that, if $l_{z+}l_{z-}>0$ then $\beta>2mf^2E$ and the classical trajectory of the particle repeatedly touches an ellipse characterized by $\cosh u_{\text{lim}}=\beta/2mf^2E$. While, if $l_{z+}l_{z-}<0$ the trajectory always lies between the focii touching a hyperbola determined by $\cos v_{\text{lim}}=\beta/2mf^2E$. In that case $\beta$ is a positive number and $l_{z+}l_{z-}$ has a lower limit given by $-2mf^2E$.

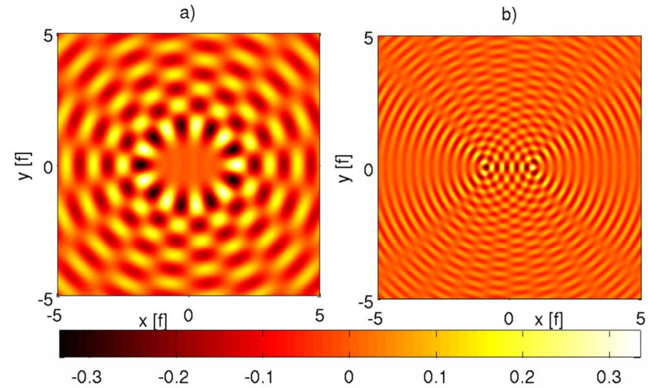


FIG. 1. (Color online) Normalized even Mathieu's functions (a) $\psi(u,v)^{(p=e,n=9,q=10)}$ and (b) $\psi(u,v)^{(p=e,n=9,q=80)}$. The corresponding values of the parameter $b$ are 81.6283 and 124.1067, so that $l_{[z+}\,l_{z-]}=61.6283$ and $l_{[z+}\,l_{z-]}=-44.1067$, respectively. The half focal distance establishes length units, $f=1$.

The normalized amplitude of the scalar wave function $\psi(u,v)$ is illustrated in Fig. 1 for positive and negative values of $l_{[z+}\,l_{z-]}$. Notice that, in analogy to the classical particle problem, for $l_{[z+}\,l_{z-]}>0$ ellipses can be observed where the amplitude is approximately constant while for $l_{[z+}\,l_{z-]}<0$ hyperbolic patters of similar amplitude are clearly distinguished.

## III. DYNAMICAL VARIABLES OF AN ELLIPTIC ELECTROMAGNETIC MODE

Given a complete set of scalar solutions $\{\Psi_\kappa\}$ of the wave equation, a joint set of complete electromagnetic modes can be obtained by considering the scalar functions $\Psi_\kappa$ as Hertz potentials [13,14]. In Coulomb gauge, any solution of the wave equation for the vector electromagnetic field $\vec{A}(\vec{r},t)$

$$\nabla^2\vec{A}(\vec{r},t)=\partial_{ct}^2\vec{A}(\vec{r},t) \tag{10}$$

can be written as a superposition of modes

$$\vec{A}_\kappa(\vec{r},t)=\mathcal{A}_\kappa^{(\text{TE})}\partial_{ct}\vec{\nabla}\times(\hat{e}_z\Psi_\kappa)+\mathcal{A}_\kappa^{(\text{TM})}[\vec{\nabla}_\perp(\vec{\nabla}\cdot\hat{e}_z\Psi_\kappa)-\hat{e}_z\nabla_\perp^2\Psi_\kappa], \tag{11}$$

where $\kappa$ denotes the labels that characterize a given scalar solution $\Psi_\kappa$. The constants $\mathcal{A}_\kappa^{(\text{TE})}$ and $\mathcal{A}_\kappa^{(\text{TM})}$ are proportional to the amplitude of the transverse electric (TE) and transverse magnetic (TM) EM fields as can be directly seen from their connection with the associated electric $\vec{E}$ and magnetic $\vec{B}$ fields,

$$\vec{E}=-\partial_{ct}\vec{A},\quad \vec{B}=\vec{\nabla}\times\vec{A}. \tag{12}$$

As usual, although expressions may involve complex functions, just the real part of them defines the corresponding physical quantities. In the case of propagation invariant electromagnetic fields in elliptic coordinates

$$\vec{A}_\kappa(\vec{r},t) = A^{(\mathrm{TE})}\vec{\mathbb{M}}\Psi_\kappa + \mathcal{A}^{(\mathrm{TM})}\vec{\mathbb{N}}\Psi_\kappa, \tag{13}$$

where the vector operators are given by the expressions

$$\vec{\mathbb{M}} = \frac{1}{h}\partial_{ct}(\hat{e}_u\partial_v - \hat{e}_v\partial_u), \quad \vec{\mathbb{N}} = \frac{1}{h}\partial_z(\hat{e}_u\partial_u + \hat{e}_v\partial_v) - \hat{e}_z\nabla_\perp^2. \tag{14}$$

As a consequence,

$$\vec{E}_\kappa(\vec{r},t) = -\mathcal{A}^{(\mathrm{TE})}\partial_{ct}\vec{\mathbb{M}}\Psi_\kappa - \mathcal{A}^{(\mathrm{TM})}\partial_{ct}\vec{\mathbb{N}}\Psi_\kappa,$$

$$\vec{B}_\kappa(\vec{r},t) = \mathcal{A}^{(\mathrm{TE})}\partial_{ct}\vec{\mathbb{N}}\Psi_\kappa - \mathcal{A}^{(\mathrm{TM})}\partial_{ct}\vec{\mathbb{M}}\Psi_\kappa. \tag{15}$$

In general, it is expected that the electromagnetic modes given by Eq. (11) inherit symmetries of the scalar field with analogous dynamical variables. For the elliptic-cylindrical case, invariance under space reflection with respect to the $Y$ axis leads to parity conservation. Meanwhile, invariance under spatial translation along the main direction of propagation of the mode is reflected in the fact that the field momentumlike integral

$$P_z^{(i,\kappa,\kappa')} = \frac{1}{4\pi c}\int d^3x[(\vec{E}_\kappa^{(i)} \times \vec{B}_{\kappa'}^{(i)})_z], \quad i = \mathrm{TE,TM} \tag{16}$$

integrated over the whole space is independent of time. Time homogeneity implies that the energylike integral

$$\mathcal{E}^{(i,\kappa,\kappa')} = \frac{1}{4\pi}\int d^3x[\vec{E}_\kappa^{(i)}(\vec{r},t)\cdot\vec{E}_{\kappa'}^{(i)}(\vec{r},t) + \vec{B}_\kappa^{(i)}(\vec{r},t)\cdot\vec{B}_{\kappa'}^{(i)}(\vec{r},t)] \tag{17}$$

is also constant. In fact, $P_z^{(i,\kappa,\kappa')}$ and $\mathcal{E}^{(i,\kappa,\kappa')}$ are proportional with $ck_z/\omega$, the constant of proportionality. By construction $\hbar k_\perp = \hbar\sqrt{\omega^2/c^2 - k_z^2}$ would yield the magnitude of the transverse component of the momentum which determines the separation constant $q$, Eq. (7).

Standard quantization rules require a proper normalization selection for the EM modes so that each photon carries an energy $\hbar\omega$. Using Eq. (12a) from Ref. [15]

$$\int_{-\infty}^{\infty} dz \int_0^\infty du \int_0^{2\pi} dv h_u h_v \Psi_\kappa(u,v,z)\Psi'_\kappa(u,v,z) = 2\pi^2 f^2 s_\kappa \delta(k_z - k'_z)\delta(q-q')\delta_{n,n'}, \tag{18}$$

$$s_{e,2n,q} = \frac{V_{e,2n,q}(0)V_{e,2n,q}(\pi/2)}{A_0^{(2n)}},$$

$$s_{e,2n+1,q} = -\frac{V_{e,2n+1,q}(0)V'_{e,2n+1,q}(\pi/2)}{q^{1/2}A_1^{(2n+1)}},$$

$$s_{o,2n+2,q} = \frac{V'_{o,2n+2,q}(0)V'_{o,2n+2,q}(\pi/2)}{qB_2^{(2n+2)}},$$

$$s_{o,2n+1,q} = \frac{V'_{o,2n+1,q}(0)V_{o,2n+1,q}(\pi/2)}{q^{1/2}B_1^{(2n+1)}}, \tag{19}$$

where $A_m^{(n)}$ and $B_m^{(n)}$ are the standard Mathieu coefficients [6,7], this normalization is trivially performed. Defining now the generalized *number operator*:

$$\hat{N}_\kappa^{(i)} = \frac{1}{2}(\hat{a}_\kappa^{(i)\dagger}\hat{a}_\kappa^{(i)} + a_\kappa^{(i)}\hat{a}_\kappa^{(i)\dagger}), \quad [a_\kappa^{(i)}, a_{\kappa'}^{(j)\dagger}] = \delta_{i,j}\delta_{\kappa,\kappa'}, \tag{20}$$

the quantum energy and the momentum along $z$ operators take the form

$$\hat{\mathcal{E}} = \sum_{i,\kappa}\hbar\omega\hat{N}_\kappa^{(i)}, \quad \hat{\mathcal{P}}_z = \sum_{i,\kappa}\hbar k_z\hat{N}_\kappa^{(i)}, \tag{21}$$

allowing the identification of $\hbar k_z$ and $\hbar\omega$ with the photon momentum along $z$ and the photon energy, in that order.

For scalar fields and in the case of space-time continuous symmetries, the generators of infinitesimal transformations become good realizations of the corresponding dynamical operator. In that sense, the rotationlike operator $B$ can be related to the product of angular momenta $l_{[z+}\, l_{z-]}$ and the eigenvalue equation $\mathbb{B}\Psi_{\omega,k_z,p,n} = b\Psi_{\omega,k_z,p,n}$ is interpreted as a manifestation of the scalar wave function $\Psi_{\omega,k_z,p,n}$ carrying a well-defined value of that angular momentum product.

The standard procedure to find the electromagnetic analog of $l_{[z+}\, l_{z-]}$ would be to apply Noether theorem to the field Lagrangian

$$\mathcal{L}_{EM} = (1/4)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{22}$$

for the transformation generated by $\mathbb{B}$ on space variables and on the electromagnetic fields $A_\mu$. Standard Noether theorem [16] concerns first order differential operators as generators of continuous symmetries, so that, if under an infinitesimal transformation that modifies the coordinates and field functions by

$$\delta x_\mu = \sum_\nu X_\mu^\nu \delta\omega_\nu, \quad \delta A_\mu = \sum_\nu \varphi_\mu^\nu \delta\omega_\nu, \tag{23}$$

the Lagrangian is left invariant, the current

$$\Theta_\rho^\nu = -\sum_\lambda \frac{\partial\mathcal{L}}{\partial A_{\lambda,\nu}}\left(\varphi_{\lambda\rho} - \sum_\sigma A_{\lambda,\sigma}X_\rho^\sigma\right) - \mathcal{L}X_\rho^\nu \tag{24}$$

has zero divergence $\partial_\nu\Theta_\rho^\nu = 0$. As a consequence $\Theta_\rho^0$ can be considered as the density of a dynamical variable whose integrated value over a volume can change only due to the flux of the current $\Theta_\rho^i$ through a surface. For the circular cylindrical problem, the assumption of isotropy of space via the effects of the infinitesimal rotation generator $[\vec{r}\times\vec{\nabla}]_z$ on $A_\mu$ and $x_\mu$ leads to the identification of

$$J_z = \frac{1}{4\pi c}\int_{\mathcal{V}}\sum_i E_i[\vec{r}\times\vec{\nabla}]_z A_i d^3x + \frac{1}{4\pi c}\int_{\mathcal{V}}(\vec{E}\times\vec{A})_z d^3x, \tag{25}$$

as the $z$ component of the angular momentum of the electromagnetic field in a volume $\mathcal{V}$ [16]. The first integral involves the anti-Hermitian differential operator $[\vec{r}\times\vec{\nabla}]_z = \partial_\varphi$ and is associated to the orbital angular momentum (notice that the standard Hermitian angular momentum operator is $\hat{L}_z = -\imath\hbar\partial_\varphi$). The second integral is independent of the choice of origin, arises from the field variation $\delta A_\mu$, and is directly related to the polarization of the field. It has been identified with the field helicity [17,18]. It can be shown that in the Coulomb gauge [19]

$$\vec{J} = \frac{1}{4\pi c}\int_{\mathcal{V}} \vec{r}\times(\vec{E}\times\vec{B})d^3x - \frac{1}{4\pi c}\oint_{\mathcal{S}} \vec{E}[(\vec{r}\times\vec{A})\cdot\hat{n}]d^2x,$$

where $\mathcal{S}$ is the surface enclosing the volume $\mathcal{V}$.

Since the electromagnetic field $A_\mu$ has a well-defined transformation rule under rotations which is independent of the origin of space coordinates, the second term in Eq. (25),

$$S_z = \frac{1}{4\pi c}\int_{\mathcal{V}} (\vec{E}\times\vec{A})_z d^3x, \tag{26}$$

is the fourth dynamical variable associated to the Mathieu electromagnetic field. This can be directly verified by substituting the general expression for the vectors $\vec{E}$ and $\vec{A}$ in terms of the elliptic modes. As usual, for a given value of mode indices $\kappa$, the helicity $S_z$ is different from zero only if the amplitudes $\mathcal{A}^{(\mathrm{TM})}$ and $\mathcal{A}^{(\mathrm{TE})}$ are complex. The concept of circular and linear polarization of Mathieu waves is analyzed in Ref. [20] classically. The quantum analysis can be carried out in complete analogy with the study in Ref. [18] for Bessel fields so that

$$\hat{S}_z = \hbar\sum_\kappa \frac{\imath k_z c}{2\omega}(\hat{a}_m^{(\mathrm{TE})\dagger}\hat{a}_m^{(\mathrm{TM})} - \hat{a}_m^{(\mathrm{TE})}\hat{a}_m^{(\mathrm{TM})\dagger}). \tag{27}$$

In the problem treated here, the generator of the transformation $\mathbb{B}$ is a Hermitian second order differential operator obviously related to the isotropy of space. It depends on the position of the two focii of the elliptic transversal coordinates and in that sense should be analogous to $L_z$. The proposal is to identify the electromagnetic dynamical variable related to $\mathbb{B}$ with the integral

$$\mathcal{B} = \frac{1}{4\pi c}\int_{\mathcal{V}}\sum_i E_i\imath\mathbb{B}A_i d^3x, \tag{28}$$

in complete analogy with Eq. (25).

In order to corroborate that $\mathcal{B}$ is the fifth dynamical variable directly associated to Mathieu electromagnetic waves, notice that

$$\sum_i A_i^{\kappa'}\mathbb{B}A_i^{\kappa} = b\vec{A}^{\kappa'}\cdot\vec{A}^{\kappa} + [k'kA_{\kappa'}^{(\mathrm{TE})}\mathcal{A}_\kappa^{(\mathrm{TE})} + k_z'k_z\mathcal{A}_{\kappa'}^{(\mathrm{TM})}\mathcal{A}_\kappa^{(\mathrm{TM})}]$$
$$\times[\vec{\nabla}\cdot\vec{C} - \Psi_{\kappa'}\nabla_\perp^2\Psi_\kappa], \tag{29}$$

where the vector $\vec{C} = -2(\mathbb{M}\Psi_{\kappa'})(\vec{r}\times\vec{\mathbb{N}}_\perp\Psi_\kappa) + \Psi_{\kappa'}\vec{\mathbb{N}}_\perp\Psi_\kappa$ —with $\vec{\mathbb{N}}_\perp = \frac{1}{h}\partial_z(\hat{e}_u\partial_u + \hat{e}_v\partial_v)$.

Evaluating the integral in Eq. (28) over a volume $\mathcal{V}$, using Eq. (29), the first resulting term is proportional to $b\vec{E}^{\kappa'}\cdot\vec{A}^{\kappa}$ involving the same integrals appearing in the energylike density, Eq. (17). The second term defines a flux of $\mathcal{B}$ through the surface around the integration volume $\mathcal{V}$. The third term adds up to the last term in the expression of operator $\mathbb{B}$, Eq. (9). Equation (29) supports the identification of $\mathcal{B}$ as the electromagnetic dynamical variable linked to the generator $\mathbb{B}$.

In the quantum realm the corresponding operator is written as

$$\hat{\mathcal{B}} = \sum_\kappa \hbar(b+1)\hat{N}_\kappa^{(\mathrm{TE})} + \hbar\left(b + \frac{k_z^2c^2}{\omega^2}\right)\hat{N}_\kappa^{(\mathrm{TM})}. \tag{30}$$

In the paraxial limit, the helicitylike factor $k_zc/\omega \sim 1$ so that $\hat{\mathcal{B}} \approx \sum_{i,\kappa}\hbar(b+1)\hat{N}_\kappa^{(i)}$.

Notice that the dynamical variable $\hat{\mathcal{B}}$ for a photon has units of $\hbar$, although for material particles the quantum variable associated to $l_{z+}l_{z-}$ has as a natural unit $\hbar^2$. The interpretation of a dynamical variable for the EM field is usually linked to the interchange of this mechanical variable with charged particles or atomic systems. The measurement of $\hat{\mathcal{B}}$ is expected to be related to changes in values of $l_{z+}l_{z-}$ although they have different units. It is thus essential to clarify how the absorption and/or emission of a Mathieu photon by a particle alters its motion.

## IV. MECHANICAL EFFECTS ON ATOMS

Theoretical and experimental analysis on the interaction between light and microscopic particles, as well as analysis on the interaction of light and cold atoms, have yielded very important results in the last 3 decades. In these areas, the use of structured light beams with peculiar dynamical properties plays an increasingly important role [21]. In the particular case of Mathieu-like beams, it is possible to generate an elliptical orbital motion of trapped microscopic particles [22]. A detailed theoretical description of this phenomenon requires the exhibition of a clear link between the observed motion and the parameters that characterize the beam, which are directly related to the mechanical properties of the field described here.

The mechanical effects of a Mathieu electromagnetic wave on a cold atom shall briefly be described under the assumption that the atom kinetic energy is low enough to be sensitive to light forces but large enough to admit a description in terms of Newton equations. The standard semiclassical approach is taken, as in the pioneering works by Letokhov [23] and Gordon and Ashkin [24]. In this approximation, a monochromatic electromagnetic wave describable by a coherent state couples to the dipole moment of an atom. This dipole moment $\vec{\mu}_{12}$ is related to the electromagnetic transitions between the atom levels that, for simplicity sake, will be taken to have just two accessible options. The gradient of the coupling $g = \imath\vec{\mu}_{12}\cdot\vec{E}/\hbar = |g|e^{\imath\phi}$ determines the explicit expression for the average semiclassical velocity dependent force through the vectors $\vec{\alpha} = \vec{\nabla}\log(|g|)$ and $\vec{\beta} = \vec{\nabla}\phi$ [24]. The nonlinear Newton equations for Mathieu waves have a rich structure that deserves a deep study on its own. Here, just some results that illustrate the relevance of the parameter $b$ are reported. Atomic transitions with changes in the atomic internal angular momentum $\delta\mathrm{m} = \pm 1$ are proportional to $\hat{e}_{\pm}\cdot\vec{E}(\vec{r},t)$. For Mathieu waves, this factor is proportional to $(\omega/c)\mathcal{A}^{(\mathrm{TE})}\pm\imath k_z\mathcal{A}^{(\mathrm{TM})}$ reinforcing the interpretation of the latter expression as a signature of circular polarization. In order to induce transitions with $\delta\mathfrak{m}=\pm1$ with equal probability, and to avoid any complication arising from a component of the electric field along $z$, it will be assumed that $\mathcal{A}^{(\mathrm{TM})}=0$. Under these conditions,

$$\vec{\alpha}=\frac{\hat{e}_u}{h}\left(\frac{\partial_v\Psi\partial^2_{uv}\Psi+\partial_u\Psi\partial^2_u\Psi}{(\partial_u\Psi)^2+(\partial_v\Psi)^2}-\frac{\sinh 2u}{\cosh 2u-\cos 2v}\right)+\frac{\hat{e}_v}{h}\left(\frac{\partial_u\Psi\partial^2_{uv}\Psi+\partial_v\Psi\partial^2_v\Psi}{(\partial_u\Psi)^2+(\partial_v\Psi)^2}-\frac{\sin 2v}{\cosh 2u-\cos 2v}\right),\tag{31}$$

$$\vec{\beta}=\frac{\hat{e}_u}{h}\left(\frac{\partial_u\Psi\partial^2_{uv}\Psi-\partial_v\Psi\partial^2_u\Psi}{(\partial_u\Psi)^2+(\partial_v\Psi)^2}-\frac{\sin 2v}{\cosh 2u-\cos 2v}\right)-\frac{\hat{e}_v}{h}\left(\frac{\partial_v\Psi\partial^2_{uv}\Psi-\partial_u\Psi\partial^2_v\Psi}{(\partial_u\Psi)^2+(\partial_v\Psi)^2}-\frac{\sinh 2u}{\cosh 2u-\cos 2v}\right).\tag{32}$$

The expression for the average semiclassical velocity dependent force [24] valid for both propagating and standing beams is

$$\langle\vec{f}\rangle=\frac{\hbar\Gamma\{[(\vec{v}\cdot\vec{\alpha})(1-p)(1+p)^{-1}+\Gamma/2]\vec{\beta}+[(\vec{v}\cdot\vec{\beta})-\delta\omega]\vec{\alpha}\}}{(1-p')p'^{-1}\Gamma+2\vec{v}\cdot\vec{\alpha}(1-p/p'-p)(1+p)^{-1}}\tag{33}$$

with $\Gamma$ the Einstein coefficient, $\Gamma=4k^3|\vec{\mu}_{12}|^2/3\hbar$, $\delta\omega$ the detuning between the wave frequency $\omega$ and the transition frequency $\omega_0$, $\delta\omega=\omega-\omega_0$, $p=2|\vec{g}|/[(\Gamma/2)^2+\delta\omega^2]$ a parameter linked to the difference $D$ between the populations of the atom two levels, $D=1/(1+p)$, and finally $p'=2|g|^2/|\gamma'|^2$, with $\gamma'=(\vec{v}\cdot\vec{\alpha})(1-p)(1+p)^{-1}=\Gamma/2+\imath[-\delta\omega+(\vec{v}\cdot\vec{\beta})]$.

This brief study is focused on the red detuned far-off resonance light case so that nonconservative terms arising from the velocity dependence of the force are not dominant [25]. This regime is particularly relevant in the context of optical lattices [26]. Following Ref. [25], the laser beam is considered with a 67 nm detuning to the red of the $5\,^2S_{1/2}$–$5\,^2P_{1/2}$ transition at 795 nm; a $6\times10^5$ W/cm$^2$ irradiance is assumed. The trajectories of the atoms are described taking the laser wavelength as the unit of length and as the unit of time the inverse of the Einstein coefficient $\Gamma$ which is $3.7\times10^7$ s$^{-1}$ for the state $5\,^2P_{1/2}$ of $^{85}$Rb.

Since Newton equations are highly nonlinear, it is expected that the motion of the particles will not have a simple structure. Nevertheless, there are some general characteristics of the atom motion that can be predicted without making a numerical analysis. Thus, since the $z$-dependence of the beam is just on its phase, $k_z z$, no longitudinal confinement is expected. In the standard paraxial regime, $k_\perp \ll k_z$, a correspondingly small transfer of radial momentum to the atom will occur. Under these conditions, the atom would be confined by transversal light potential wells whenever its height is larger than the initial atom kinetic energy. Conversely, if $k_\perp\sim k_z$ the atom may dwell around different transverse wells. Typical trajectories are shown in Figs. 2 and 3 [31].

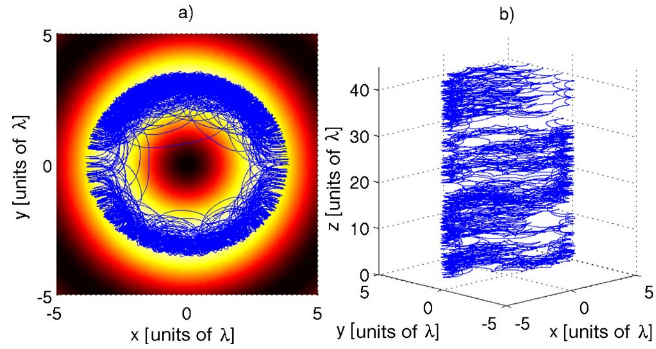


FIG. 2. (Color online) Trajectory of an atom driven by an even Mathieu beam of order $n=0$ in the paraxial regime with half focal distance $f=\lambda$, and beam parameters $q=0.0984$ and $b=-0.0048$. The initial conditions for the atom are $u_0=1.5\lambda$, $v_0=\pi/4$, $z_0=0\lambda$, $\dot{u}_0=0.1\lambda\Gamma$, $\dot{v}_0=-0.001\Gamma$, and $\dot{z}_0=0.001\lambda\Gamma$.

Using several numerical simulations [28], the correlations between the time average value of the atomic

$$l_+l_-=\langle l_{z+}l_{z-}\rangle_t=m^2\langle[(\vec{r}+f\hat{e}_x)\times\vec{v}]_z[(\vec{r}-f\hat{e}_x)\times\vec{v}]_z\rangle_t$$

and the $b$ value of the light mode were studied.

As illustrated in Fig. 4, in general, the variable $l_{z+}l_{z-}$ exhibits large fluctuations in an initial transitory stage. For paraxial beams, there is a time $T$ such that the time average $l_+l_-$ over any interval $t_0<t<t_0+T$ becomes independent of $t_0$ whenever $t_0>T$. The order of magnitude of $T$ is typically $10^4\Gamma$. A rich structure, consistent with the frequent atomic recoils due to the light potential wells, could be observed at lower scales. As expected, the specific numerical results depend on all the involved variables. Once $q$ and $f$ are fixed, and for the same atomic initial conditions, a monotonic nonlinear increase of $l_+l_-$ as a function of $b$ is observed in general [29] as illustrated in Fig. 5. For nonparaxial modes the time average $l_+l_-$ is highly dependent on the time interval considered due to the persistent increase of the radii of the atom motion illustrated in Fig. 3.

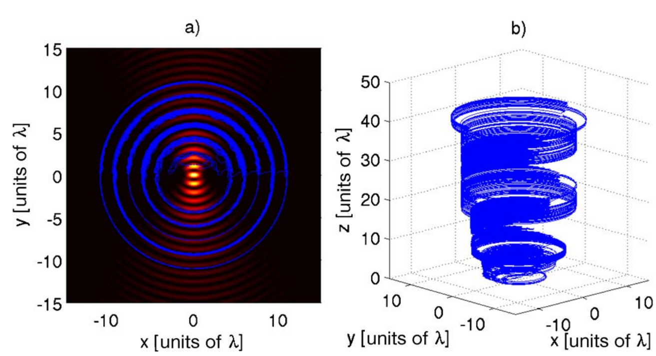


FIG. 3. (Color online) Trajectory of an atom driven by an odd Mathieu beam or order $n=1$ with half focal distance $f=\lambda$, and beam parameters $q=3.5531$ and $b=-3.5924$. The initial conditions for the atom are $u_0=1.5\lambda$, $v_0=\pi/4$, $z_0=0\lambda$, $\dot{u}_0=0.1\lambda\Gamma$, $\dot{v}_0=-0.001\Gamma$, and $\dot{z}_0=0.001\lambda\Gamma$. Notice that the particle dwells on bright zones of the field.

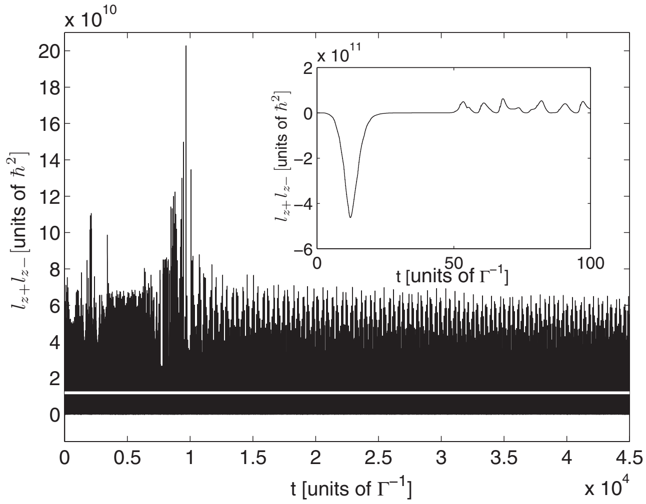


FIG. 4. Typical behavior of $l_{z+}l_{z-}$ as a function of time (black) and its average (white) for paraxial beams. The initial conditions for the atom are $u_0=1.2\lambda$, $v_0=\pi/4$, $z_0=0\lambda$, $\dot{u}_0=0.2\lambda\Gamma$, $\dot{v}_0=-0.001\Gamma$, and $\dot{z}_0=0.001\lambda\Gamma$. The odd Mathieu beam is of order $n=7$ with half focal distance $f=\lambda$ and parameter $q=0.1$. The inset graphic is a zoom focused on the initial behavior of $l_{z+}l_{z-}$.

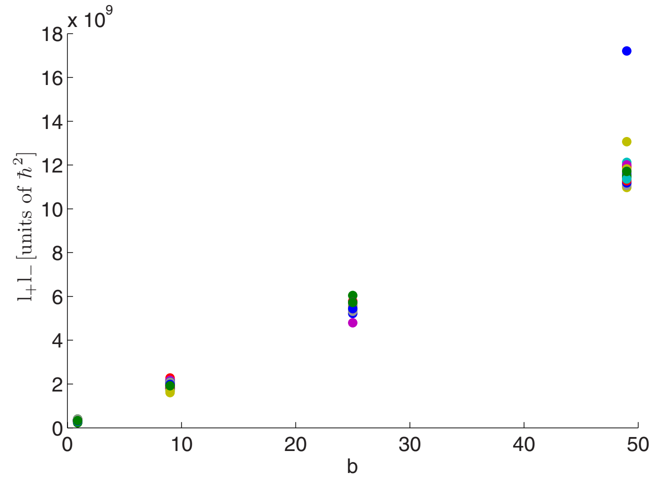


FIG. 5. (Color online) Average $l_+l_-$ of the atom, in units of $\lambda^2\Gamma^{-1}$, as a function of the $b$ parameter of an odd Mathieu beam of order $n=1,3,5,7$ with half focal distance $f=\lambda$ and parameter $q=0.1$. The initial conditions for the atom that yield $l_+l_-$ within the range plotted here are $u_0\in[0.9,1.5]\lambda$, $v_0\in[\pi/16,\pi/4]$, $\dot{u}_0\in[0.1,0.2]\lambda\Gamma$, $\dot{v}_0=-0.001\Gamma$, and $\dot{z}_0=0.001\lambda\Gamma$.

## V. CONCLUSIONS

The electromagnetic modes with elliptic-cylindrical symmetry are characterized by their polarization and the extended set of parameters $\kappa=\{\omega,k_z,p,q,b\}$, that is the field frequency $\omega$ and wave vector axial component $k_z$ related to the energy $\mathcal{E}$ and $z$ component of the linear momentum $P_z$; the parity $p$ of Mathieu functions and the parameter $q$ which, as in the scalar case, is related to the perpendicular component of the wave vector in units of the focal distance. It has been proposed and shown that the transformation generator $\mathbb{B}$ arising from elliptic symmetry is related to the dynamical EM variable $\mathcal{B}$ of the electromagnetic field, giving a physical significance to $b$.

It has been exhibited that the motion of cold atoms in Mathieu beams can be used to "measure" $\mathcal{B}$ since a strong correlation between the particle product of angular momenta $l_+l_-$ and the parameter $b$ can be approximately isolated by using paraxial TE modes in the far off-resonance regime.

The use of light beams with elliptical-cylindrical symmetry has potential applications in controlling the mechanical motion of atomic systems which could include nanoparticles. Nowadays, it is well-recognized that laser-driven nanoparticles have a variety of uses in nanofluidics, nanobiotechnology, and biomedicine. However, although the possibility of optically trapping gold nanoparticles was demonstrated in 1994 [30], developing tweezers for nanoparticles is not straightforward. The gradient forces with conventional beams fall off with particle size. The geometry of elliptical beams adds the focal distance $2f$ as a parameter to tailor gradient forces besides opening the possibility of using the mechanical variable $\mathcal{B}$ to select over a wider kind of motions.

The inclusion and further study of this new dynamical variable could also help elucidate modern concerns on the "twisted" properties of light, such as why Mathieu functions are intelligent states for the conjugate pair exponential of the angle-angular momentum [27].

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