Low-frequency atomic stabilization and dichotomy in superintense laser fields from the high-intensity high-frequency Floquet theory

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The Floquet problem for the interaction of an atom with a monochromatic laser field of frequency ω was studied long ago for the case of high ω and arbitrary intensity using the "high-frequency Floquet theory" (HFFT). The two parameters of the theory are the frequency ω and the classical excursion parameter $\alpha_0 \equiv E_0\omega^{-2}$, where E_0 is the electric field strength. HFFT solves the Floquet system by successive iterations. Convergence of the iteration procedure was shown to be ensured by the condition that ω be suffiently large with respect to some typical atomic excitation energy. We now establish that the same iteration procedure is capable of handling the case of low frequency at sufficiently high intensity. This leads to the conclusion that in this case the ionization rates display quasistationary stabilization also at low ω. The concept is thus not exclusively related to high frequencies, as widely assumed. In addition, it suggests that a more appropriate designation for the theory should be "high-intensity, high-frequency Floquet theory" (HIHFFT). Our general results are applied to a frequently used one-dimensional (1D) soft-core potential model, for which explicit analytic results can be obtained for the quasienergies and wave functions from the general HIHFFT formulas. The relevance of these quasistationary results for the case of laser pulses is pointed out.

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I. INTRODUCTION

Atomic stabilization is a well-documented phenomenon [1], occurring both for short laser pulses ["dynamic stabilization" (DS)] and for long, quasimonochromatic pulses ["quasistationary stabilization" (QS) [2]]. In the first case it is displayed by the decrease with intensity of the ionization probabilities at the end of the pulse, calculated with the time-dependent Schrödinger equation (TDSE), in the other, by the decrease of the decay rates with increasing intensity, calculated with the Floquet formalism. Much of the work was carried out using the space-translated version of the Schrödinger equation. Its Floquet version, used in the following, has as basic parameters the frequency ω and the excursion of the classical motion, $\alpha_0=E_0\omega^{-2}$, instead of the frequency ω and electric field strength E_0 used in the laboratory frame. Within this approach, a formal iteration procedure was developed to handle the high-frequency case [3]. To lowest order, the approach extracts a "structure equation" from the differential Floquet system, which describes a stable, nonionizing atom, and imposes dichotomy on its eigenfunctions. The following iteration yields decay rates and level shifts. The procedure was shown qualitatively to converge at high frequency (high with respect to some average atomic excitation energy) and arbitrary intensity. The formalism was termed "high-frequency Floquet theory" (HFFT) [4]. The rates calculated are decreasing functions at large α_0 (sometimes in an oscillatory manner), which is per definition QS [1]. This has tacitly accredited the idea that stabilization is an essentially high-frequency phenomenon, and all studies made pertain to this case.

However, it was realized long ago that HFFT was capable of treating also cases of low frequency and high intensity, although no proofs were given [5,6]. In the following, we undertake a systematic study of this possibility, including the issue of stabilization. We show formally that the iteration procedure of HFFT converges indeed at low frequency ω if the intensity (or α_0) is sufficiently high. This leads to the existence of QS for the ionization rates, thus showing that QS is not exclusively a high-frequency phenomenon. It also suggests that the proper designation of the formalism should be "high-intensity, high-frequency Floquet theory" (HIHFFT), rather than simply HFFT.

Our results are presented as follows. Section II is devoted to our Floquet results in three dimensions (3D). In Sec. II A we recall those aspects of Floquet theory needed to apply the iteration procedure of HIHFFT. In Sec. II B we develop the formal iteration procedure. In Sec. II C we analyze the convergence of the theory and show that the criterion needed at high ω and arbitrary intensity (or α_0) actually covers also the case of large intensity (α_0) and low ω. In order to gain more physical insight, we apply HIHFFT to a popular 1D model with soft-core potential for which the theory can be worked out analytically to a great extent; see Sec. III. The eigenvalues and eigenfunctions of the structure equation are derived in Sec. III A, the first-order correction to the quasienergy is obtained in Sec. III B, and the expression of the Floquet components in Sec. III C. Conclusions are drawn in Sec. IV.

We emphasize that our results are based entirely on the nonrelativistic Schrödinger equation, with no allowance for relativistic corrections, which become important at high intensities. Our large-α_0 limits are to be understood in the mathematical sense, within nonrelativistic theory.

II. LARGE-α_0 HIHFFT IN 3D

Our starting point is the space-translated Schrödinger equation for a one-electron atom in a laser field $\mathbf{A}(t)$:

$$\left(\frac{1}{2}\mathbf{P}^2 + V[\mathbf{r} + \boldsymbol{\alpha}(t)]\right)\Psi = i\frac{\partial\Psi}{\partial t}, \tag{1}$$

where

$$\boldsymbol{\alpha}(t) \equiv \frac{1}{c}\int_0^t \mathbf{A}(t')dt'. \tag{2}$$

We shall be using atomic units (a.u.) throughout the paper.

A. Floquet theory background

For simplicity, we shall consider in the following the case of a linearly polarized field:

$$\mathbf{E}(t) = \mathrm{E}_0\mathbf{e}\cos\omega t, \quad \boldsymbol{\alpha}(t) = \alpha_0\mathbf{e}\cos\omega t, \tag{3}$$

where $\mathbf{e}$ is the polarization vector, and α_0 is the excursion parameter defined above.

We write Floquet solutions as

$$\psi(\mathbf{r},t) = e^{-iEt}\sum_n \phi_n(\mathbf{r})e^{-in\omega t}. \tag{4}$$

The Floquet components $\{\phi_n(\mathbf{r})\}$ need to satisfy the system of coupled differential equations

$$[H - (E + n\omega)]\phi_n(\mathbf{r}) = -\sum_m^{(m\neq n)} V_{n-m}(\alpha_0;\mathbf{r})\phi_m(\mathbf{r}), \tag{5}$$

$$H \equiv \frac{1}{2}\mathbf{P}^2 + V_0(\alpha_0;\mathbf{r}), \tag{6}$$

where $V_n(\alpha_0;\mathbf{r})$ are the Fourier components of $V[\mathbf{r}+\boldsymbol{\alpha}(t)]$:

$$V_n(\alpha_0;\mathbf{r}) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\chi}V(\mathbf{r} + \alpha_0\mathbf{e}\cos\chi)d\chi. \tag{7}$$

Since $V(\mathbf{r})$ is real and assumed to be even, we have

$$V_n(-\mathbf{r}) = (-1)^n V_n(\mathbf{r}), \quad V_{-n}(\mathbf{r}) = V_n(\mathbf{r}). \tag{8}$$

Note that, for linear polarization, Eq. (3), $V_n(\mathbf{r})$ is real. The conditions Eq. (8) lead to the possibility of finding solutions with generalized parity $P=0$ (even) or 1 (odd), such that $\phi_n(-\mathbf{r}) = (-1)^{P+n}\phi_n(\mathbf{r})$ for all n.

If the column $\{\phi_n\}$ represents a solution of the Floquet system Eq. (5) for quasienergy E and parity P, it is easy to check that the column $\{\phi_n^{(q)}\}$, where $\phi_n^{(q)} = \phi_{n+q}$, $E^{(q)} = E + q\omega$, $P^{(q)} = P + q$, and q is an integer, represents a solution for quasienergy $E^{(q)}$ and parity $P^{(q)}$. Although the two solutions are mathematically distinct, they are physically equivalent, as they lead to the same Floquet solution Eq. (4) of the TDSE ("Floquet redundancy"). It is desirable to select a definite representative for each set, if possible. As, at vanishing intensity, the Floquet system reduces to the field-free Schrödinger equation, we shall select as representative that solution for which at $\alpha_0 \to 0$, Re E coincides with the unperturbed atomic energy (the "natural normalization convention".

Boundary conditions need to be imposed on the components (channel functions) ϕ_n in order to ensure the uniqueness of the solution $\{\phi_n\}$. These are expressed in terms of the channel momenta k_n, defined by $k_n^2/2 \equiv (E + n\omega)$. To study bound states and ionization, the boundary conditions are chosen of the Gamow-Siegert type. Two classes of solutions emerge. The first one is that of *physical solutions*, which have outgoing asymptotic waves in all open channels n (for which Re $k_n^2 > 0$) and have asymptotic decay in all closed channels n (for which Re $k_n^2 < 0$). This leads to

$$\mathrm{Re}\, k_n < 0, \quad \mathrm{Im}\, k_n > 0, \quad |\mathrm{Re}\, k_n| < \mathrm{Im}\, k_n \ \text{(closed channels)}, \tag{9}$$

$$\mathrm{Re}\, k_n > 0, \quad \mathrm{Im}\, k_n < 0, \quad \mathrm{Re}\, k_n > |\mathrm{Im}\, k_n| \quad \text{(open channels)}. \tag{10}$$

Equation (9) forces a closed channel function $\phi_n(\mathbf{r})$ to decay exponentially at infinity, whereas Eq. (10) forces an open channel function $\phi_n(\mathbf{r})$ to blow up exponentially. The second class is that of *nonphysical solutions* (also called "shadows"), which do not satisfy all of these conditions. Imposing the boundary conditions Eqs. (9) and (10) leads to an eigenvalue problem: the "quasienergy" E in Eq. (5) needs to be a well-defined complex number $E \equiv W - (i/2)\Gamma$, in order that $\{\phi_n\}$ satisfy Eq. (5). From Eq. (9) it follows that $\Gamma > 0$.

Individual Floquet states can be used for the description of atomic behavior in long, quasimonochromatic pulses, under certain limitations (the "single-state Floquet theory"). W can be interpreted as the mean energy of the atom in the field (binding energy $|W|$), and Γ as the total ionization rate, with the condition that Γ be sufficiently small, in essence $\Gamma/|W| \ll 1$. Since the neutral atom in the field is a resonant decaying state, Γ is at the same time the width associated with W. Only physical solutions can be used in single-state Floquet theory, because only these give rise to the correct outgoing currents in the open channels.

When followed continuously in α_0, and all channel boundary conditions are kept the same, Floquet solutions may change their character, from physical to nonphysical, or vice versa. For example, assuming the natural normalization convention mentioned above, $W(\alpha_0) \equiv \mathrm{Re}\, E(\alpha_0)$ for a physical state starts at $\alpha_0 = 0$ from the unperturbed eigenvalue $W_0 < 0$ located between two consecutive ω thresholds $-(p-1)\omega$ and $-p\omega$, where p is a positive integer, i.e., $-(p-1)\omega < W_0 < -p\omega$. As α_0 grows it may happen that $W(\alpha_0)$ hits either the upper or lower energy threshold. At that point, either $\mathrm{Re}\, k_{p-1}^2/2 \equiv W(\alpha_0) + (p-1)\omega = 0$ or $\mathrm{Re}\, k_p^2/2 \equiv W(\alpha_0) + p\omega = 0$, which means that either channel $p-1$, which was closed to ionization before, now opens (channel opening), or channel p, which was open to ionization before, now closes (channel closure). By keeping across the threshold a closed channel boundary condition for channel $p-1$, or alternatively an open channel boundary condition for channel p, the solution becomes nonphysical. In both cases we are dealing with a state suppressed by the variation of the intensity, a "light-suppressed state" (LSS).

On the other hand, there are nonphysical solutions starting from W_0 at $\alpha_0 = 0$, for which $W(\alpha_0)$ may cross one of the adjacent thresholds $-(p-1)\omega$ or $-p\omega$, and thereafter satisfy

all of Eqs. (9) and (10), thereby becoming physical. If this happens the atom acquires a new state, called a "light-induced state" (LIS).

B. High-intensity, high-frequency Floquet theory

We now review the *iteration procedure* which is the basis of our approach to Floquet theory ([3]; for details, see Sec. IV B of [4]). We write the system of equations Eq. (5) in an alternative way, by singling out the $n=0$ case,

$$(H-E)\phi_0 = -\sum_m^{(m\neq 0)} V_{-m}\phi_m. \tag{11}$$

The equations for $n\neq 0$ we rewrite in integral form using the Green's operator associated with the Hamiltonian H:

$$G(E) \equiv \frac{1}{E-H}. \tag{12}$$

Thus,

$$\phi_n = G(E+n\omega)\sum_m^{(m\neq n)} V_{n-m}\phi_m \quad (n\neq 0). \tag{13}$$

A double *iteration procedure* was applied to Eqs. (11) and (13); see [4], Sec. IV B. It was thus possible to extract an equation for ϕ_0 alone, the equations for $\phi_{n\neq 0}$ being expressed in terms of ϕ_0. Let us denote by $E^{(\nu)}$ and $\phi_n^{(\nu)}$ the approximations of order ν to E and ϕ_n, obtained by stopping after ν iterations. It was found that $\phi_0^{(\nu)}$ satisfies

$$(H+\bar{V}^{(\nu)}-E^{(\nu)})\phi_0^{(\nu)} = 0, \tag{14}$$

where the optical potential $\bar{V}^{(\nu)}$ is:

$$\bar{V}^{(\nu)} \equiv \bar{V}_1 + \cdots + \bar{V}_\nu, \tag{15}$$

$$\bar{V}_\mu \equiv \sum_{m_1}\sum_{m_2}\cdots\sum_{m_\mu} V_{-m_1}G(E^{(\nu)}+m_1\omega)V_{m_1-m_2}G(E^{(\nu)} + m_2\omega)\cdots V_{m_{\mu-1}-m_\mu}G(E^{(\nu)}+m_\mu\omega)V_{m_\mu}. \tag{16}$$

The sums above are written with the understanding that all terms containing $G(E^{(\nu)}+m\omega)$ or V_m with $m=0$ should be omitted. For the expression of $\phi_n^{(\nu)}$, see Sec. IV B of [4].

To zeroth order ($\nu=0$), the procedure gives the *structure equation*

$$\left(\frac{1}{2}\mathbf{P}^2 + V_0(\alpha_0;\mathbf{r}) - E^{(0)}\right)u = 0, \tag{17}$$

where we have introduced the explicit form of H, Eq. (6), and have denoted $\phi_0^{(0)} \equiv u$. The equation has real eigenvalues $E^{(0)} \equiv W(\alpha_0)$ that are ω independent.

The first iteration ($\nu=1$) contributes to Eq. (14) a correction equal to $\Delta H^{(1)} \equiv \bar{V}_1$. Assuming this to give a small contribution $\Delta E^{(1)}$ to the quasienergy, it can be handled by perturbation theory. Note that $\bar{V}_1$ contains $G(E^{(1)}+m\omega)$. To lowest order one can replace $E^{(1)}$ by $E^{(0)} \equiv W(\alpha_0)$. We need to recall, however, that the exact E should be located in the complex energy plane on the unphysical sheet, under the cut along the positive energy axis. To retain this feature correctly in our approximation, we need to replace $E^{(1)}$ by $W(\alpha_0)+i\varepsilon$. Hence, to lowest order,

$$\Delta H^{(1)} \equiv \bar{V}_1 = \sum_m^{(m\neq 0)} V_m G^{(+)}(W+m\omega)V_m, \tag{18}$$

which yields

$$\Delta E^{(1)} = \langle u|\Delta H^{(1)}|u\rangle \equiv \Delta W - (i/2)\Gamma. \tag{19}$$

The result for $E^{(1)}$ is then

$$\mathrm{Re}\, E^{(1)} = W(\alpha_0) + \Delta W, \quad \mathrm{Im}\, E^{(1)} = -(1/2)\Gamma. \tag{20}$$

On the other hand, by considering the asymptotic form of ϕ_n, Eq. (13), we get the n-photon ionization amplitude f_n. This, and the corresponding differential ionization rate $d\Gamma_n/d\Omega$, can be written

$$f_n = -\frac{1}{2\pi}\langle u_{\mathbf{k}_n}^{(-)}|V_n|u\rangle, \quad d\Gamma_n/d\Omega = k_n|f_n|^2, \tag{21}$$

where $u_{\mathbf{k}_n}^{(-)}$ is a continuum state of Eq. (17), with asymptotic momentum $\mathbf{k}_n$ contained in $d\Omega$, and having incoming spherical waves; it is normalized so as to have asymptotic amplitude 1.

In order that the *formal iteration* described above have a meaning, it needs to converge in some pragmatic sense. By this we mean that consecutive corrections of the iteration to $E^{(0)} \equiv W(\alpha_0)$, as well as the expression for the remainder, should decrease in ranges of interest of the frequency and intensity. The issue of convergence of the theory will be discussed in the following section.

The basic equations of HIHFFT, like Eqs. (17) and (19), require a numerical evaluation, albeit one much simpler than that of the original Floquet system. However, in the limits of very high ω or very large α_0, these equations are amenable to rather simple analytic formulas (see for example Sec. III for the 1D case).

C. Convergence of HIHFFT

We first recall a few facts about the structure equation, which is the starting point of the iteration procedure (for details see [4]). This has been extensively studied numerically for 3D hydrogen [7,8], and 1D models (references are contained in [1], Sec. 2). Its eigenvalues $W(\alpha_0)$ and eigenfunctions were determined for low-lying as well as for Rydberg states, at α_0 extending from 0 to large values (into the thousands). Dichotomy was established. It has been predicted analytically, and confirmed numerically, that the bound-state energy spectrum is compressed to 0 at growing α_0. For all excited states $\nu\neq 0$ and at all α_0 we have the sequence of inequalities

$$0 < |W_\nu(\alpha_0)| < |W_0(\alpha_0)| \leqslant |W_0(0)| = 0.5\text{a.u.} \tag{22}$$

We are using the "natural" Floquet normalization convention defined in Sec. II.

Let us now consider the problem of the *convergence of*

the iteration procedure. The most obvious case of convergence is that of sufficiently high frequencies, which was developed within HFFT. At very high ω, using Eq. (12), we may replace the operators $G(E+m\omega)$ by

$$G(E+m\omega) \simeq \frac{1}{m\omega} I \quad (m \neq 0). \tag{23}$$

Consequently, to dominant order Eq. (16) becomes [9]

$$\bar{V}_\mu \simeq \frac{1}{\omega^\mu} \sum_{m_1} \sum_{m_2} \cdots \sum_{m_\mu} \frac{1}{m_1 m_2 \cdots m_\mu} V_{-m_1} V_{m_1-m_2} \cdots V_{m_\mu}. \tag{24}$$

The case of $\mu=1$ is an exception to this estimate because of a cancellation of terms:

$$\bar{V}_1 \simeq \frac{1}{\omega} \sum_m \frac{1}{m} V_{-m} V_m = 0, \tag{25}$$

where we have used Eq. (8). $\bar{V}_1$ is thus smaller than estimated by Eq. (24), and a better approximation than Eq. (23) is needed to establish its correct order of magnitude.

It should be noted, however, that in actual calculations operators such as $\bar{V}_\mu$ appear in matrix elements (e.g., average values, transition elements) like $\langle u|\bar{V}_\mu|u'\rangle$, evaluated for example in the basis set of eigenstates of the zeroth-order structure equation. The order of magnitude we have ascribed to the operators $\bar{V}_\mu$ can be transferred to the matrix elements if their integrals remain convergent after applying the approximation Eq. (23). This depends on the singularities of $V_m(\alpha_0, \mathbf{r})$, which in turn depend on the polarization and α_0. For hydrogen and linear polarization, V_m behaves as $r^{-1/2}$ at the end points $\pm\alpha_0\mathbf{e}$, for all α_0 [see Eq. (129) of [4]]; for the special case of large α_0, [see Eqs. (30) and (36) below]. Therefore, for $\mu \geqslant 5$ the matrix elements of $\bar{V}_\mu$ [Eq. (24)] become singular. This means that their true order of magnitude is larger than the $\omega^{-\mu}$ we have estimated and a more refined evaluation would be needed to assess it. However, the cases $\mu \leqslant 4$ are sufficient for our needs. Thus, at high enough ω, we have "convergence" in the sense that the successive iterations for the optical potential $\bar{V}^{(\nu)}$, Eq. (15), up to $\mu \leqslant 4$ certainly improve its value, while their error decreases; see Sec. IV B of [4].

As opposed to the linear case, V_m for elliptic (circular) polarization has only logarithmic singularities along the classical trajectory [see Eq. (130) of [4]]. These singularities are integrable, and Eq. (24) holds for any μ.

The order of magnitude of the first iteration $\Delta E^{(1)}$ was left open by the crude estimate Eq. (23). In fact, Γ could be evaluated explicitly, and has been extensively studied; for references, see [1], Sec. 2. Its dependence on α_0 has revealed *quasistationary stabilization* [10]. Although no systematic study has been made in 3D for ΔW, Eq. (19), order of magnitude estimates have been given in [4], Sec. IV D.

The issue of a formal condition of convergence was discussed in [4], Sec. IV D, with the conclusion that it was sufficient that ω be large with respect to some average excitation energy $W_{\text{exc}}(\alpha_0)$ for the manifold of states containing the state considered:

$$\omega \gg W_{\text{exc}}(\alpha_0). \tag{26}$$

$W_{\text{exc}}(\alpha_0)$ is of the order of magnitude of the largest binding energy of the manifold. Taking into account the sequence of inequalities Eq. (22), in order to apply HFFT to hydrogen at *all intensities* (α_0) and all states, it should be sufficient to require that $\omega > 0.5$ a.u. Practically all studies of QS (and DS as well) have been carried out in this frequency range.

A closer look at the condition Eq. (26) shows that ω need not be high at all if α_0 is sufficiently large, in fact it can be arbitrarily small. This is because $W_{\text{exc}}(\alpha_0)$ vanishes at large α_0. To understand this, we need to consider the structure equation (17) at large α_0. We shall show that, for linear polarization, the eigenfunctions concentrate around the "end points" of the classical motion, $\pm\alpha_0\mathbf{e}$, and the motion is determined by the form of the dressed potential $V_0(\mathbf{r})$ in the vicinity of these points.

Indeed, let us first find the form of the Fourier components $V_n(\alpha_0;\mathbf{r})$ near the end points. For example, in the vicinity of $+\alpha_0\mathbf{e}$ we can rewrite Eq. (7) as

$$V_n(\mathbf{r}) = \frac{(-1)^n}{\pi} \int_0^\pi (\cos n\chi) V[\mathbf{r}_- + \alpha_0\mathbf{e}(1-\cos\chi)] d\chi, \tag{27}$$

where, $n \leqslant 0$,

$$\mathbf{r}_\pm \equiv \mathbf{r} \pm \alpha_0\mathbf{e}. \tag{28}$$

By changing the integration variable according to $\xi = \alpha_0(1-\cos\chi)$, we get

$$V_n(\mathbf{r}) = \frac{(-1)^n}{\pi} \int_0^{2\alpha_0} (\cos n\chi) V(\mathbf{r}_- + \xi\mathbf{e}) \frac{d\xi}{\sqrt{2\alpha_0\xi}\sqrt{1-(\xi/2\alpha_0)}}. \tag{29}$$

Assuming $V(\mathbf{r})$ to be singular at the origin, the main contribution to the integral comes from $\mathbf{r}_- + \xi\mathbf{e} \approx \mathbf{0}$. As at large enough α_0 the (essentially bounded) values of interest of r_- will satisfy $r_- \ll \alpha_0$; this implies that the relevant values of ξ are $\xi \ll \alpha_0$. Hence, $\sqrt{1-(2\xi/\alpha_0)}$ can be replaced by 1, $\cos n\chi$ by 1, and we may extend without consequences the upper limit of the integral to ∞. This gives

$$V_n(\mathbf{r}) \cong (-1)^n \tilde{V}_0(\mathbf{r}_-) \quad \text{for } (r_-/\alpha_0) \ll 1 \text{ and } n \leqslant 0, \tag{30}$$

$$\tilde{V}_0(\mathbf{r}) \equiv \frac{1}{\pi\sqrt{2\alpha_0}} \int_0^\infty V(\mathbf{r}+\xi\mathbf{e}) \frac{d\xi}{\sqrt{\xi}}. \tag{31}$$

Proceeding similarly, near the other end point $-\alpha_0\mathbf{e}$, we have

$$V_n(\mathbf{r}) \cong \tilde{V}_0(-\mathbf{r}_+) \quad \text{for } (r_+/\alpha_0) \ll 1 \text{ and } n \lesssim 0. \tag{32}$$

Thus, $V_n(\mathbf{r})$ is expressible in the vicinity of the end points in terms of the "end-point potential" $\tilde{V}_0(\mathbf{r})$.

With Eqs. (30)–(32), the structure equation reduces in the vicinity of the end points to

$$\left(-\frac{1}{2}\Delta + \tilde{V}_0(\mp \mathbf{r}_\pm)\right)u(\mathbf{r}_\pm) = W(\alpha_0)u(\mathbf{r}_\pm). \tag{33}$$

The change of variables $\boldsymbol{\rho} \equiv \mathbf{r}_-$ and $\boldsymbol{\rho} \equiv -\mathbf{r}_+$, respectively, transforms the two equations into the following one

$$\left(-\frac{1}{2}\Delta_\rho + \tilde{V}_0(\boldsymbol{\rho})\right)s(\boldsymbol{\rho}) = W(\alpha_0)s(\boldsymbol{\rho}). \tag{34}$$

The eigenfunctions for the two alternatives in Eq. (33) can be expressed as

$$u(\mathbf{r}_+) = s(-\mathbf{r}_+), \quad u(\mathbf{r}_-) = s(\mathbf{r}_-). \tag{35}$$

For hydrogen, $V(r) = -(1/r)$, Eq. (31) gives [11]

$$\tilde{V}_0(\mathbf{r}) = \frac{1}{\sqrt{\alpha_0 r}}U(\hat{\mathbf{r}}), \tag{36}$$

$$U(\hat{\mathbf{r}}) \equiv -\frac{\sqrt{2}}{\pi}K\left(\frac{(1-\hat{\mathbf{r}}\cdot\mathbf{e})^{1/2}}{2^{1/2}}\right), \tag{37}$$

where $K(x)$ is the complete elliptic integral of the first kind [see [12], Eq. (8.113.3)]. $\tilde{V}_0$ has a $r^{-1/2}$ singularity at the origin, and a logarithmic singularity along the line $\hat{\mathbf{r}} = \mathbf{e}$. It has been shown [13] that the corrective terms to Eq. (30) are of order $O(\alpha_0^{-3/2})$ for the Coulomb case.

Introducing Eq. (36) in Eq. (34) and scaling the variable according to $\boldsymbol{\xi} = \alpha_0^{-1/3}\boldsymbol{\rho}$, we find

$$\left(-\frac{1}{2}\Delta_\xi + \frac{1}{\xi^{1/2}}U(\hat{\boldsymbol{\xi}})\right)v(\boldsymbol{\xi}) = \mathcal{W}v(\boldsymbol{\xi}), \tag{38}$$

where

$$W(\alpha_0) \equiv \alpha_0^{-2/3}\mathcal{W}, \tag{39}$$

$$s(\boldsymbol{\rho}) \equiv v(\boldsymbol{\rho}/\alpha_0^{1/3}). \tag{40}$$

Equation (39) displays the $\alpha_0^{-2/3}$ dependence of the bound-state eigenvalues $W(\alpha_0)$ at large α_0.

The eigenfunctions of the end-point equations (33) are $u(\mathbf{r}_-) = v(\mathbf{r}_-/\alpha_0^{1/3})$ and $u(\mathbf{r}_+) = v(-\mathbf{r}_+/\alpha_0^{1/3})$. Since they expand with α_0 as $\alpha_0^{1/3}$, and the separation of their centers is $2\alpha_0$, it follows that their overlap tends to zero as α_0 increases. This means that any linear combination of them corresponding to the same $\mathcal{W}$ is an eigensolution (degeneracy). In particular, if we choose

$$u^P(\mathbf{r}) \simeq \frac{1}{\sqrt{2\alpha_0}}[v(\mathbf{r}_-/\alpha_0^{1/3}) + (-1)^P v(-\mathbf{r}_+/\alpha_0^{1/3})], \tag{41}$$

$u^P(\mathbf{r})$ has parity $P = 0, 1$. If $v(\boldsymbol{\xi})$ is normalized to 1, so will be $u^P(\mathbf{r})$. Physically speaking, the charge distribution of the state represented by $u^P(\mathbf{r})$ splits into two separate lobes centered on the end points (*dichotomy*).

With Eq. (39), the convergence condition Eq. (26) can be expressed at large α_0 as

$$R \equiv \frac{|\mathcal{W}_{\text{exc}}|}{\omega\alpha_0^{2/3}} \ll 1, \tag{42}$$

where $|\mathcal{W}_{\text{exc}}|$ is a number comparable to 1. It now is obvious that, at sufficiently large α_0, ω can be allowed to be small. In the rest of the paper we shall be concerned with this regime. Let us start with the convergence of the iteration procedure.

Considering the higher corrections to the quasienergy $E^{(\nu)}$ in Eqs. (14)–(16), for example, we are interested in matrix elements such as $\langle u|\bar{V}_\mu|u'\rangle$. The multiple sum of $\bar{V}_\mu$ yields terms like [see Eqs. (23) and (24)]

$$\langle u|V_{-m_1}V_{m_1-m_2}\cdots V_{m_\mu}|u'\rangle, \tag{43}$$

which contains $\mu+1$ Fourier components V_m. The main contribution to the integral Eq. (43) comes from the vicinity of the end points $\pm\alpha_0\mathbf{e}$, where the V_m are singular, and given by Eqs. (30), (32), and (36). For $u(\mathbf{r})$ we shall use the dichotomous form Eq. (41). The dominant contribution to Eq. (43) from the end point $+\alpha_0\mathbf{e}$ is then

$$\langle u|(V_0)^{\mu+1}|u'\rangle \sim \frac{1}{2\alpha_0}\left\langle v(\mathbf{r}_-/\alpha_0^{1/3})\left|\left(\frac{1}{\sqrt{\alpha_0 r_-}}U(\hat{\mathbf{r}}_-)\right)^{\mu+1}\right|v'(\mathbf{r}_-/\alpha_0^{1/3})\right\rangle. \tag{44}$$

By changing the integration variable according to $\mathbf{t} \equiv \alpha_0^{-1/3}\mathbf{r}_-$, we get

$$\langle u|(V_0)^{\mu+1}|u'\rangle \sim \frac{1}{2(\alpha_0)^{(2\mu+2)/3}}\int v(\mathbf{t})\left(\frac{1}{\sqrt{t}}U(\hat{\mathbf{t}})\right)^{\mu+1}v'(\mathbf{t})d\mathbf{t}. \tag{45}$$

As the other end point $-\alpha_0\mathbf{e}$ gives an identical contribution, we can write

$$\langle u|V_{-m_1}V_{m_1-m_2}\cdots V_{m_\mu}|u'\rangle \sim \frac{\mathcal{V}_{\mu+1}}{\alpha_0^{(2\mu+2)/3}}, \tag{46}$$

where $\mathcal{V}_{\mu+1}$ is an α_0-independent constant. With Eq. (16), for $\mu \neq 1$ this leads to

$$\langle u|\bar{V}_\mu|u'\rangle = \frac{1}{\omega^\mu}O\left(\frac{1}{\alpha_0^{(2\mu+2)/3}}\right) = \alpha_0^{-2/3}O(R^\mu). \tag{47}$$

The case $\mu = 1$ is an exception, as noted earlier in Eq. (25). Thus, if $\omega\alpha_0^{-2/3}$ is sufficiently large and the criterion Eq. (42) is well satisfied, the higher μ, the smaller the matrix element is at large α_0; this applies to $\langle u|\bar{V}^{(\nu)}|u'\rangle$ as well. This means that the iteration procedure indeed yields decreasing corrections.

We have considered above the case of linear polarization. A similar discussion could be made for general elliptic polarization, with some changes regarding the order of magnitude of the terms, without affecting the conclusions on convergence.

We shall illustrate the predictions of HIHFFT in more detail on the case of a 1D model. Not only can the HIHFFT formulas be numerically computed in this case, but they can also be evaluated analytically, allowing a more transparent interpretation. This evaluation is presented in the next section.

III. LARGE-α_0 HIHFFT FOR 1D MODEL

We consider the popular 1D model with the "soft-core" Coulomb-potential

$$V(x) = -\frac{1}{\sqrt{a^2 + x^2}}. \tag{48}$$

At large $|x|$, this has a Coulomb tail supporting an infinite set of Rydberg states. The constant a^2 under the square root "softens" the $1/|x|$ singularity at the origin, which is too strong for a consistent mathematical treatment of the 1D case. By choosing $a=1.414$ the energy of the ground state is $W_0=-0.500$. In the following we present analytic HIHFFT results for the *large-*α_0, *all-*ω regime of this model. Their accuracy is confirmed by numerical results, see Sec. IV.

A. Eigenvalues $W(\alpha_0)$ of structure equation

Let us consider the structure equation for the potential Eq. (48). As in the 3D case, at large α_0 it is sufficient to solve it in the vicinity of one of the end points, say $+\alpha_0$. Starting from the analog of Eq. (33), after changing the variable according to $x_-=\lambda\xi$, we get

$$\left(-\frac{1}{2}\frac{d^2}{d\xi^2} + \lambda^2 \tilde{V}_0(\lambda\xi)\right) v(\xi) = \mathcal{W} v(\xi), \tag{49}$$

where we have denoted

$$W(\alpha_0) = \lambda^{-2}\mathcal{W}, \tag{50}$$

$$u(x_-) \equiv v(x_-/\lambda). \tag{51}$$

The end point potential corresponding to Eq. (48) is

$$\tilde{V}_0(x) = -\frac{1}{\pi\sqrt{2\alpha_0}} \int_0^\infty \frac{1}{\sqrt{(x+\xi)^2 + a^2}} \frac{d\xi}{\sqrt{\xi}}. \tag{52}$$

This can be expressed in terms of the complete elliptic integral $\mathbf{K}(k)$, or the hypergeometric function ${}_2F_1$ [14]:

$$\tilde{V}_0(x) = -\frac{1}{\pi}\sqrt{\frac{2}{\alpha_0}}\mathbf{K}(k) = \frac{1}{\sqrt{2\alpha_0}} \frac{1}{(x^2+a^2)^{1/4}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \tag{53}$$

where

$$k^2 = \frac{(x^2+a^2)^{1/2} - x}{2(x^2+a^2)^{1/2}}. \tag{54}$$

In order that Eq. (49) have solutions, $\lambda^2\tilde{V}_0(\lambda\xi)$ needs to stay finite in the limit of large α_0. By inspection of Eq. (53) it is easy to see that this can happen only if λ increases with α_0. Consequently, at finite ξ, where the eigenfunction $v(\xi)$ is significantly different from zero, the variable of $\tilde{V}_0(\lambda\xi)$ lies in the asymptotic range $\lambda\xi \to \infty$, [15]. The asymptotic form of $\tilde{V}_0(x)$, Eq. (53), is [16]

$$\tilde{V}_0^{\mathrm{as}}(x) = \begin{cases} -\dfrac{2}{\pi\sqrt{2\alpha_0}} \dfrac{1}{\sqrt{|x|}} (\ln 8|x|/a)\left[1 + O\left(\dfrac{1}{x^2}\right)\right], & x < 0, \\ -\dfrac{1}{\sqrt{2\alpha_0}} \dfrac{1}{\sqrt{x}}\left[1 + O\left(\dfrac{1}{x^2}\right)\right], & x > 0. \end{cases} \tag{55}$$

We now choose λ such that the coefficient of the term $-(1/\sqrt{|\xi|})$ of $\lambda^2\tilde{V}_0^{\mathrm{as}}(\lambda\xi)$ for $x<0$ be 1. This means

$$\frac{2\lambda^{3/2}\ln(8\lambda/a)}{\pi\sqrt{2\alpha_0}} = 1. \tag{56}$$

With this choice we can write

$$\lambda^2\tilde{V}_0^{\mathrm{as}}(\lambda\xi) \equiv U(\xi) + U'(\alpha_0, \xi), \tag{57}$$

where

$$U(\xi) = \begin{cases} -\dfrac{1}{\sqrt{|\xi|}}, & \xi < 0, \\ 0, & \xi > 0, \end{cases} \tag{58}$$

and

$$U'(\alpha_0, \xi) = \begin{cases} -\dfrac{1}{\ln(8\lambda/a)} \dfrac{\ln|\xi|}{\sqrt{|\xi|}}, & \xi < 0, \\ -\dfrac{\pi}{2\ln(8\lambda/a)} \dfrac{1}{\sqrt{\xi}}, & \xi > 0. \end{cases} \tag{59}$$

Equation (49) thereby becomes

$$\left(-\frac{1}{2}\frac{d^2}{d\xi^2}+U(\xi)+U'(\alpha_0,\xi)\right)v(\xi)=\mathcal{W}v(\xi). \tag{60}$$

Let us consider now the solution $\lambda(\alpha_0)$ of the algebraic equation Eq. (56). The equation can be solved numerically, but also an analytic solution can be obtained to dominant order in α_0, which is sufficient for our purposes. This is [17]

$$\lambda=\alpha_0^{1/3}\left(\frac{\pi}{\sqrt{2}\ln(8\alpha_0^{1/3}\mu_0/a)}\right)^{2/3},\quad \mu_0\equiv\left(\frac{\pi}{\sqrt{2}\ln(8\alpha_0^{1/3}/a)}\right)^{2/3}. \tag{61}$$

The logarithmic dependence of λ on α_0 is a consequence of the logarithmic term in the potential Eq. (55). The behavior of λ is dominated by the factor $\alpha_0^{1/3}$, present also in the 3D Coulomb case. By comparing Eq. (61) to the numerical solution of Eq. (56), we find an agreement at the level of 1% on the interval $100<\alpha_0<3000$. From Eq. (56), we have also

$$\ln(8\lambda/a)\simeq\ln(8\alpha_0^{1/3}\mu_0/a). \tag{62}$$

As at large α_0 the quantity in Eq. (62) is a small, $U'(\xi,\alpha_0)$ of Eq. (59) can be considered a corrective term in Eq. (60). Neglecting it, by combining Eqs. (50) and (61), we find for $W(\alpha_0)$ to lowest order

$$W(\alpha_0)\simeq\alpha_0^{-2/3}\left(\frac{\sqrt{2}}{\pi}\ln(8\alpha_0^{1/3}\mu_0/a)\right)^{4/3}\mathcal{W}^{(0)}, \tag{63}$$

where $\mathcal{W}^{(0)}$ is the eigenvalue of Eq. (60) without $U'(\alpha_0,\xi)$. Treating the latter by perturbation theory gives the energy correction to $\mathcal{W}^{(0)}$

$$\mathcal{W}^{(1)}(\alpha_0)=-\frac{1}{\ln(8\alpha_0^{1/3}\mu_0/a)}\left(\frac{\pi}{2}\int_{-\infty}^{0}\frac{\ln|\xi|}{\sqrt{|\xi|}}v^2(\xi)d\xi+\int_0^{\infty}\frac{1}{\sqrt{\xi}}v^2(\xi)d\xi\right). \tag{64}$$

Collecting terms yields

$$W(\alpha_0)\simeq[\lambda(\alpha_0)]^{-2}[\mathcal{W}^{(0)}+\mathcal{W}^{(1)}(\alpha_0)]. \tag{65}$$

Equation (63) indicates that, to lowest order in α_0 [i.e., with $U'(\alpha_0,\xi)$ neglected], we have a scaling law similar to the 3D case, Eq. (39), albeit more complicated. The analytic approximation for $\lambda(\alpha_0)$, Eq. (61), shows that it increases with α_0 slightly more slowly than $\alpha_0^{1/3}$. It follows that $W(\alpha_0)$ decreases with increasing α_0 slightly more slowly than $\alpha_0^{-2/3}$, which is the 3D case Eq. (39) [18].

Using Eq. (51), the complete eigenfunction corresponding to $W(\alpha_0)$ is given by

$$u^P(x)\simeq\frac{1}{\sqrt{2\lambda}}[v(x_-/\lambda)+(-1)^Pv(-x_+/\lambda)]. \tag{66}$$

The functions $v(x_\pm/\lambda)$ expand in the vicinity of the end points proportionally to λ. Since λ behaves roughly as $\alpha_0^{1/3}$, we conclude that we have dichotomy for the 1D model as well.

B. First-order quasienergy correction $\Delta E^{(1)}$

We now evaluate the first iteration correction $\Delta E^{(1)}$, Eqs. (18) and (19). Our emphasis is on the case of large α_0, with the convergence condition Eq. (42) satisfied. We make use of the dichotomous structure of the wave function u, Eq. (66). This gives to dominant order in α_0

$$\Delta E^{(1)}=\frac{1}{\lambda}\sum_m^{m\neq0}\{\langle v(x_-/\lambda)|V_mG^{(+)}[W(\alpha_0)+m\omega]V_m|v(x_-/\lambda)\rangle+(-1)^P\langle v(x_-/\lambda)|V_mG^{(+)}[W(\alpha_0)+m\omega]V_m|v(-x_+/\lambda)\rangle\}, \tag{67}$$

where we have used the parity property of $V_m(x)$, Eq. (8).

We shall designate the first term in Eq. (67) as the *direct term* $\Delta E^{(d)}$. It is convenient to evaluate it by using the general decomposition formula for the Green's function $G^{(+)}(W)$ associated with the Hamiltonian H:

$$G^{(+)}(W)=P\left(\frac{1}{W-H}\right)-i\pi\delta(W-H), \tag{68}$$

where P and δ are the principal value and delta operators. The principal value contribution is

$$\Delta E_P^{(d)}\equiv\frac{1}{\lambda}\sum_m^{(m\neq0)}\left\langle v(x_-/\lambda)\left|V_mP\frac{1}{[W(\alpha_0)+m\omega]-H}V_m\right|v(x_-/\lambda)\right\rangle. \tag{69}$$

Using Eq. (8) the sum over $m\lessgtr0$ can be converted into one over $m>0$ only:

$$\Delta E_P^{(d)}=\frac{2}{\lambda}\sum_{m>0}\left\langle v(x_-/\lambda)\left|PV_m\frac{W(\alpha_0)-H}{[W(\alpha_0)-H]^2-(m\omega)^2}V_m\right|v(x_-/\lambda)\right\rangle. \tag{70}$$

The contribution of the operator H in Eq. (70) is of the order of a typical excitation energy $W_{\mathrm{exc}}(\alpha_0)$, a fact that can be easily ascertained by inserting the completeness equation for the eigenfunctions of H, $\mathcal{S}_\alpha|u_\alpha\rangle\langle u_\alpha|=I$. By virtue of the conditions Eqs. (42) and (26), we can approximate the denominator by $-(m\omega)^2$. Moreover, by using the Schrödinger equation (33), we can reexpress the numerator, to get

$$\Delta E_P^{(d)} \simeq \frac{1}{\lambda\omega^2}\sum_{m>0}\frac{1}{m^2}\langle v(x_-/\lambda)|[V_m,[H,V_m]]|v(x_-/\lambda)\rangle. \tag{71}$$

Note the cancellation of the terms of order $1/\omega$, already signaled in Eq. (25).

With $P_x \equiv -id/dx$, we have

$$[H,V_m]=\{P_xV_m\}P_x+\frac{1}{2}\{P_x^2V_m\}, \tag{72}$$

where the curly brackets indicate that P_x acts only on the function inside it. This leads to

$$[V_m,[H,V_m]]=[V_m,\{P_xV_m\}P_x]=-\{P_xV_m\}^2. \tag{73}$$

Thus

$$\Delta E_P^{(d)} \simeq \frac{1}{\lambda\omega^2}\sum_{m>0}\frac{1}{m^2}\langle v(x_-/\lambda)|\{dV_m(x)/dx\}^2|v(x_-/\lambda)\rangle. \tag{74}$$

The dominant contributions to the integrals of the matrix elements in Eq. (71) comes at large α_0 (or λ) from the vicinity of the end point $+\alpha_0$. Therefore, as in the 3D case, we can use Eqs. (30) to replace $V_m(x)$ by $\tilde{V}_0(x_-)$ to get

$$\Delta E_P^{(d)} \simeq \frac{1}{\lambda\omega^2}\langle v(x_-/\lambda)|\{d\tilde{V}_0(x_-)/dx_-\}^2|v(x_-/\lambda)\rangle\sum_{m>0}\frac{1}{m^2}. \tag{75}$$

Further, by changing the integration variable in the matrix element according to $x_-=\lambda\xi$, we find

$$\langle v(x_-/\lambda)|\{d\tilde{V}_0(x_-)/dx_-\}^2|v(x_-/\lambda)\rangle \simeq (1/\lambda)\int_{-\infty}^{\infty}[d\tilde{V}_0(\lambda\xi)/d\xi]^2[v(\xi)]^2d\xi. \tag{76}$$

The last integral can be decomposed in two parts, for $|\xi|<q$ and $|\xi|>q$, where $q\gg a/\lambda$. For $|\xi|>q$, the integral remains finite in the limit $(a/\lambda)\to 0$ so that we can use the dominant asymptotic behavior of $\tilde{V}_0(\lambda\xi)$, given by Eqs. (55), (57), and (58) to get a contribution $O(\lambda^{-5})$. With Eq. (53) we have further

$$\frac{1}{\lambda}\int_{|\xi|<q}[d\tilde{V}_0(\lambda\xi)/dx]^2[v(\xi)]^2d\xi = \frac{1}{2\alpha_0\lambda^2}\int_{|\xi|<q}\left(\frac{d}{d\xi}\frac{f(\xi)}{[\xi^2+(a/\lambda)^2]^{1/4}}\right)^2[v(\xi)]^2d\xi, \tag{77}$$

where $f(\xi)\equiv {}_2F_1(1/2,1/2,1;k^2)$. The square of the derivative gives rise to three terms containing, respectively, $\xi^n[\xi^2+(a/\lambda)^2]^{-1/2-n}$, with $n=0,1,2$. The most singular of these for $\lambda\to\infty$ or $(a/\lambda)\to 0$ is that for $n=2$, and hence also the one giving the dominant contribution at large λ. Moreover, because $f(\xi)$ and $v(\xi)$ are smooth functions at $\xi=0$, they can be extracted from the integral with their value at this point. The remaining integral over ξ is elementary and proportional to $(\lambda/a)^2$. Returning to Eq. (76), this gives the dominant behavior:

$$\left\langle v(x_-/\lambda)\middle|\{d\tilde{V}_0(x_-)/dx_-\}^2\middle|v(x_-/\lambda)\right\rangle \simeq \frac{1}{12a^2\alpha_0}[f(0)v(0)]^2, \tag{78}$$

where $f(0)={}_2F_1(1/2,1/2,1;1/2)$. As $\Sigma_{m>0}(1/m^2)\equiv\zeta(2)=\pi^2/6$, we find for $\Delta E_P^{(d)}$:

$$\Delta E_P^{(d)} \simeq \frac{\pi^2}{6}\frac{1}{12a^2}\frac{1}{\alpha_0\lambda\omega^2}[f(0)v(0)]^2. \tag{79}$$

In the evaluation of $\Delta E_P^{(d)}$ we have used the complete Green's function $G^{(+)}(W)$ associated with the full Hamiltonian H because it was simpler to do so. However, we recall that due to our condition Eq. (42), at increasing α_0, the arguments of the Green's functions $G^{(+)}[W(\alpha_0)+m\omega]$, for $m\neq 0$, become sufficiently large [i.e., large compared to the extension of the bound state spectrum Eq. (63)], to be entitled to apply the first Born approximation to the Green's function, see [19]. In 1D this is

$$G_0^{(+)}[W(\alpha_0)+m\omega;x-x']\equiv\frac{1}{ik_m}e^{ik_m|x-x'|}, \tag{80}$$

with k_m derived from $k_m^2/2\equiv(E+m\omega)$. With our approximations

$$\frac{1}{2}k_m^2 \simeq W(\alpha_0)+m\omega+i\varepsilon \simeq m\omega \quad \text{for } m\neq 0, \tag{81}$$

for both $m\gtrless 0$. Open channels correspond to $m>0$, closed ones to $m<0$. Equations (9) and (10) then give for k_m:

$$k_m>0 \quad \text{for } m>0; \quad k_m=i|k_m| \quad \text{for } m<0. \tag{82}$$

In our further evaluations of $\Delta E^{(1)}$ we will replace $G^{(+)}$ by $G_0^{(+)}$, and consequently also H by $H_0=P_x^2/2$ in the δ term of Eq. (68). The latter contributes then to $\Delta E^{(d)}$ with

$$\Delta E_\delta^{(d)} \equiv -i\frac{\pi}{\lambda}\sum_m^{(m\neq 0)}\langle v(x_-/\lambda)|V_m \times\delta\{[W(\alpha_0)+m\omega]-H_0\}V_m|v(x_-/\lambda)\rangle. \tag{83}$$

Alternatively, this can be written as

$$\Delta E_\delta^{(d)} = -i\frac{1}{2\lambda}\sum_{m>0}\int_{-\infty}^{+\infty}dk|\langle e^{-ikx}|V_m|v(x_-/\lambda)\rangle|^2\delta\left[\frac{1}{2}(k_m^2-k^2)\right], \tag{84}$$

with k_m given by Eq. (82). Note that we need to have $m>0$ (i.e., open channels) in the sum, so that the δ function does not vanish. Using Eq. (30) and a known property of the δ function, Eq. (84) becomes

$$\Delta E_{\delta}^{(d)} = -\frac{i}{\lambda}\sum_{m>0}\frac{1}{k_m}|I(k_m)|^2, \tag{85}$$

where we have defined

$$I(k) \equiv \int_{-\infty}^{+\infty} e^{-ikx}\tilde{V}_0(x)v(x/\lambda)dx, \tag{86}$$

where k can be taken complex. As the extension of $v(\mathbf{r})$ is practically finite, the integral $I(k)$ is convergent.

We now calculate $I(k)$ for k real. As, by increasing λ, each integration point gets a contribution from $v(x/\lambda)$, which is close to $v(0)$, the latter can be extracted from the integral in Eq. (86). The remaining integral represents the Fourier transform of the end-point potential, denoted $\tilde{\mathcal{V}}_0(k)$, which is well defined [20,21]. To dominant order in $1/\lambda$ we then get

$$I(k) \simeq (2\pi)\tilde{\mathcal{V}}_0(k)v(0). \tag{87}$$

Using Eq. (52), one finds

$$\tilde{\mathcal{V}}_0(k) = \frac{1}{\pi\sqrt{2\alpha_0}}h(-k)\mathcal{V}(k), \tag{88}$$

where $\mathcal{V}(k)$ is the Fourier transform of the original potential, Eq. (48), which is real [21]. $h(a)$ is defined by [22]:

$$h(a) \equiv \int_0^{+\infty} e^{-ia\xi}\frac{d\xi}{\sqrt{\xi}} = \begin{cases} \sqrt{\pi/a}e^{-i(\pi/4)} & (a>0) \\ \sqrt{\pi/|a|}e^{+i(\pi/4)} & (a<0) \end{cases}. \tag{89}$$

Consequently, we find for k real, to dominant order in $1/\lambda$:

$$I(k) \simeq \sqrt{\frac{2\pi}{\alpha_0|k|}}e^{\pm i(\pi/4)}\mathcal{V}(k)v(0), \tag{90}$$

where the plus sign in the exponent corresponds to $k>0$, and the minus sign to $k<0$. $I(k)$ is independent of λ to dominant order. By inserting $I(k_m)$ in Eq. (85) we get k_m^2 in the denominator, for which we use Eq. (81). This gives [23]

$$\Delta E_{\delta}^{(d)} \simeq -i\frac{\pi}{\alpha_0\lambda\omega}[v(0)]^2\sum_{m>0}\frac{1}{m}[\mathcal{V}(k_m)]^2. \tag{91}$$

We now turn to the evaluation of the second sum of $\Delta E^{(1)}$ in Eq. (67), which we shall designate as the *exchange term* $\Delta E^{(\mathrm{ex})}$. It is now convenient to use the explicit form of the 1D free-particle Green's function Eq. (80). Further, we split the sum over m into two parts, for $m>0$ and $m<0$. For $m>0$ (open channels), we write using Eq. (82)

$$\Delta E_{>}^{(\mathrm{ex})} \equiv \frac{(-1)^P}{i\lambda}\sum_{m>0}\frac{(-1)^m}{k_m}\int_{-\Delta/2}^{+\Delta/2}dx_-\int_{-\Delta/2}^{+\Delta/2}dx'_+ \times v(x_-/\lambda)\tilde{V}_0(x'_-)e^{ik_m|x-x'|}\tilde{V}_0(-x'_+)v(-x'_+/\lambda). \tag{92}$$

We have made use here of Eqs. (30) and (32). The integration over dx_- and dx'_+ should extend in principle from $-\infty$ to $+\infty$, but it is limited by the extension of the functions v, which is practically finite; we have denoted it by Δ. Recall that $\Delta = O(\lambda)$ and, as $\lambda \approx \alpha_0^{1/3}$, Δ is much smaller than the separation of the centers of the two end points $2\alpha_0$. Thus, the difference

$$x - x' = 2\alpha_0 + x_- - x'_+ \tag{93}$$

in the exponent of Eq. (92) can be taken as positive. The double integral in Eq. (92) splits then into the product of two equal integrals, such that

$$\Delta E_{>}^{(\mathrm{ex})} \simeq \frac{(-1)^P}{i\lambda}\sum_{m>0}\frac{(-1)^m}{k_m}e^{2ik_m\alpha_0}[I(-k_m)]^2, \tag{94}$$

with $I(k)$ defined by Eq. (86). Inserting Eq. (90) into Eq. (94) this will contain k_m^2 in the denominator, for which we use Eq. (81). We find then to dominant order in α_0 [23]

$$\Delta E_{>}^{(\mathrm{ex})} \simeq (-1)^{P+1}\frac{\pi}{\lambda\alpha_0\omega}[v(0)]^2\sum_{m>0}\frac{(-1)^m}{m}e^{2ik_m\alpha_0}[\mathcal{V}(k_m)]^2. \tag{95}$$

Let us consider now the $m<0$ sum of $\Delta E^{(\mathrm{ex})}$, which we denote $\Delta E_{<}^{(\mathrm{ex})}$. In view of Eq. (93) [see also Eq. (82)], the exponential in Eq. (92) is of order $\exp(ik_m|x-x'|) \simeq \exp(-2\alpha_0|k_m|)$. Thus, the terms contained in $\Delta E_{<}^{(\mathrm{ex})}$ are negligible with respect to those of $\Delta E_{>}^{(\mathrm{ex})}$, Eq. (95), and $\Delta E^{(\mathrm{ex})} \simeq \Delta E_{>}^{(\mathrm{ex})}$.

Finally, $\Delta E^{(1)}$ is the sum of contributions from Eqs. (79), (91), and (95). In view of Eq. (19) we get

$$\Delta W = \frac{\pi^2}{6}\frac{[f(0)]^2}{12a^2}[v(0)]^2\frac{1}{\alpha_0\lambda\omega^2} + (-1)^{P+1}\frac{\pi}{\alpha_0\lambda\omega}[v(0)]^2\sum_{m>0}\frac{(-1)^m}{m}\cos 2k_m\alpha_0[\mathcal{V}(k_m)]^2, \tag{96}$$

$$\Gamma = \frac{2\pi}{\alpha_0\lambda\omega}[v(0)]^2\sum_{m>0}\frac{1}{m}[1+(-1)^{m+P}\sin 2k_m\alpha_0][\mathcal{V}(k_m)]^2. \tag{97}$$

Let us now compare the order of magnitude in α_0 and ω of the correction $\Delta E^{(1)}$ with the lowest-order approximation for the quasienergy $W(\alpha_0)$. Consider the first term of ΔW. We note that the constant a it contains can be expressed in terms of the value of the end-point potential $\tilde{V}_0(x)$ at the origin $\tilde{V}_0(0)$ [see Eq. (53)] as $\tilde{V}_0(0) = (2\alpha_0 a)^{-1/2}f(0)$, where $f(0)$ is of order 1. With $\lambda \approx \alpha_0^{1/3}$ [see Eq. (61)], the estimate for the first term of ΔW is $\alpha_0^{2/3}[\tilde{V}_0(0)]^4/\omega^2$. As $\tilde{V}_0(0)$ is of the order of the binding energy $W(\alpha_0)$ [see Eq. (34)], the estimate gives $\alpha_0^{2/3}[W(\alpha_0)]^2[W(\alpha_0)/\omega]^2$. Taking into account that $|W(\alpha_0)| \sim \alpha_0^{-2/3}$, the estimate gives further $R^2|W(\alpha_0)| \ll |W(\alpha_0)|$, where we have used Eq. (42). Consider now the second term of ΔW, which has the same order of magnitude as Γ. For both, the order is $(\alpha_0\lambda\omega)^{-1}[\mathcal{V}(k_m)]^2 = R|W(\alpha_0)|[\mathcal{V}(k_m)]^2 \ll |W(\alpha_0)|[\mathcal{V}(k_m)]^2 \ll |W(\alpha_0)|$ (see [23]). Thus, under our conditions, ΔW and Γ are indeed small corrections to $W(\alpha_0)$. The fact that Γ is decreasing at large α_0 represents QS (see [1], Sec. I).

ΔW and Γ contain sums over m involving $\cos 2k_m\alpha_0$ and

$\sin 2k_m\alpha_0$, respectively. These sums are *dephased in* α_0 by $\pi/2$ for $P=0$ and by $3\pi/2$ for $P=1$.

C. Floquet components

We now investigate the behavior of the Floquet components $\phi_n(x)$ to first order in the HIHFFT iteration, obtained by setting $\phi_m \simeq u\delta_{0m}$ on the right-hand side of Eq. (13) and using the first Born approximation of the Green's function, Eq. (80):

$$\phi_n^{(1)}(x) = \int_\infty G_0^{(+)}[W(\alpha_0) + n\omega; x - x']V_n(x')u(x')dx'. \tag{98}$$

Inserting here Eqs. (30), (32), and (66), we get

$$\phi_n^{(1)}(x) = \frac{1}{\sqrt{2\lambda}ik_n}\left((-1)^n\int_{-\Delta/2}^{+\Delta/2} e^{ik_n|x_- - x_-'|}\tilde{V}_0(x_-')v(x_-'/\lambda)dx_-' + (-1)^P\int_{-\Delta/2}^{+\Delta/2} e^{ik_n|x_+ - x_+'|}\tilde{V}_0(-x_+')v(-x_+'/\lambda)dx_+'\right). \tag{99}$$

k_n is given for the open ($n>0$) and closed ($n<0$) channels by Eq. (82). Concerning the integration limits in Eq. (99), we have proceeded as for Eq. (92). Note that we have by definition

$$u^P(\alpha_0 \pm \Delta/2) = (-1)^P u^P(-\alpha_0 \mp \Delta/2) \simeq 0, \quad v(\pm\Delta/2\lambda) \simeq 0. \tag{100}$$

To study the behavior of $\phi_n^{(1)}(x)$, we distinguish three intervals on the x axis: (A) the interval between the lobes of the dichotomous atom; (B) the interval covering the lobes; (C) the interval lying outside both lobes, and extending to $\pm\infty$. As $\phi_n^{(1)}(x)$, Eq. (99) has parity properties (see Sec. II), it is sufficient to consider $x>0$.

On *interval A*, characterized by $0 \leqslant x \leqslant \alpha_0 - \Delta/2$, we can write, in view of Eqs. (86) and (99):

$$\phi_n^{(1)}(x) = \frac{1}{\sqrt{2\lambda}ik_n} I(-k_n)e^{ik_n\alpha_0}[(-1)^n e^{-ik_n x} + (-1)^P e^{ik_n x}]. \tag{101}$$

For open channels we get

$$|\phi_n^{(1)}(x)| = \sqrt{\frac{2}{\lambda}}\frac{1}{k_n}|I(-k_n)|\left|\cos\left(k_n x + \frac{\pi}{2}(P+n)\right)\right|, \tag{102}$$

and for closed ones

$$|\phi_n^{(1)}(x)| = \sqrt{\frac{2}{\lambda}}\frac{1}{|k_n|}e^{-|k_n|\alpha_0}|I(-i|k_n|)| \times \begin{cases} \cosh|k_n|x & \text{if } P+n = \text{even}, \\ \sinh|k_n|x & \text{if } P+n = \text{odd}. \end{cases} \tag{103}$$

Hence, for closed channels $|\phi_n^{(1)}(x)|$ increases exponentially when $|x|$ grows from 0 to α_0, which means that we have dichotomy, just as for $n=0$.

Interval B is characterized by $\alpha_0 - \Delta/2 \leqslant x \leqslant \alpha_0 + \Delta/2$. In the second integral of Eq. (99), $|x_+ - x_+'| = (x_+ - x_+')$, and the integral is proportional to $I(-k_n)$. Because in the first integral $(x_- - x_-')$ can change sign, we split it into two, so that Eq. (99) becomes

$$\phi_n^{(1)}(x) = \frac{1}{\sqrt{2\lambda}ik_n}\left[(-1)^n\left(e^{ik_n x_-}\int_{-\Delta/2}^{x_-} e^{-ik_n x_-'}\tilde{V}_0(x_-')v(x_-'/\lambda)dx_-' + e^{-ik_n x_-}\int_{x_-}^{+\Delta/2} e^{ik_n x_-'}\tilde{V}_0(x_-')v(x_-'/\lambda)dx_-'\right) + (-1)^P e^{ik_n x_+}I(-k_n)\right]. \tag{104}$$

We next apply an integration by parts to the integrals in this equation. For the first one we get

$$\int_{-\Delta/2}^{x_-} e^{-ik_n x_-'}\tilde{V}_0(x_-')v(x_-'/\lambda)dx_- = -\frac{1}{ik_n}[e^{-ik_n x_-}\tilde{V}_0(x_-)v(x_-/\lambda) - e^{+ik_n\Delta/2}\tilde{V}_0(-\Delta/2)v(-\Delta/2\lambda)] + T. \tag{105}$$

Note that the second term vanishes in view of Eq. (100). The term T is an integral that can be again integrated by parts to show that it is of order $1/\lambda k_n$, and hence negligible. A similar result holds for the second integral of Eq. (104), and we find eventually

$$\phi_n^{(1)}(x) \simeq \frac{1}{\sqrt{2\lambda}ik_n}\left((-1)^n\frac{2i}{k_n}\tilde{V}_0(x_-)v(x_-/\lambda) + (-1)^P e^{ik_n(x_- + 2\alpha_0)}I(-k_n)\right). \tag{106}$$

For open channels this leads to

$$|\phi_n^{(1)}(x)|^2 \simeq \frac{1}{2\lambda k_n^2}\left[\frac{4}{k_n^2}\tilde{V}_0^2(x_-)v^2(x_-/\lambda) + \frac{4}{k_n}(-1)^{P+n} \times |I(-k_n)|\tilde{V}_0(x_-)v(x_-/\lambda)\sin\left(k_n(x_- + 2\alpha_0) - \frac{\pi}{4}\right) + |I(-k_n)|^2\right]. \tag{107}$$

If the last term in the square brackets is small with respect to the first one, $|\phi_n^{(1)}(x)|^2$ has practically the same nodes as the dichotomous state $u^P(x)$ in the vicinity of the end point $+\alpha_0$ [recall Eq. (66)]. The smooth dependence on x of the first term is then *modulated* by the second term, with the same frequency k_n that $|\phi_n^{(1)}(x)|$ has on interval A; moreover, the amplitude of the oscillations is small. Note that $\tilde{V}_0(x_-)$ has no zeros.

For closed channels, k_n should be replaced everywhere in Eq. (106) by $i|k_n|$. Then the second term in the square brackets becomes of the order of $e^{-2|k_n|\alpha_0}$ and is negligible. This leads to

$$|\phi_n^{(1)}(x)|^2 \simeq \frac{2}{\lambda|k_n|^4}\tilde{V}_0^2(x_-)v^2(x_-/\lambda). \tag{108}$$

Now $|\phi_n^{(1)}(x)|^2$ has precisely the same nodes as $u^P(x)$. As opposed to the open channel case, there is no modulation.

On *interval C*, characterized by $\alpha_0+\Delta/2<x<+\infty$, we can write

$$\phi_n^{(1)}(x) = \frac{1}{\sqrt{2\lambda}ik_n}[(-1)^n e^{ik_n x_-}I(k_n) + (-1)^P e^{ik_n x_+}I(-k_n)]. \tag{109}$$

For open channels we get with the help of Eq. (90) [24]

$$|\phi_n^{(1)}(x)| = \sqrt{\frac{4\pi}{\lambda\alpha_0}}\frac{1}{k_n^{3/2}}|v(0)||\mathcal{V}(k_n)| \times \left|\cos\left(k_n\alpha_0 + \frac{\pi}{2}(P+n) - \frac{\pi}{4}\right)\right|, \tag{110}$$

which is constant in x. For closed channels, it follows from Eq. (109) that $\phi_n^{(1)}(x)$ is real and vanishes exponentially for $x\to+\infty$.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have shown that the iteration procedure developed earlier within nonrelativistic Floquet theory at high frequency and arbitrary intensity converges also in the case of low frequency and high intensity. From the physical point of view this means that, contrary to a widespread belief, QS and dichotomy are phenomena that can occur also at low ω (compared to the unperturbed ground state energy $\sim$0.5 a.u.), provided that the intensity is sufficiently large. Therefore, the theory should be more appropriately designated as high-intensity, high-frequency Floquet theory. We have then applied the general 3D HIHFFT formulas to the case of a 1D Coulomb soft-core potential model, where analytic calculations could be pushed surprisingly far, enabling results to be obtained for the dominant behavior of the energy levels, ionization amplitudes, etc.

We have considered here the case of individual Floquet states, which, from the physical point of view are apt to describe atomic behavior in long quasimonochromatic laser pulses, of adiabatically varying field amplitude. However, our findings bear also on atomic behavior in short laser pulses, because wave packets can be expressed as superpositions of Floquet states ("multistate Floquet theory;" see [25–27]). If the wave packet of the system contains a superposition of bound Floquet states of the type we have considered, which undergo QS at high α_0, this would have as consequence the fact that the wave packet would undergo DS. Indeed, the decrease with α_0 of the $\Gamma(\alpha_0)$ of the individual states of the wave packet would translate into an ionization probability P_{ion} that is decreasing with the peak field strength, i.e., DS. Thus, the superintense regime appears like a haven of stability for the neutral atom at all frequencies, low and high, to the extent that relativistic effects can be neglected. The critical issue is, however, to devise strategies for bringing the atom into this regime, without ionizing it during the rise of the pulse, i.e., to create physically the aforementioned superposition of bound Floquet states.

In a following presentation we shall approach the low-frequency regime for our 1D model from the numerical point of view [28]. On that occasion we shall show that the HIHFFT results obtained in Sec. III for our model are in good agreement with the numerical results, within the expected errors.

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[14] See [12], Eqs. 3.137.7, 8.112.1, and 8.121.3.
[15] It may seem contradictory that, on the one hand, we claim to be operating in the vicinity of the end point $+\alpha_0$ when replacing $V_m(x)$ by $\tilde{V}_0(\alpha_0;x_-)$, and, on the other hand, we propose to

use the asymptotic form of the end-point potential $\tilde{V}_0^{as}(\lambda\xi)$ in this area; see Eq. (55). There is no contradiction, however, because the asymptotic distances of interest from the end point are $O(\lambda)\approx\alpha_0^{1/3}$, which are vanishingly small in comparison to the distance between the end points, $2\alpha_0$.

[16] See [12], Eq. 8.113.1 for $x>0$, and Eq. 8.113.3 for $x<0$.

[17] The form of λ, Eq. (61), can be obtained by making use of the substitution $\lambda=\alpha_0^{1/3}\ \mu_0\ \zeta$ in Eq. (56), and solving the equation for ζ, on the assumption that $\ln\zeta$ is negligible (i.e., ζ is not far from 1), which is confirmed *a posteriori*.

[18] In the case of the 1D model with a potential $V(x)=-V_0\delta(x)$, the scaling law of the levels is exactly $\alpha_0^{-2/3}$ as in the 3D case; see T. P. Grozdanov, P. S. Krstic, and M. H. Mittleman, Phys. Lett. A **149**, 144 (1990).

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[22] See [12], Eq. 3.381.5, with $p=0$, $\nu=1/2$.

[23] The Fourier transform $\mathcal{V}(k)$ tends to a constant for $k\to0$, and decreases rapidly as $k\to\infty$ for reasons of satisfaction of Parseval's equation. Hence, $[\mathcal{V}(k)]^2$ is at all k of order 1 or smaller. As $k_m\simeq\sqrt{2m\omega}\to\infty$ with increasing m, this ensures the convergence of the sums over m even at small ω.

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