Ground-state hyperfine structure of the muonic helium atom

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On the basis of perturbation theory using the fine structure constant α and the ratio of the electron to muon masses, we calculate the one-loop vacuum polarization, electron vertex corrections, and nuclear structure corrections to the hyperfine splitting of the ground state of the muonic helium atom ($\mu e\, ^4_2$He). We obtain the total result for the ground-state hyperfine splitting $\Delta\nu^{\rm HFS}$=4465.526 MHz, which improves the previous calculation of Lakdawala and Mohr due to the additional corrections taken into account. The remaining difference between the theoretical result and experimental value of the hyperfine splitting, equal to 0.522 MHz, lies in the range of theoretical error and requires subsequent investigation of higher-order corrections.

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I. INTRODUCTION

Muonic helium atom ($\mu e\, ^4_2$He) represents the simplest three-body atomic system. The interaction between magnetic moments of the muon and electron leads to the hyperfine structure (HFS) of the energy levels. The investigation of the energy spectrum of this three-particle bound state is important for a further check of quantum electrodynamics. Hyperfine splitting of the ground state of muonic helium was measured many years ago with sufficiently high accuracy [1,2]:

$$\Delta\nu^{\rm HFS}_{\rm expt} = 4465.004(29)\ \text{MHz}. \tag{1}$$

In contrast to the energy levels of two-particle bound states, which have been accurately calculated in quantum electrodynamics [3–9], the hyperfine splitting of the ground state in the muonic helium atom was calculated on the basis of perturbation theory (PT) and the variational method with significantly less accuracy [10–21]. Indeed, the theoretical errors of the results obtained in Refs. [10–21] lie in the interval 0.05–1.8 MHz. The variational method gives high numerical accuracy of the calculation, as was demonstrated in Refs. [15,18,20,21]. But higher-order corrections are taken into account in this approach less precisely. So, for instance, the theoretical uncertainty 0.05 MHz in Ref. [18] is estimated only from the numerical convergence of the results obtained by the variational method for the lowest-order hyperfine splitting. A nonvariational calculation of the lowest-order contribution to the hyperfine structure was performed using the hyperspherical harmonic method in Ref. [22]. But numerous important corrections to the hyperfine splitting, which are necessary for a successful comparison with the experimental data, were not considered in [22].

Many theoretical efforts have been focused on the calculation of different corrections to the Fermi energy which is of fourth order in the fine structure constant α. The first of the calculations were devoted to the recoil corrections which contain the ratio of the electron and muon masses [12,17]. The second group was connected with the relativistic and QED effects, which include another small parameter α [10,11,14]. Note that several authors [10,11,14] take the electron coordinate relative to the muon–α-particle center of mass. Another possibility realized in Refs. [12,17] consists in the choice of the lepton coordinates relative to the α particle. So, the numerical results for the corrections of different order in the fine structure constant α and the ratio of the particle masses obtained in these papers are difficult to compare directly.

The bound particles in the muonic helium atom have different masses $m_e \ll m_\mu \ll m_\alpha$. As a result the muon and α particle compose the pseudonucleus $(\mu\, ^4_2{\rm He})^+$ and the muonic helium atom looks like a two-particle system in the first approximation. The perturbation theory approach to the investigation of the hyperfine structure of muonic helium based on the nonrelativistic Schrödinger equation was developed previously by Lakdawala and Mohr in Refs. [12,17]. The three-particle bound system ($\mu e\, ^4_2$He) is described by the Hamiltonian

$$H = H_0 + \Delta H + \Delta H_{\rm rec}, \quad H_0 = -\frac{1}{2M_\mu}\nabla^2_\mu - \frac{1}{2M_e}\nabla^2_e - \frac{2\alpha}{x_\mu} - \frac{\alpha}{x_e}, \tag{2}$$

$$\Delta H = \frac{\alpha}{x_{\mu e}} - \frac{\alpha}{x_e}, \quad \Delta H_{\rm rec} = -\frac{1}{m_\alpha}\boldsymbol{\nabla}_\mu\cdot\boldsymbol{\nabla}_e, \tag{3}$$

where $\mathbf{x}_\mu$ and $\mathbf{x}_e$ are the coordinates of the muon and electron relative to the helium nucleus, and $M_e = m_e m_\alpha/(m_e+m_\alpha)$ and $M_\mu = m_\mu m_\alpha/(m_\mu+m_\alpha)$ are the reduced masses of the subsystems $(e\, ^4_2{\rm He})^+$ and $(\mu\, ^4_2{\rm He})^+$ [12,17]. The hyperfine part of the Hamiltonian is

$$\Delta H^{\rm HFS} = -\frac{8\pi\alpha}{3m_e m_\mu}\frac{(\boldsymbol{\sigma}_e\boldsymbol{\sigma}_\mu)}{4}\delta(\mathbf{x}_\mu - \mathbf{x}_e), \tag{4}$$

where $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_\mu$ are the spin matrices of the electron and muon, the κ_e and κ_μ are the electron and muon anomalous magnetic moments. In the initial approximation the wave function of the ground state has the form [12,17]

$$\Psi_0(\mathbf{x}_e,\mathbf{x}_\mu) = \psi_e(\mathbf{x}_e)\psi_\mu(\mathbf{x}_\mu) = \frac{1}{\pi}(2\alpha^2 M_e M_\mu)^{3/2} e^{-2\alpha M_\mu x_\mu} e^{-\alpha M_e x_e}. \tag{5}$$

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TABLE I. Hyperfine singlet-triplet splitting of the ground state of the muonic helium atom.

Contribution to the HFS	$\Delta\nu^{hfs}$ (MHz)	Reference
Fermi splitting	4516.915	Eq. (6), [12]
Recoil correction of order $\alpha^4(m_e/m_\mu)$	−33.525	Eq. (6), [12]
Correction of muon anomalous magnetic moment of order α^5	5.244	[11,17]
Recoil correction of order $\alpha^4(M_e/m_\alpha)\sqrt{(M_e/M_\mu)}$	0.079	[12,17]
Correction due to the perturbation (3) in second-order PT of order $\alpha^4 M_e/M_\mu$	−29.650	[12,17]
Relativistic correction of order α^6	0.040	[10]
One-loop VP contribution in 1γ electron-muon interaction of order $\alpha^5 M_e/M_\mu$	0.035	Eq. (14)
One-loop VP contribution in the electron-muon interaction in second-order PT of order $\alpha^5 M_e/M_\mu$	−0.145	Eqs. (23), (27), (28), and (30), (31), (40), and (44)
One-loop VP contribution in the electron-nucleus interaction in second-order PT of order $\alpha^5 M_e/M_\mu$	0.151	Eq. (18)
One-loop VP contribution in the muon-nucleus interaction in second-order PT of order $\alpha^5 M_e/M_\mu$	0.048	Eq. (20)
Nuclear structure correction in second-order PT of order α^6	−0.010	Eqs. (47) and (49)
Recoil correction of order $\alpha^5(m_e/m_\mu)\ln(m_e/m_\mu)$	0.812	Eq. (51), [18]
Vertex correction of order α^6	−0.606	[5,31–33]
Electron vertex contribution of order α^5	6.138	Eqs. (53), (56), and (58), (59), (61), and (62)
Summed contribution	4465.526	

Then the basic contribution to the singlet-triplet hyperfine splitting can be calculated analytically from the contact interaction (4):

$$\Delta\nu_0^{\rm HFS} = \left\langle \frac{8\pi\alpha}{3m_e m_\mu}\delta(\mathbf{x}_\mu - \mathbf{x}_e)\right\rangle = \frac{\nu_F}{(1+M_e/2M_\mu)^3},$$

$$\nu_F = \frac{8\alpha^4 M_e^3}{3m_e m_\mu}. \quad (6)$$

Numerically the Fermi splitting is ν_F=4516.915 MHz. From now on we express the hyperfine splitting contributions in frequency units using the relation $\Delta E^{\rm HFS}=2\pi\hbar\Delta\nu^{\rm HFS}$. The recoil correction determined by the ratio M_e/M_μ in Eq. (6) amounts to $\Delta\nu_{\rm rec}^{\rm HFS}=-33.525$ MHz [12]. Modern numerical values of fundamental physical constants are taken from the paper [23]: the electron mass m_e =0.510 998 918(44) × 10^{-3} GeV, the muon mass m_μ =0.105 658 369 2(94) GeV, the fine structure constant α^{-1} =137.035 999 11(46), the helium mass $m(^4_2\mathrm{He})$ =3.727 379 04(15) GeV, the electron anomalous magnetic moment κ_e=1.159 652 186 9(41) × 10^{-3}, and the muon anomalous magnetic moment κ_μ=1.165 919 81(62) × 10^{-3}.

Analytical and numerical calculations of the corrections which are determined by the Hamiltonians ΔH and $\Delta H_{\rm rec}$ in second-order perturbation theory were performed in Refs. [12,17]. Their results and the order of the calculated contributions are presented in Table I. In this work we aim to refine the calculation of Lakdawala and Mohr using their approach to the description of the muonic helium atom. A feature that distinguishes light muonic atoms among the simplest atoms is that the structure, of their energy levels depends strongly on the vacuum polarization, nuclear structure, and recoil effects [3–9]. So we investigate the contributions of the one-loop electron vacuum polarization of order $\alpha^5 M_e/M_\mu$ and the nuclear structure of order α^6 which are significant for the improvement of the theoretical value of the hyperfine splitting. Another purpose of our study consists in the improved calculation of the electron one-loop vertex corrections to the HFS of order α^5 using the analytical expressions for the Dirac and Pauli form factors of the electron.

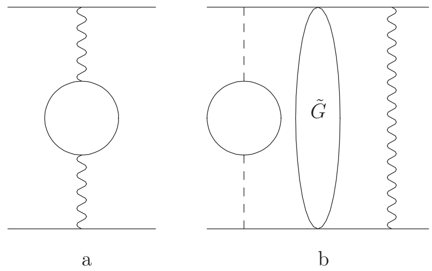

FIG. 1. Vacuum polarization effects. The dashed line represents the Coulomb photon. The wavy line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

II. EFFECTS OF THE VACUUM POLARIZATION

The vacuum effects change the interaction (2) and (3) between particles in the muonic helium atom. One of the most important contributions to the HFS is determined by the one-loop vacuum polarization (VP) and the electron vertex operator. Indeed, the vacuum loop leads to an additional factor α/π in the interaction operator, so that the corresponding correction to the HFS is of fifth order in the fine structure constant. At the same time the electron vacuum polarization and vertex corrections to the hyperfine splitting of the ground state contain a parameter equal to the ratio of the Compton wavelength of the electron and the radius of the Bohr orbit in the subsystem $(\mu\,{}^4_2\mathrm{He})^+$: $m_\mu\alpha/m_e=1.508\,86\ldots$. It appears in the matrix elements using the bound-state wave functions in which the characteristic momentum is of order $m_\mu\alpha$. It is impossible to use expansion over α for such contributions to the energy spectrum. So we calculate them the performing analytical or numerical integration over the particle coordinates and other parameters without an expansion in α. The effect of the electron vacuum polarization leads to the appearance of a number of additional terms in the interaction operator which we present in the form [5,24]

$$\Delta V^{e\alpha}_{\mathrm{VP}}(x_e)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\left(-\frac{2\alpha}{x_e}\right)e^{-2m_e\xi x_e}d\xi,$$

$$\rho(\xi)=\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4},\qquad(7)$$

$$\Delta V^{\mu\alpha}_{\mathrm{VP}}(x_\mu)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\left(-\frac{2\alpha}{x_\mu}\right)e^{-2m_e\xi x_\mu}d\xi,\qquad(8)$$

$$\Delta V^{e\mu}_{\mathrm{VP}}(|\mathbf{x}_e-\mathbf{x}_\mu|)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\frac{\alpha}{x_{e\mu}}e^{-2m_e\xi x_{e\mu}}d\xi,\qquad(9)$$

where $x_{e\mu}=|\mathbf{x}_e-\mathbf{x}_\mu|$. They give contributions to the hyperfine splitting in second-order perturbation theory and are discussed below. In first-order perturbation theory, the contribution of the vacuum polarization is connected with the modification of the hyperfine splitting part of the Hamiltonian (4) [Fig. 1(a)]. In the coordinate representation it is determined by the integral expression [25–27]

$$\Delta V^{\mathrm{HFS}}_{\mathrm{VP}}(\mathbf{x}_{e\mu})=-\frac{8\alpha}{3m_em_\mu}\frac{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)}{4}\frac{\alpha}{3\pi}\times\int_1^\infty\rho(\xi)d\xi\left(\pi\delta(\mathbf{x}_{e\mu})-\frac{m_e^2\xi^2}{x_{e\mu}}e^{-2m_e\xi x_{e\mu}}\right).\qquad(10)$$

Averaging the potential (10) over the wave function (5), we obtain the following contribution to the hyperfine splitting:

$$\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP}}=\frac{8\alpha^2}{9m_em_\mu}\frac{(\alpha M_e)^3(2\alpha M_\mu)^3}{\pi^3}\int_1^\infty\rho(\xi)d\xi\int d\mathbf{x}_e\times\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\times\left(\pi\delta(\mathbf{x}_\mu-\mathbf{x}_e)-\frac{m_e^2\xi^2}{|\mathbf{x}_\mu-\mathbf{x}_e|}\right)e^{-2m_e\xi|\mathbf{x}_\mu-\mathbf{x}_e|}.\qquad(11)$$

There are two integrals over the muon and electron coordinates in Eq. (11) which can be calculated analytically:

$$I_1=\int d\mathbf{x}_e\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\pi\delta(\mathbf{x}_\mu-\mathbf{x}_e)=\frac{\pi^2}{8\alpha^3M_\mu^3(1+M_e/2M_\mu)^3},\qquad(12)$$

$$I_2=\int d\mathbf{x}_e\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\frac{1}{|\mathbf{x}_\mu-\mathbf{x}_e|}e^{-2m_e\xi|\mathbf{x}_\mu-\mathbf{x}_e|}=\frac{32\pi^2}{(4\alpha M_\mu)^5}\frac{\left[\frac{M_e^2}{4M_\mu^2}+\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2+\frac{M_e}{2M_\mu}\left(3+\frac{m_e\xi}{M_\mu\alpha}\right)\right]}{\left(1+\frac{M_e}{2M_\mu}\right)^3\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2\left(\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\right)^2}.\qquad(13)$$

They are divergent separately in the subsequent integration over the parameter ξ. But their sum is finite and can be written in the integral form,

$$\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP}}=\nu_F\frac{\alpha M_e}{6\pi M_\mu(1+M_e/2M_\mu)^3}\int_1^\infty\rho(\xi)d\xi\times\frac{\left[\frac{M_e}{2M_\mu}+2\frac{m_e\xi}{2M_\mu\alpha}\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\left(2+\frac{m_e\xi}{2M_\mu\alpha}\right)\right]}{\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2\left(\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\right)^2}=0.035\ \mathrm{MHz}.\qquad(14)$$

The order of this contribution is determined by two small parameters α and M_e/M_μ which are written explicitly. The correction $\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP}}$ is of fifth order in α and first order in the ratio of the electron and muon masses. The contribution of the muon vacuum polarization to the hyperfine splitting is extremely small ($\sim10^{-6}$ MHz). One should expect that two-loop vacuum polarization contributions to the hyperfine structure are suppressed relative to the one-loop VP contribution by the factor α/π. This means that at the present level of accuracy we can neglect these corrections because their numerical value does not exceed 0.001 MHz. Higher orders of perturbation theory which contain one-loop vacuum polarization and the Coulomb interaction (3) lead to recoil corrections of order $\nu_F\alpha(M_e^2/M_\mu^2)\ln(M_\mu/M_e)$. Such terms, which can contribute 0.004 MHz, are included in the theoretical error.

It is useful to compare the obtained result (14) with a calculation of the VP contribution to the HFS in which the expansion in α is used. Instead of the potential (10) we obtain the following operator in the coordinate representation:

$$\Delta\tilde{V}_{\rm VP}^{\rm HFS}(\mathbf{x}_{e\mu}) = -\frac{8\pi\alpha}{3m_e m_\mu}\frac{\alpha}{15\pi}\frac{1}{m_e^2}\nabla^2\delta(\mathbf{x}_{e\mu}). \quad (15)$$

Then the contribution of (15) to the hyperfine structure can be derived in the analytical form

$$\Delta\tilde{\nu}_{\rm VP}^{\rm HFS} = \nu_F\frac{8\alpha}{15\pi}\left(\frac{\alpha M_\mu}{m_e}\right)^2\frac{M_e}{M_\mu}\frac{1}{(1+M_e/2M_\mu)^3} = 0.060\ \text{MHz}. \quad (16)$$

This calculation demonstrates the need to employ the exact potentials (7)–(9) for the study of the electron vacuum polarization corrections.

Let us consider the corrections of the electron vacuum polarization (7)–(9) in second-sorder perturbation theory (SOPT) [Fig. 1(b)]. The contribution of the electron-nucleus interaction (7) to the hyperfine splitting can be written as follows:

$$\Delta\nu_{\rm VP\ SOPT\ }^{\rm HFS}{}_{e\alpha} = \frac{16\pi\alpha}{3m_e m_\mu}\int d\mathbf{x}_1\int d\mathbf{x}_2 \times\int d\mathbf{x}_3\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\psi_{\mu0}^*(\mathbf{x}_3)\psi_{e0}^*(\mathbf{x}_3) \times\sum_{n,n'\neq0}^\infty\frac{\psi_{\mu n}(\mathbf{x}_3)\psi_{en'}(\mathbf{x}_3)\psi_{\mu n}^*(\mathbf{x}_2)\psi_{en'}^*(\mathbf{x}_1)}{E_{\mu0}+E_{e0}-E_{\mu n}-E_{en'}} \times e^{-2m_e\xi x_1}\psi_{\mu0}(\mathbf{x}_2)\psi_{e0}(\mathbf{x}_1). \quad (17)$$

Here the summation is carried out over the complete system of the eigenstates of the electron and muon, excluding the state with $n,n'=0$. The computation of the expression (17) is simplified with the use of the orthogonality condition for the muon wave functions:

$$\Delta\nu_{\rm VP\ SOPT\ }^{\rm HFS}{}_{e\alpha} = \nu_F\frac{32\alpha M_e^2}{3\pi M_\mu^2}\int_1^\infty\rho(\xi)d\xi\int_0^\infty x_3^3dx_3\int_0^\infty x_1dx_1e^{-x_1(M_e/M_\mu)(1+m_e\xi/\alpha M_\mu)}e^{-2x_3(1+M_e/2M_\mu)} \times\left[\frac{M_\mu}{M_ex_>}-\ln\left(\frac{M_e}{M_\mu}x_<\right)-\ln\left(\frac{M_e}{M_\mu}x_>\right)+\text{Ei}\left(\frac{M_e}{M_\mu}x_<\right)+\frac{7}{2}-2C-\frac{M_e}{2M_\mu}(x_1+x_3)+\frac{1-e^{(M_e/M_\mu)x_<}}{(M_e/M_\mu)x_<}\right] = 0.151\ \text{MHz}, \quad (18)$$

where $x_<=\min(x_1,x_3)$, $x_>=\max(x_1,x_3)$, $C=0.577\,216\ldots$ is Euler's constant, and $\text{Ei}(x)$ is the exponential-integral function. It is necessary to emphasize that the transformation of the expression (17) into (18) is carried out with the use of the compact representation for the electron reduced Coulomb Green's function obtained in Refs. [12,28]:

$$G_e(\mathbf{x}_1,\mathbf{x}_3) = \sum_{n\neq0}^\infty\frac{\psi_{en}(\mathbf{x}_3)\psi_{en}^*(\mathbf{x}_1)}{E_{e0}-E_{en}} = -\frac{\alpha M_e^2}{\pi}e^{-\alpha M_e(x_1+x_3)}\left(\frac{1}{2\alpha M_ex_>}-\ln(2\alpha M_ex_>)-\ln(2\alpha M_ex_<)+\text{Ei}(2\alpha M_ex_<)+\frac{7}{2}-2C-\alpha M_e(x_1+x_3)+\frac{1-e^{2\alpha M_ex_<}}{2\alpha M_ex_<}\right). \quad (19)$$

The contribution (18) has the same order of magnitude $O(\alpha^5M_e/M_\mu)$ as the previous correction (14) in first-order perturbation theory. A similar calculation can be performed in the case the of muon-nucleus vacuum polarization operator (8). The intermediate electron state is the $1S$ state and the reduced Coulomb Green's function of the system appearing in the second-order PT transforms to the Green's function of the muon. The correction of the operator (8) to the hyperfine splitting is obtained in the following integral form:

$$\Delta\nu_{\rm VP\ SOPT\ }^{\rm HFS}{}_{\mu\alpha} = \nu_F\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\int_0^\infty x_3^2dx_3\int_0^\infty x_2dx_2e^{-x_3(1+M_e/2M_\mu)}e^{-x_2(1+m_e\xi/2M_\mu\alpha)} \times\left(\frac{1}{x_>}-\ln x_>-\ln x_<+\text{Ei}(x_<)+\frac{7}{2}-2C-\frac{x_2+x_3}{2}+\frac{1-e^{x_<}}{x_<}\right) = 0.048\ \text{MHz}. \quad (20)$$

The vacuum polarization correction to the HFS which is determined by the operator (9) in second-order perturbation theory is the most difficult for the calculation. Indeed, in this case we have to consider the intermediate excited states for both the muon and electron. Following Ref. [12], we have divided the total contribution into two parts. The first part, in which the intermediate muon is in the $1S$ state, can be written as

$$\Delta\nu_{\rm VP\ SOPT\ }^{\rm HFS}{}_{\mu e}(n=0) = \frac{256\alpha^2(\alpha M_e)^3(2\alpha M_\mu)^3}{9}\int_0^\infty x_3^2dx_3 \times\int_0^\infty x_1^2dx_1e^{-\alpha(M_e+4M_\mu)x_3} \times\int_1^\infty\rho(\xi)d\xi\Delta V_{\rm VP\ \mu}(x_1)G_e(x_1,x_3), \quad (21)$$

where the function $V_{\rm VP\ \mu}(x_1)$ is given by

$$\Delta V_{\mathrm{VP}\ \mu}(x_1) = \int d\mathbf{x}_2 e^{-4\alpha M_\mu x_2}\frac{(2\alpha M_\mu)^3}{\pi}\frac{\alpha}{|\mathbf{x}_1-\mathbf{x}_2|}e^{-2m_e\xi|\mathbf{x}_1-\mathbf{x}_2|}$$
$$= \frac{32\alpha^4 M_\mu^3}{x_1(16\alpha^2 M_\mu^2 - 4m_e^2\xi^2)^2}[8\alpha M_\mu \times(e^{-2m_e\xi x_1} - e^{-4\alpha M_\mu x_1}) + x_1(4m_e^2\xi^2 - 16\alpha^2 M_\mu^2)e^{-4\alpha M_\mu x_1}]. \quad (22)$$

After the substitution of (22) in (21) numerical integration gives the result

$$\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP\ SOPT}\ \mu e}(n=0) = -0.030\ \mathrm{MHz}. \quad (23)$$

The second part of the vacuum polarization correction to the hyperfine splitting due to the electron-muon interaction (9) can be presented as follows:

$$\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP\ SOPT}\ \mu e}(n\neq 0) = -\frac{16\alpha^2}{9m_e m_\mu}\int d\mathbf{x}_3\int d\mathbf{x}_2\int_1^\infty \times\rho(\xi)d\xi\psi^*_{\mu 0}(\mathbf{x}_3)\psi^*_{e0}(\mathbf{x}_3) \times\sum_{n\neq 0}\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)\frac{M_e}{2\pi}\frac{e^{-b|\mathbf{x}_3-\mathbf{x}_1|}}{|\mathbf{x}_3-\mathbf{x}_1|} \times\frac{\alpha}{|\mathbf{x}_2-\mathbf{x}_1|}e^{-2m_e\xi|\mathbf{x}_2-\mathbf{x}_1|}\psi_{\mu 0}(\mathbf{x}_2)\psi_{e0}(\mathbf{x}_1). \quad (24)$$

In the expression (24) we have replaced the exact electron Coulomb Green's function by the free electron Green's function, which contains $b=[2M_e(E_{\mu n}-E_{\mu 0}-E_{e0})]^{1/2}$ (see a more detailed discussion of this approximation in Refs. [12,17]). We also replace the electron wave functions by their values at the origin as in Ref. [12], neglecting higher-order recoil corrections. After that the integration over $\mathbf{x}_1$ can be done analytically:

$$J = \int d\mathbf{x}_1\frac{e^{-b|\mathbf{x}_3-\mathbf{x}_1|}}{|\mathbf{x}_3-\mathbf{x}_1|}\frac{e^{-2m_e\xi|\mathbf{x}_2-\mathbf{x}_1|}}{|\mathbf{x}_2-\mathbf{x}_1|}$$
$$= -\frac{4\pi}{|\mathbf{x}_3-\mathbf{x}_2|}\frac{1}{b^2-4m_e^2\xi^2}(e^{-b|\mathbf{x}_3-\mathbf{x}_2|} - e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|})$$
$$= 2\pi\left(\frac{(1-e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|})}{2m_e^2\xi^2|\mathbf{x}_3-\mathbf{x}_2|} - \frac{b}{2m_e^2\xi^2} + \frac{(1-e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|})b^2}{8m_e^4\xi^4|\mathbf{x}_3-\mathbf{x}_2|} + \frac{b^2|\mathbf{x}_3-\mathbf{x}_2|}{4m_e^2\xi^2} - \frac{b^3}{8m_e^4\xi^4} - \frac{b^3(\mathbf{x}_3-\mathbf{x}_1)^2}{12m_e^2\xi^2} + \cdots\right), \quad (25)$$

where we have performed the expansion of the first exponential in parentheses over powers of $b|\mathbf{x}_3-\mathbf{x}_2|$. As discussed in Ref. [12], one can treat this series as an expansion over the recoil parameter $\sqrt{M_e/M_\mu}$. For the further transformation the completeness condition is useful:

$$\sum_{n\neq 0}\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2) = \delta(\mathbf{x}_3-\mathbf{x}_2) - \psi_{\mu 0}(\mathbf{x}_3)\psi^*_{\mu 0}(\mathbf{x}_2). \quad (26)$$

The wave function orthogonality leads to zero results for the second and fifth terms in large parentheses of Eq. (25). The first term in Eq. (25) gives the leading order contribution in two small parameters α and M_e/M_μ:

$$\Delta\nu^{\mathrm{HFS}}_{\mathrm{VP\ SOPT}\ \mu e}(n\neq 0) = \Delta\nu_{11} + \Delta\nu_{12}, \quad \Delta\nu_{11} = -\frac{3\alpha^2 M_e}{8m_e}\nu_F, \quad (27)$$

$$\Delta\nu_{12} = \nu_F\frac{2\alpha^2}{3\pi m_e/M_e}\int_1^\infty\rho(\xi)\frac{d\xi}{\xi}\frac{M_\mu^4\alpha^4}{(4\alpha M_\mu+2m_e\xi)^4} \times\left(256 + 232\frac{m_e\xi}{M_\mu\alpha} + 80\frac{m_e^2\xi^2}{M_\mu^2\alpha^2} + 10\frac{m_e^3\xi^3}{M_\mu^3\alpha^3}\right). \quad (28)$$

The numerical value of the sum $\Delta\nu_{11}+\Delta\nu_{12}$ is included in Table I. It is important to calculate also the contributions of other terms of the expression (25) to the hyperfine splitting. Taking the fourth term in the second equality of Eq. (25), which is proportional to $b^2=2M_e(E_{\mu n}-E_{\mu 0})$, we have performed the following sequence of transformations in coordinate representation:

$$\sum_{n=0}^\infty E_{\mu n}\int d\mathbf{x}_2\int d\mathbf{x}_3\psi^*_{\mu 0}(\mathbf{x}_2)\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)|\mathbf{x}_3-\mathbf{x}_2|\psi_{\mu 0}(\mathbf{x}_2)$$
$$= \int d\mathbf{x}_2\int d\mathbf{x}_3\delta(\mathbf{x}_3-\mathbf{x}_2) \times\left(-\frac{\nabla_3^2}{2M_\mu}|\mathbf{x}_3-\mathbf{x}_2|\psi^*_{\mu 0}(\mathbf{x}_3)\right)\psi_{\mu 0}(\mathbf{x}_2). \quad (29)$$

Evidently, we have a divergent expression in Eq. (29) due to the presence of the δ function. The same divergence occurs in the other term containing b^2 in the second equality of Eq. (25). But their sum is finite and can be calculated analytically, with the result

$$\Delta\nu^{\mathrm{HFS}}_{b^2} = \nu_F\frac{\alpha^2 M_e^2}{m_e M_\mu}\left(18 - 5\frac{\alpha^2 M_\mu^2}{m_e^2}\right). \quad (30)$$

The numerical value of this correction, 0.0002 MHz, is essentially smaller than the leading order term. Let us consider also the nonzero term in Eq. (25) proportional to b^3. First of all, it can be transformed to the following expression after integration over ξ:

$$\Delta\nu^{\mathrm{HFS}}_{b^3} = -\nu_F\frac{4\alpha^3}{45\pi}\sqrt{\frac{M_e}{M_\mu}}\frac{M_e^2}{m_e^2}S_{3/2}, \quad (31)$$

where the sum $S_{3/2}$ is defined as follows:

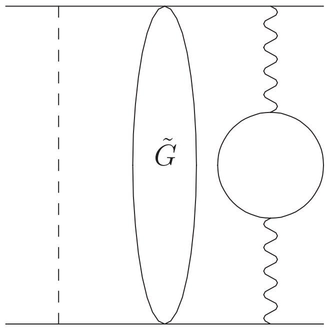

FIG. 2. Vacuum polarization effects in second-order perturbation theory. The dashed line represents the first part of the potential ΔH (3). The wavy line represents the hyperfine part of the Breit potential.

$$S_p=\sum_n\left[\left(\frac{E_{\mu n}-E_{\mu 0}}{R_\mu}\right)^p\right]\left|\langle\psi_{\mu 0}|\frac{\mathbf{x}}{a_\mu}|\psi_{\mu n}\rangle\right|^2, \quad (32)$$

where $R_\mu=2\alpha^2M_\mu$ and $a_\mu=1/2\alpha M_\mu$. Using the known analytical expressions for the dipole matrix elements entering in Eq. (32) in the case of the discrete and continuous spectrum [3,29] we can write their contributions to the sum $S_{3/2}$ separately in the form

$$S^d_{3/2}=\sum_{n=0}^{\infty}\frac{2^8n^4(n-1)^{2n-7/2}}{(n+1)^{2n+7/2}}=1.509\,89\ldots, \quad (33)$$

$$S^c_{3/2}=\int_0^\infty k\,dk\frac{2^8}{(1-e^{-2\pi/k})}\frac{1}{(1+k^2)^{7/2}}\left|\left(\frac{1+ik}{1-ik}\right)^{i/k}\right|^2 = 1.762\,36\ldots. \quad (34)$$

As a result $S_{3/2}=3.2722\ldots$. A similar calculation of the sum $S_{1/2}$ relating to this problem (see Ref. [12]) gives $S_{1/2}=2.9380\ldots$. The numerical value (31) is taken into account in the total result presented in Table I.

There exists another contribution of second-order perturbation theory in which we have the vacuum polarization perturbation connected with the hyperfine splitting part of the Breit potential (10) (see Fig. 2). Another perturbation potential in this case is determined by the first term of relation (3). We can divide this correction into two parts as previously. One part with $n=0$ corresponds to the ground-state muon. The other part with $n\neq 0$ accounts for the excited muon states. The δ-function term in Eq. (10) gives the following contribution to the HFS at $n=0$ (compare with Ref. [12]):

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 11}(n=0)=\nu_F\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\frac{11M_e}{16M_\mu}. \quad (35)$$

Obviously, this integral in the variable ξ is divergent. So we have to consider the contribution of the second term of the potential (10) to the hyperfine splitting, which is determined by the more complicated expression

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 12}(n=0) = \frac{16\alpha^2m_e^2}{9\pi m_em_\mu}\int_1^\infty\rho(\xi)\xi^2d\xi\int d\mathbf{x}_3\psi_{e0}(\mathbf{x}_3)\Delta V_1(\mathbf{x}_3) \times\sum_{n'\neq 0}\frac{\psi_{en'}(\mathbf{x}_3)\psi^*_{en'}(\mathbf{x}_1)}{E_{e0}-E_{en'}}\Delta V_2(\mathbf{x}_1)\psi_{e0}(\mathbf{x}_1), \quad (36)$$

where

$$\Delta V_1(\mathbf{x}_3)=\int d\mathbf{x}_4\psi^*_{\mu 0}(\mathbf{x}_4)\frac{e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_4|}}{|\mathbf{x}_3-\mathbf{x}_4|}\psi_{\mu 0}(\mathbf{x}_4) = \frac{4(2\alpha M_\mu)^3}{x_3[(4\alpha M_\mu)^2-(2m_e\xi)^2]^2}[8\alpha M_\mu e^{-2m_e\xi x_3} + e^{-4\alpha M_\mu x_3}(-8\alpha M_\mu-16\alpha^2M_\mu^2x_3+4m_e^2\xi^2x_3)], \quad (37)$$

$$\Delta V_2(\mathbf{x}_1)=\int d\mathbf{x}_2\psi_{\mu 0}(\mathbf{x}_2)\left(\frac{\alpha}{|\mathbf{x}_2-\mathbf{x}_1|}-\frac{\alpha}{x_1}\right)\psi_{\mu 0}(\mathbf{x}_2) = -\frac{\alpha}{x_1}(1+2\alpha M_\mu x_1)e^{-4\alpha M_\mu x_1}. \quad (38)$$

Nevertheless, integrating over all coordinates in Eq. (36), we obtain the following result in the leading order with respect to the ratio (M_e/M_μ):

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 12}(n=0) = \nu_F\frac{m_e}{M_e}\frac{M_e^2}{96\pi M_\mu^2}\int_1^\infty\rho(\xi)\xi d\xi\frac{32+63\gamma+44\gamma^2+11\gamma^3}{(1+\gamma)^4}, \quad (39)$$

where $\gamma=m_e\xi/2\alpha M_\mu$. This integral also has a divergence at large values of the parameter ξ. But the sum of the integrals (35) and (39) is finite:

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 11}(n=0)+\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 12}(n=0)=0.008\ \text{MHz}. \quad (40)$$

Let us consider now the terms with $n\neq 0$. The δ-like term of the potential (10) gives the contribution to the HFS known from the calculation of Ref. [12]:

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 21}(n\neq 0)=\nu_F\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\left(-\frac{35M_e}{16M_\mu}\right). \quad (41)$$

Another correction from the second term of the expression (10) can be simplified after the replacement of the exact electron Green's function by the free electron Green's function:

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 22}(n\neq 0) = -\frac{16\alpha^3 M_e m_e^2}{9m_e m_\mu}\int_1^\infty \rho(\xi)\xi^2 d\xi \int d\mathbf{x}_2 \int d\mathbf{x}_3 \times \int d\mathbf{x}_4 \psi^*_{\mu 0}(\mathbf{x}_4)\frac{e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_4|}}{|\mathbf{x}_3-\mathbf{x}_4|}\sum_{n\neq 0}^{\infty}\psi_{\mu n}(\mathbf{x}_4)\psi_{\mu n}(\mathbf{x}_2) \times |\mathbf{x}_3-\mathbf{x}_2|\psi_{\mu 0}(\mathbf{x}_2). \quad (42)$$

The analytical integration in Eq. (42) over all coordinates leads to the result

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 22}(n\neq 0) = -\nu_F\frac{\alpha M_e}{3\pi M_\mu}\int_1^\infty \rho(\xi)d\xi\left[\frac{1}{\gamma}-\frac{1}{(1+\gamma)^4} \times\left(4+\frac{1}{\gamma}+10\gamma+\frac{215\gamma^2}{16}+\frac{35\gamma^4}{16}\right)\right]. \quad (43)$$

The sum of expressions (41) and (43) gives again a finite contribution to the hyperfine splitting:

$$\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 21}(n\neq 0)+\Delta\nu^{\rm HFS}_{\rm VP\ SOPT\ 22}(n\neq 0) = -\nu_F\frac{\alpha M_e}{3\pi M_\mu}\int_1^\infty \rho(\xi)d\xi\frac{35+76\gamma+59\gamma^2+16\gamma^3}{16(1+\gamma)^4} = -0.062\ \text{MHz}. \quad (44)$$

Despite the fact that the absolute values of the calculated VP corrections (23), (27), (28), (30), (31), (40), and (44) are sufficiently large, their summed contribution to the hyperfine splitting (see Table I) is small because they have different signs.

III. NUCLEAR STRUCTURE AND RECOIL EFFECTS

Other significant corrections to the hyperfine splitting of muonic helium atoms that we study in this work are determined by the nuclear structure effects. They are specific for any muonic atom. In the leading order over α they are described by the charge radius of the α particle r_α. If we consider the interaction between the muon and the nucleus, then the nuclear structure correction to the interaction operator has the form [5]

$$\Delta V_{{\rm str},\mu}(\mathbf{r}_\mu) = \frac{2}{3}\pi Z\alpha\langle r_\alpha^2\rangle\delta(\mathbf{r}_\mu). \quad (45)$$

The contribution of the operator $\Delta V_{{\rm str},\mu}$ to the hyperfine splitting appears in second-order perturbation theory (see the diagram in Fig. 3). First we can write it in the integral form

$$\Delta\nu^{\rm HFS}_{{\rm str},\mu} = \frac{64\pi^2\alpha^2}{9m_e m_\mu}r_\alpha^2\frac{1}{\sqrt{\pi}}(2\alpha M_\mu)^{3/2}\int d\mathbf{x}_3\psi^*_{\mu 0}(\mathbf{x}_3) \times|\psi_{e0}(\mathbf{x}_3)|^2 G_\mu(\mathbf{x}_3,0,E_{\mu 0}). \quad (46)$$

After that the analytical integration over the coordinate $\mathbf{x}_3$ in Eq. (46) can be carried out using a representation of the muon Green's function similar to expression (17). The result

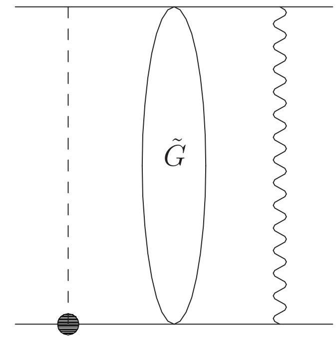

FIG. 3. Nuclear structure effects in second-order perturbation theory. The bold point represents the nuclear vertex operator. The wavy line represents the hyperfine part of the Breit potential.

of the integration of order $O(\alpha^6)$ is written as an expansion in the ratio M_e/M_μ:

$$\Delta\nu^{\rm HFS}_{{\rm str},\mu} = -\nu_F\frac{8}{3}\alpha^2 M_\mu^2 r_\alpha^2\left(3\frac{M_e}{M_\mu}-\frac{11}{2}\frac{M_e^2}{M_\mu^2}+\cdots\right) = -0.007\ \text{MHz}. \quad (47)$$

The numerical value of the contribution $\Delta\nu^{\rm HFS}_{{\rm str},\mu}$ is obtained by means of the charge radius of the α particle, r_α=1.676 fm. The same approach can be used in the study of the electron-nucleus interaction. The electron feels as well the distribution of the electric charge of α particle. The corresponding contribution of the nuclear structure effect to the hyperfine splitting is determined by the expression

$$\Delta\nu^{\rm HFS}_{{\rm str},e} = \frac{64\pi^2\alpha^2}{9m_e m_\mu}r_\alpha^2\int d\mathbf{x}_1\int d\mathbf{x}_3|\psi^*_{\mu 0}(\mathbf{x}_3)|^2\psi_{e0}(\mathbf{x}_3)G_\mu \times(\mathbf{x}_3,\mathbf{x}_1,E_{e0})\psi_{e0}(\mathbf{x}_1)\delta(\mathbf{x}_1). \quad (48)$$

Performing the analytical integration in Eq. (46), we obtain the following series:

$$\Delta\nu^{\rm HFS}_{{\rm str},e} = -\nu_F\frac{4}{3}\alpha^2 M_e^2 r_\alpha^2\left[5-\ln\frac{M_e}{M_\mu}+\frac{M_e^2}{M_\mu^2}\left(3\ln\frac{M_e}{M_\mu}-7\right)+\frac{M_e^2}{M_\mu^2}\left(\frac{17}{2}-3\ln\frac{M_e}{M_\mu}\right)\cdots\right] = -0.003\ \text{MHz}. \quad (49)$$

We have included in Table I the total nuclear structure contribution, which is equal to the sum of the numerical values (45) and (47).

Special attention has to be given to the recoil corrections connected with the two-photon exchange diagrams shown in Fig. 4 in the case of the electron-muon interaction. For the singlet-triplet splitting, the leading order recoil contribution to the interaction operator between the muon and electron is determined as follows [5,18,30]:

$$\Delta V^{\rm HFS}_{{\rm rec},\mu e}(\mathbf{x}_{\mu e}) = 8\frac{\alpha^2}{m_\mu^2-m_e^2}\ln\frac{m_\mu}{m_e}\delta(\mathbf{x}_{\mu e}). \quad (50)$$

Averaging the potential $\Delta V^{\rm HFS}_{{\rm rec},\mu e}$ over the wave functions (5), we obtain the leading order recoil correction to the hyperfine splitting:

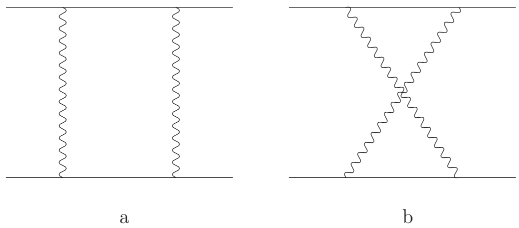

FIG. 4. Two-photon exchange amplitudes in the electron-muon hyperfine interaction.

$$\Delta \nu^{\rm HFS}_{{\rm rec},\mu e} = \nu_F \frac{3\alpha}{\pi}\frac{m_e m_\mu}{m_\mu^2 - m_e^2} \ln \frac{m_\mu}{m_e} = 0.812 \text{ MHz}. \quad (51)$$

There exist also two-photon interactions between the bound particles of the muonic helium atom when one hyperfine photon transfers the interaction from the electron to the muon and another Coulomb photon from the electron to the nucleus (or from the muon to the nucleus). Supposing that these amplitudes give smaller contributions to the hyperfine splitting, we included them in the theoretical error.

IV. ELECTRON VERTEX CORRECTIONS

In the initial approximation, the potential of the hyperfine splitting is determined by Eq. (4). It leads to energy splitting of order α^4. In QED perturbation theory there is an electron vertex correction to the potential (4) which is defined by the diagram in Fig. 5(a). In momentum representation the corresponding operator of the hyperfine interaction has the form

$$\Delta V^{\rm HFS}_{\rm vertex}(k^2) = -\frac{8\alpha^2}{3m_e m_\mu}\left(\frac{\boldsymbol{\sigma}_e \boldsymbol{\sigma}_\mu}{4}\right)[G_M^{(e)}(k^2) - 1], \quad (52)$$

where $G_M^{(e)}(k^2)$ is the electron magnetic form factor. We extracted for convenience the factor α/π from $[G_M^{(e)}(k^2)-1]$. The approximation usually used for the electron magnetic form factor, $G_M^{(e)}(k^2) \approx G_M^{(e)}(0) = 1+\kappa_e$, is not quite correct for this task. Indeed, the characteristic momentum of the exchanged photon is $k \sim \alpha M_\mu$. It is impossible to neglect it in the magnetic form factor as compared with the electron mass m_e. So we should use the exact one-loop expression for the magnetic form factor which has been obtained by many authors [24]. Let us note that the Dirac form factor of the electron is dependent on the parameter of the infrared cutoff

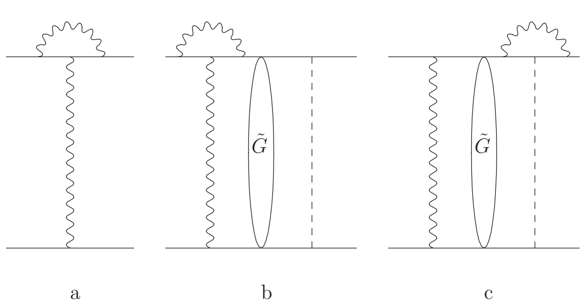

FIG. 5. Electron vertex corrections. The dashed line represents the Coulomb photon. The wavy line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

λ. We take it in the form $\lambda = m_e\alpha$, using the prescription $m_e\alpha^2 \ll \lambda \ll m_e$ from Ref. [3].

Using the Fourier transform of the potential (52) and averaging the obtained expression over the wave functions (5), we represent the electron vertex correction to the hyperfine splitting as follows:

$$\Delta \nu^{\rm HFS}_{\rm vertex} = \nu_F \frac{\alpha}{32\pi^2}\left(\frac{M_e}{M_\mu}\right)\left(\frac{m_e}{\alpha M_\mu}\right)^3 \int_0^\infty k^2 dk [G_M^{(e)}(k^2) - 1] \times \left\{\left[1 + \left(\frac{m_e}{4\alpha M_\mu}\right)^2 k^2\right]\left[\left(\frac{M_e}{2M_\mu}\right)^2 + \left(\frac{m_e}{4\alpha M_\mu}\right)^2 k^2\right]^2\right\}^{-1} = 4.214 \text{ MHz}. \quad (53)$$

Let us remark that the contribution (53) is of order α^5. The numerical value (53) is obtained after numerical integration with the one-loop expression of the electron magnetic form factor $G_M^{(e)}(k^2)$. If we use the value $G_M^{(e)}(k^2=0)$ then the electron vertex correction is equal to 5.244 MHz. So, by using the exact expression for the electron form factor in the one-loop approximation we observe a 1 MHz decrease of the vertex correction to the hyperfine splitting from 1γ interaction. Taking the expression (52) as an additional perturbation potential we have to calculate its contribution to the HFS in second-order perturbation theory [see the diagram in Fig. 5(b)]. In this case the dashed line represents the Coulomb Hamiltonian ΔH (3). Following the method of calculation formulated in the previous section (see also Refs. [12,17]), we again divide the total contribution from the amplitude in Fig. 5(b) into two parts, which correspond to the muon ground ($n=0$) and excited intermediate states ($n \neq 0$). In this way the first contribution with $n=0$ takes the form

$$\Delta \nu^{\rm HFS}_{\rm vertex}(n=0) = \frac{8\alpha^2}{3\pi^2 m_e m_\mu}\int_0^\infty k[G_M^{(e)}(k^2) - 1] \times dk \int d\mathbf{x}_1 \int d\mathbf{x}_3 \psi_{e0}(\mathbf{x}_3) \times \Delta\tilde{V}_1(k,\mathbf{x}_3) G_e(\mathbf{x}_1,\mathbf{x}_3) \Delta V_2(\mathbf{x}_1) \psi_{e0}(\mathbf{x}_1), \quad (54)$$

where $\Delta V_2(\mathbf{x}_1)$ is defined by Eq. (38) and

$$\Delta\tilde{V}_1(k,\mathbf{x}_3) = \int d\mathbf{x}_4 \psi_{\mu0}(\mathbf{x}_4) \frac{\sin(k|\mathbf{x}_3 - \mathbf{x}_4|)}{|\mathbf{x}_3 - \mathbf{x}_4|}\psi_{\mu0}(\mathbf{x}_4) = \frac{\sin(kx_3/4\alpha M_\mu)}{x_3}\frac{1}{[1 + k^2/(4\alpha M_\mu)^2]^2}. \quad (55)$$

Substituting the electron Green's function (19) in Eq. (54), we transform the desired relation to the integral form,

$$\Delta\nu^{\rm HFS}_{\rm vertex}(n=0)=\nu_F\frac{\alpha}{16\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^2\left(\frac{M_e}{M_\mu}\right)^2\int_0^\infty \times\frac{k[G_M^{(e)}(k^2)-1]dk}{[1+m_e^2k^2/(4\alpha M_\mu)^2]^2}\times\int_0^\infty x_3e^{-(M_e/2M_\mu)x_3}\sin\left(\frac{m_ek}{4\alpha M_\mu}x_3\right)dx_3\times\int_0^\infty x_1\left(1+\frac{x_1}{2}\right)e^{-x_1(1+M_e/2M_\mu)}dx_1\times\left[\frac{2M_\mu}{M_ex_>}-\ln\left(\frac{M_e}{2M_\mu}x_<\right)-\ln\left(\frac{M_e}{2M_\mu}x_>\right)+{\rm Ei}\left(\frac{M_e}{2M_\mu}x_<\right)+\frac{7}{2}-2C-\frac{M_e}{4M_\mu}(x_1+x_3)+\frac{1-e^{(M_e/2M_\mu)x_<}}{M_e/2M_\mu x_<}\right]=-0.210\ {\rm MHz}. \quad (56)$$

One integration over the coordinate x_1 is carried out analytically and two other integrations are performed numerically. The second part of the vertex contribution [Fig. 5(b)] with $n\neq 0$ can be reduced to the following form after several simplifications which are discussed in Sec. II (see also Refs. [12,17]):

$$\Delta\nu^{\rm HFS}_{\rm vertex}(n\neq 0)=\nu_F\frac{8\alpha^4M_eM_\mu^3}{\pi^3}\int e^{-2\alpha M_\mu x_2}d\mathbf{x}_2\times\int e^{-\alpha M_ex_3}d\mathbf{x}_3\int e^{-2\alpha M_\mu x_4}d\mathbf{x}_4\times\int_0^\infty k\sin(k|\mathbf{x}_3-\mathbf{x}_4|)\times[G_M^{(e)}(k^2)-1]\frac{|\mathbf{x}_3-\mathbf{x}_2|}{|\mathbf{x}_3-\mathbf{x}_4|}\times[\delta(\mathbf{x}_4-\mathbf{x}_2)-\psi_{\mu0}(\mathbf{x}_4)\psi_{\mu0}(\mathbf{x}_2)]. \quad (57)$$

We divide the expression (57) into two parts as provided by the two terms in the square brackets of the last term. After that the integration (57) over the coordinates $\mathbf{x}_1$ and $\mathbf{x}_3$ is carried out analytically. In the end we obtain ($\gamma_1=M_e/4M_\mu$, $\gamma_2=m_ek/4\alpha M_\mu$)

$$\Delta\nu^{\rm HFS}_{1,\rm vertex}(n\neq 0)=\nu_F\frac{\alpha}{32\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^3\frac{M_e}{M_\mu}\int_0^\infty\times k^2[G_M^{(e)}(k^2)-1]dk\times\left(\frac{4\gamma_1(\gamma_1^2-1)}{(1+\gamma_2^2)^3}-\frac{\gamma_1(3+\gamma_1^2)}{(1+\gamma_2^2)^2}+\frac{4\gamma_1^2(\gamma_1^2-1)}{(\gamma_1^2+\gamma_2^2)^3}+\frac{1+3\gamma_1^2}{(\gamma_1^2+\gamma_2^2)^2}\right)=2.516\ {\rm MHz}, \quad (58)$$

$$\Delta\nu^{\rm HFS}_{2,\rm vertex}(n\neq 0)=-\nu_F\frac{\alpha}{32\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^3\frac{M_e}{M_\mu}\int_0^\infty k^2\times[G_M^{(e)}(k^2)-1]dk\frac{1}{(1+\gamma_2^2)^2}\left(\frac{2}{(\gamma_1^2+\gamma_2^2)}-\frac{(\gamma_1+1)}{[(1+\gamma_1)^2+\gamma_2^2]^2}-\frac{2}{(\gamma_1+1)^2+\gamma_2^2}-\frac{\gamma_2^2-3\gamma_1^2}{(\gamma_1^2+\gamma_2^2)^3}\right)=-0.831\ {\rm MHz}. \quad (59)$$

It is necessary to emphasize that the theoretical error in the contributions $\Delta\nu^{\rm HFS}_{1,2,\rm vertex}(n\neq 0)$ is determined by the factor $\sqrt{M_e/M_\mu}$ connected with the omitted terms of an expansion similar to Eq. (25) (see also Refs. [12,17]). It can amount to 10% of the results (58) and (59) which is a value near 0.2 MHz.

Until now we have considered the electron vertex corrections connected with the hyperfine part of the interaction Hamiltonian (4). But in second-order perturbation theory we should analyze vertex corrections to the Coulomb interactions of the electron and muon and electron and nucleus. Then in the coordinate representation we have the following potential:

$$\Delta V^C_{{\rm vertex},eN}(x_e)+\Delta V^C_{{\rm vertex},e\mu}(x_{e\mu})=\frac{2\alpha^2}{\pi^2}\int_0^\infty\frac{[G_E^{(e)}(k^2)-1]}{k}dk\left(\frac{\sin(kx_{e\mu})}{x_{e\mu}}-2\frac{\sin(kx_e)}{x_e}\right), \quad (60)$$

where we extract again the factor α/π from $[G_E^{(e)}(k^2)-1]$. $G_E^{(e)}$ is the electron electric form factor. One part of the contribution in Fig. 5(c) is specified by electron-muon intermediate states in which the muon is in the ground state $n=0$. This correction is determined by both terms in large parentheses of Eq. (60) and can be presented as follows:

$$\Delta\nu^{\rm HFS}_{C,\rm vertex}(n=0)=\nu_F\frac{\alpha}{\pi^2}\left(\frac{M_e}{M_\mu}\right)^2\int_0^\infty x_3^2e^{-x_3(1+M_e/2M_\mu)}dx_3\times\int_0^\infty x_1e^{-(M_e/2M_\mu)x_1}dx_1\int_0^\infty\times\frac{[G_E^{(e)}(k^2)-1]dk}{k}\sin\left(\frac{m_ek}{4\alpha M_\mu}x_1\right)\times\left\{1-\frac{1}{2[m_e^2k^2/(4\alpha M_\mu)^2+1]^2}\right\}\times\left[\frac{2M_\mu}{M_ex_>}-\ln\left(\frac{M_e}{2M_\mu}x_<\right)-\ln\left(\frac{M_e}{2M_\mu}x_>\right)+{\rm Ei}\left(\frac{M_e}{2M_\mu}x_<\right)+\frac{7}{2}-2C-\frac{M_e}{4M_\mu}(x_1+x_3)+\frac{1-e^{(M_e/2M_\mu)x_<}}{(M_e/2M_\mu)x_<}\right]=-1.321\ {\rm MHz}. \quad (61)$$

The index C means that the vertex correction to the Coulomb part of the Hamiltonian is considered. Excited states of the muon ($n \neq 0$) contribute to another part of the matrix element [Fig. 5(c)]. By changing the Coulomb Green's function of the electron to the free Green's function (see discussion in Sec. II), we can make a coordinate integration and express the correction to the HFS as a one-dimensional integral

$$\Delta\nu^{\rm HFS}_{C,\rm vertex}(n\neq 0) = -\nu_F \frac{8\alpha}{\pi^2}\frac{M_e}{M_\mu}\left(\frac{\alpha M_\mu}{m_e}\right)\int_0^\infty \frac{[G_E^{(e)}(k^2)-1]dk}{k^2} \times\left(1-\frac{1}{[1+m_e^2k^2/(4\alpha M_\mu)^2]^4}\right) = 1.770\ \text{MHz}. \qquad (62)$$

The electron vertex corrections investigated in this section are of order α^5 in the hyperfine interval. The summed value of all the obtained contributions (53), (56), (58), (59), (61), and (62) is equal to 6.138 MHz. It differs by a significant value, 0.894 MHz, from the result 5.244 MHz which was used previously by many authors for the estimation of the electron anomalous magnetic moment contribution.

V. CONCLUSIONS

In the present study, we have performed analytical and numerical calculations of several important contributions to the hyperfine splitting of the ground state in the muonic helium atom connected with the vacuum polarization, the nuclear structure effects, and the electron vertex corrections. To solve this task we use the method of perturbation theory which was formulated previously for the description of the muonic helium hyperfine splitting in Refs. [12,17]. We have considered corrections of order α^5 of the electron vacuum polarization and electromagnetic form factors and nuclear structure effects of order α^6. The numerical values of the corresponding contributions are displayed in Table I. We present in Table I the references to the calculations of other corrections which are not considered here. The relativistic correction was obtained in Ref. [10], the vertex correction was calculated in the case of hydrogenic atoms in Refs. [5,31–33]. Basic contributions to the hyperfine splitting obtained by Lakdawala and Mohr are also included in Table I because our calculation is closely related to their approach.

Let us list a number of features of the calculation.

(1) For the muonic helium atom, the vacuum polarization effects are important and give rise to a modification of the two-particle interaction potential which provides ($\alpha^5 M_e/M_\mu$)-order corrections to the hyperfine structure. The next to leading order vacuum polarization corrections (two-loop vacuum polarization) are negligible.

(2) The electron vertex corrections should be considered with exact account of the one-loop electromagnetic form factors of the electron because the characteristic momentum incoming in the electron vertex operator is of order of the electron mass.

(3) At α^6 order the nuclear structure corrections to the ground-state hyperfine splitting are expressed in terms of the charge radius of the α particle.

(4) Analyzing the one-loop electron vacuum polarization and vertex effects and the nuclear structure contributions at each order of α, we have taken into account recoil terms proportional to the ratio of the electron and muon masses.

The resulting numerical value 4465.526 MHz of the ground-state hyperfine splitting in muonic helium is presented in Table I. It is sufficiently close both to the experimental result (1) and to the earlier performed calculations by perturbation theory, the variational approach, and the Born-Oppenheimer theory: 4464.3 ± 1.8 MHz [17], 4465.0 ± 0.3 MHz [16], 4462.9 MHz [14], 4450.4 ± 0.4 MHz [13], 4459.9 MHz [15], and 4464.87 ± 0.05 MHz [19]. The estimation of the theoretical uncertainty can be done in terms of the Fermi energy ν_F and the small parameters α and the ratio of the particle masses. In our opinion there exist several main sources of theoretical errors. First of all, as we mentioned above, a comprehensive analytical and numerical calculation of recoil corrections of orders α^4/M_eM_μ, $\alpha^4M_e^2/M_\mu^2$, $\alpha^4(M_e^2/M_\mu^2)\ln(M_\mu/M_e)$ was carried out by Lakdawala and Mohr in second-order PT in Refs. [12,17]. The error of their calculation connected with the correction $\nu_F(M_e^2/M_\mu^2)\ln(M_\mu/M_e)$ is 0.6 MHz. The second source of error is related to contributions of order $\alpha^2\nu_F\approx 0.2$ MHz, which appear both from QED amplitudes and in higher orders of perturbation theory. Another part of the theoretical error is determined by the two-photon three-body exchange amplitudes mentioned above. They are of the fifth order in α and contain the recoil parameter $(m_e/m_\alpha)\ln(m_e/m_\alpha)$, so that their possible numerical value can be ± 0.05 MHz. Finally, a part of the theoretical error is connected with our calculation of the electron vertex corrections of order α^5 in Sec. IV. It consists of at least 0.2 MHz [see the discussion after Eq. (59)]. We neglect also the electron vertex contributions of order $\nu_F\alpha M_e/M_\mu\approx 0.2$ MHz which appear in higher orders of the perturbation theory. Thereby, the total theoretical uncertainty does not exceed ± 0.7 MHz. The existing difference between the theoretical result obtained and the experimental value of the hyperfine splitting (1) equal to 0.522 MHz lies in the range of the total error. The theoretical error, which remains rather large in comparison with the experimental uncertainty, suggests further theoretical investigation of the higher-order contributions including more careful construction of the three-particle interaction operator connected with the multiphoton exchanges.

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