Quaternions, octonions, and Bell-type inequalities

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Multipartite Bell-type inequalities are derived for general systems. They involve up to eight observables with arbitrary spectra on each site. These inequalities are closely related to the algebras of quaternions and octonions.

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I. INTRODUCTION

In their famous paper Einstein, Podolsky, and Rosen (EPR) suggested a Gedankenexperiment [1]. As they believed, it would prove the incompleteness of quantum mechanics. An interesting analysis of this problem was given by Bohr [2], who did not agree with EPR. Almost 30 years later, in 1964, in his remarkable paper Bell proposed a quantitative test which should resolve the problem of completeness of quantum mechanics [3]. He showed that the assumptions of the EPR arguments lead to some restrictions on multipartite correlations, which are now referred to as Bell inequalities. If Bohr's arguments are correct, then these inequalities can be violated. In the 1980s the first experimental violations of these inequalities were demonstrated [4,5], and so the dispute was resolved in Bohr's favor.

There exist many different Bell-type inequalities. Most of them deal with dichotomic observables or with generalizations to observables with more general discrete spectra, for example, see [6]. On the other hand, in the original EPR paper the situation of observables with a continuous spectrum was considered. Presently, the theory of Bell-type inequalities is not well established for observables with arbitrary spectra.

The first multipartite Bell-type inequality, valid for arbitrary observables, has been obtained recently [7]. In the simplest, bipartite, case with two observables $\hat{A}_j$, $\hat{B}_j$ on each site $(j=1,2)$ it reads as

$$|\langle(\hat{A}_1+i\hat{B}_1)(\hat{A}_2+i\hat{B}_2)\rangle|^2 = \langle\hat{A}_1\hat{A}_2-\hat{B}_1\hat{B}_2\rangle^2+\langle\hat{B}_1\hat{A}_2+\hat{A}_1\hat{B}_2\rangle^2 \leqslant \langle(\hat{A}_1^2+\hat{B}_1^2)(\hat{A}_2^2+\hat{B}_2^2)\rangle. \quad (1)$$

The original proof is based on ignoring local commutators in the quantum mechanical analog of a classical inequality. In some sense such a procedure is ambiguous, since only those commutators appearing explicitly are ignored. Furthermore, no direct proof was given that the resulting inequality is fulfilled for any separable quantum state—only in this case it should be called Bell-type. For dichotomic observables it was shown that the inequality (1) cannot be violated for a two-qubit system [8]. More recently it has also been proven that, for measuring two quadratures on each site, this inequality can never be violated [9].

In this paper we give a strict and unambiguous proof of general Bell-type inequalities, including their relation to the separability problem. It allows us to consider up to eight arbitrary observables at each site, for a complex quantum system. The multipartite extension of the inequalities is based on the algebra of quarternions and octonions. The recently proposed inequalities [7] will appear as special cases of our approach.

The paper is structured as follows. In Sec. II we present the main idea to derive Bell-type inequalities from square identities and provide all possible square identities. Based on the Euler four-square identity, in Sec. III we obtain a bipartite Bell-type inequality with four observables per site, which can be extended to the multipartite case by using the algebra of quaternions. In Sec. IV the Degen eight-square identity is used to derive a bipartite Bell-type inequality with eight observables per site, which can be extended to many parties by applying the algebra of octonions. The relation of our inequalities to quantum nonlocality is discussed in Sec. V. In Sec. VI we summarize our results.

II. SQUARE IDENTITIES

Let us start to prove that each bipartite separable state satisfies the inequality (1). This approach also allows us to obtain Bell-type inequalities with four and eight observables on each site. First, we prove a more general statement: If $\hat{F}_1,\ldots,\hat{F}_N$ and $\hat{G}$ are Hermitian operators such that some (in general, multipartite) states $\hat{\varrho}_m$, $m=0,1,\ldots$, satisfy the inequality

$$\langle\hat{F}_1\rangle^2+\cdots+\langle\hat{F}_N\rangle^2 \leqslant \langle\hat{G}\rangle, \quad (2)$$

then arbitrary mixtures (i.e., convex combinations) of these states also satisfy this inequality. The proof is based on the inequality

$$\left|\sum_{m=0}^{+\infty} p_m x_m\right|^2 \leqslant \sum_{m=0}^{+\infty} p_m |x_m|^2, \quad (3)$$

where x_m are arbitrary complex numbers and p_m is a probability distribution, i.e., $p_m \geqslant 0$ and $\Sigma_{m=0}^{+\infty} p_m = 1$. This inequality simply expresses the non-negativity of the variance of a random variable.

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Let us now take a convex combination $\hat{\varrho}=\sum_{m=0}^{+\infty}p_m\hat{\varrho}_m$ and estimate the left-hand side of the inequality (2). According to the inequality (3) we have

$$\langle \hat{F}_l\rangle^2 = \left(\sum_{m=0}^{+\infty} p_m \langle \hat{F}_l\rangle_m\right)^2 \leqslant \sum_{m=0}^{+\infty} p_m \langle \hat{F}_l\rangle_m^2, \quad (4)$$

where $\langle \hat{F}_l\rangle_m$ is the average value on the state $\hat{\varrho}_m$. Now we can estimate the left-hand side of (2) as follows:

$$\sum_{l=1}^{N} \langle \hat{F}_l\rangle^2 \leqslant \sum_{m=0}^{+\infty} p_m \sum_{l=1}^{N} \langle \hat{F}_l\rangle_m^2 \leqslant \sum_{m=0}^{+\infty} p_m \langle \hat{G}\rangle_m = \langle \hat{G}\rangle. \quad (5)$$

So, we have obtained the desired result.

Now we can easily prove the inequality (1) for separable states. Here we have $k=2$ and $\hat{F}_1=\hat{A}_1\hat{A}_2-\hat{B}_1\hat{B}_2$, $\hat{F}_2=\hat{B}_1\hat{A}_2+\hat{A}_1\hat{B}_2$, and $\hat{G}=(\hat{A}_1^2+\hat{B}_1^2)(\hat{A}_2^2+\hat{B}_2^2)$. We must show that the inequality (1) is valid for all factorizable states. Then we obtain

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2-\hat{B}_1\hat{B}_2\rangle^2 + \langle \hat{B}_1\hat{A}_2+\hat{A}_1\hat{B}_2\rangle^2 &= (\langle \hat{A}_1\rangle^2+\langle \hat{B}_1\rangle^2)(\langle \hat{A}_2\rangle^2+\langle \hat{B}_2\rangle^2)\\ &\leqslant \langle (\hat{A}_1^2+\hat{B}_1^2)(\hat{A}_2^2+\hat{B}_2^2)\rangle. \quad (6)\end{aligned}$$

The equality in this chain is valid since only squares remain in the sum and mixed terms cancel each other. The last step, the inequality, is valid since it just expresses the fact that the variance of an observable is non-negative. We conclude that each convex combination of factorizable states, i.e., each separable state, satisfies the inequality (1).

We see that the key point in our proof of the inequality (1) is the estimation (6), which can be divided into two steps. The first step, the equality, can be expressed as the following square identity:

$$(a_1a_2-b_1b_2)^2+(a_1b_2+b_1a_2)^2=(a_1^2+b_1^2)(a_2^2+b_2^2). \quad (7)$$

The second step is valid due to the non-negativity of the variance of observables. Having an identity of the form

$$(a_1^2+b_1^2+\cdots)(a_2^2+b_2^2+\cdots)=x^2+y^2+\cdots, \quad (8)$$

where all the sums contain the same number n of terms and $x,y,\ldots$ are bilinear functions of $a_l,b_l,\ldots,l=1,2$, we can immediately write a Bell-type inequality

$$\langle \hat{X}\rangle^2+\langle \hat{Y}\rangle^2+\cdots \leqslant \langle (\hat{A}_1^2+\hat{B}_1^2+\cdots)(\hat{A}_2^2+\hat{B}_2^2+\cdots)\rangle, \quad (9)$$

where $\hat{X},\hat{Y},\ldots$ are the Hermitian operators obtained by replacing $a_l,b_l,\ldots$ by arbitrary Hermitian operators $\hat{A}_l,\hat{B}_l,\ldots,l=1,2$ in the bilinear forms $x,y,\ldots$, respectively. Let us again formulate the reasons why this inequality is valid for all separable states. First, it is of the form (2), i.e., if it is valid for some states, it is also valid for their mixtures. Second, it is valid for all factorizable states due to the identity (8) and non-negativity of the variances of observables. It follows that it is valid for all mixtures of any factorizable states, i.e., for all separable states.

Which square identities exist? The case of $n=2$ was considered above. There are also square identities for $n=4$ and $n=8$, for more details we refer to [10]. The first one is known as the Euler four-square identity, which reads as follows:

$$\begin{aligned}&(a_1a_2-b_1b_2-c_1c_2-d_1d_2)^2+(b_1a_2+a_1b_2-d_1c_2+c_1d_2)^2+(c_1a_2+d_1b_2+a_1c_2-b_1d_2)^2+(d_1a_2-c_1b_2+b_1c_2+a_1d_2)^2\\ &\quad=(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2). \quad (10)\end{aligned}$$

The second one is referred to as the Degen eight-square identity, which is given by

$$\begin{aligned}&(a_1a_2-b_1b_2-c_1c_2-d_1d_2-e_1e_2-f_1f_2-g_1g_2-h_1h_2)^2+(b_1a_2+a_1b_2+d_1c_2-c_1d_2+f_1e_2-e_1f_2-h_1g_2+g_1h_2)^2\\ &\quad+(c_1a_2-d_1b_2+a_1c_2+b_1d_2+g_1e_2+h_1f_2-e_1g_2-f_1h_2)^2+(d_1a_2+c_1b_2-b_1c_2+a_1d_2+h_1e_2-g_1f_2+f_1g_2-e_1h_2)^2\\ &\quad+(e_1a_2-f_1b_2-g_1c_2-h_1d_2+a_1e_2+b_1f_2+c_1g_2+d_1h_2)^2+(f_1a_2+e_1b_2-h_1c_2+g_1d_2-b_1e_2+a_1f_2-d_1g_2+c_1h_2)^2\\ &\quad+(g_1a_2+h_1b_2+e_1c_2-f_1d_2-c_1e_2+d_1f_2+a_1g_2-b_1h_2)^2+(h_1a_2-g_1b_2+f_1c_2+e_1d_2-d_1e_2-c_1f_2+b_1g_2+a_1h_2)^2\\ &=(a_1^2+b_1^2+c_1^2+d_1^2+e_1^2+f_1^2+g_1^2+h_1^2)(a_2^2+b_2^2+c_2^2+d_2^2+e_2^2+f_2^2+g_2^2+h_2^2). \quad (11)\end{aligned}$$

The famous Hurwitz theorem states that there are no other identities of such a form [10].

III. BELL-TYPE INEQUALITIES WITH FOUR OBSERVABLES

The inequality corresponding to the identity (10) is

$$\begin{aligned}&\langle \hat{A}_1\hat{A}_2-\hat{B}_1\hat{B}_2-\hat{C}_1\hat{C}_2-\hat{D}_1\hat{D}_2\rangle^2+\langle \hat{B}_1\hat{A}_2+\hat{A}_1\hat{B}_2-\hat{D}_1\hat{C}_2+\hat{C}_1\hat{D}_2\rangle^2+\langle \hat{C}_1\hat{A}_2+\hat{D}_1\hat{B}_2+\hat{A}_1\hat{C}_2-\hat{B}_1\hat{D}_2\rangle^2\\ &\quad+\langle \hat{D}_1\hat{A}_2-\hat{C}_1\hat{B}_2+\hat{B}_1\hat{C}_2+\hat{A}_1\hat{D}_2\rangle^2 \leqslant \langle (\hat{A}_1^2+\hat{B}_1^2+\hat{C}_1^2+\hat{D}_1^2)(\hat{A}_2^2+\hat{B}_2^2+\hat{C}_2^2+\hat{D}_2^2)\rangle. \quad (12)\end{aligned}$$

TABLE I. The multiplication table of imaginary units of octonions.

	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	-1	i_4	i_7	$-i_2$	i_6	$-i_5$	$-i_3$
i_2	$-i_4$	-1	i_5	i_1	$-i_3$	i_7	$-i_6$
i_3	$-i_7$	$-i_5$	-1	i_6	i_2	$-i_4$	i_1
i_4	i_2	$-i_1$	$-i_6$	-1	i_7	i_3	$-i_5$
i_5	$-i_6$	i_3	$-i_2$	$-i_7$	-1	i_1	i_4
i_6	i_5	$-i_7$	i_4	$-i_3$	$-i_1$	-1	i_2
i_7	i_3	i_6	$-i_1$	i_5	$-i_4$	$-i_2$	-1

This is a bipartite Bell-type inequality with four observables on each site. To extend it to the general multipartite case it is natural to use the algebra of quaternions. Here we follow the quaternionic quantum mechanics (QQM) approach [11]. Applications of QQM to entanglement in two-qubit systems can be found in [12]. The algebra of quaternions has dimension 4 over the reals, so each quaternion q can be written as $q=x+iy+ju+kv$ in a unique way, where x, y, u, and v are reals. The multiplication rules for the imaginary units i, j, and k are $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$. The conjugation q^* of the quaternion q is defined as $q^*=x-iy-ju-kv$. The norm of q is defined in the standard way as $|q|=\sqrt{q^*q}$. The identity (10) simply expresses the fact that the norm is multiplicative: $|q'q''|=|q'\|q''|$ for arbitrary quaternions q' and q''. The norm also satisfies the triangle inequality $|q_1+q_2|\leq|q_1|+|q_2|$. The inequality (1) is formulated using the operators of the form $\hat{f}=\hat{A}+i\hat{B}$, which is a general form of a non-Hermitian operator. We extend the class of operators acting on the state space of the system to quaternionic operators of the form $\hat{q}=\hat{A}+i\hat{B}+j\hat{C}+k\hat{D}$, where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are ordinary Hermitian operators. Since the algebra of quaternions is noncommutative, care must be taken when defining the product of quaternions with operators. We define this product such that, if $\hat{q}_m=\hat{A}_m+i\hat{B}_m+j\hat{C}_m+k\hat{D}_m$ are quaternionic operators acting on different degrees of freedom $m=1,\ldots,n$, then

$$\langle\hat{q}_1\cdots\hat{q}_n\rangle=\langle\hat{q}_1\rangle\cdots\langle\hat{q}_n\rangle \tag{13}$$

for each completely factorizable state. Let us calculate the average value of the product $j\hat{f}$, where the operator $\hat{f}$ has been defined above, resulting in

$$\langle j\hat{f}\rangle=j\langle\hat{f}\rangle=j(\langle\hat{A}\rangle+i\langle\hat{B}\rangle)=j\langle\hat{A}\rangle-ij\langle\hat{B}\rangle=\langle\hat{A}-i\hat{B}\rangle j=\langle\hat{f}^\dagger j\rangle. \tag{14}$$

Here we have used the fact that the numbers $\langle\hat{A}\rangle$ and $\langle\hat{B}\rangle$ are real and due to this they commute with j. The same is valid with respect to the other imaginary unit k. We see that the natural way to define the product of the quaternion q with the operator $\hat{f}$ is $q\hat{f}=\hat{f}(x+iy)+\hat{f}^\dagger(ju+kv)$. In particular, if $\hat{F}$ is a Hermitian operator, then we have $q\hat{F}=\hat{F}q$, so the quaternionic operators $\hat{q}$ defined above behave as if the operators $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ were ordinary real numbers. This guarantees that the equality (13) is fulfilled for all factorizable states.

The same idea we used to prove the inequality (2) allows us to prove the following statement: If $\hat{q}_m$ are quaternionic operators and $\hat{F}_m$ are Hermitian operators, acting on different degrees of freedom, such that $|\langle\hat{q}_m\rangle|^2\leq\langle\hat{F}_m\rangle$, $m=1,\ldots,n$, then each completely separable state satisfies the inequality

$$|\langle\hat{q}_1\cdots\hat{q}_m\rangle|^2\leq\langle\hat{F}_1\cdots\hat{F}_n\rangle. \tag{15}$$

Here we need the factorization property (13), multiplicativity of the norm and the triangle inequality. Since we can estimate $|\langle\hat{q}_m\rangle|^2$ as

$$|\langle\hat{q}_m\rangle|^2=\langle\hat{A}_m\rangle^2+\langle\hat{B}_m\rangle^2+\langle\hat{C}_m\rangle^2+\langle\hat{D}_m\rangle^2 \leq\langle\hat{A}_m^2+\hat{B}_m^2+\hat{C}_m^2+\hat{D}_m^2\rangle, \tag{16}$$

we can take $\hat{F}_m=\hat{A}_m^2+\hat{B}_m^2+\hat{C}_m^2+\hat{D}_m^2$. Upon taking the product of quaternionic operators $\hat{q}_1,\ldots,\hat{q}_n$, the left-hand side of the inequality (15) will be a sum of four squares of average values of some observables. Then the inequality (15) is a multipartite Bell-type inequality with four observables on each site. In the case of $n=2$ it is the inequality (12).

IV. BELL-TYPE INEQUALITIES WITH EIGHT OBSERVABLES

Based on the Degen eight-square identity (11), we obtain the inequality

$$\begin{aligned}
&\langle\hat{A}_1\hat{A}_2-\hat{B}_1\hat{B}_2-\hat{C}_1\hat{C}_2-\hat{D}_1\hat{D}_2-\hat{E}_1\hat{E}_2-\hat{F}_1\hat{F}_2-\hat{G}_1\hat{G}_2-\hat{H}_1\hat{H}_2\rangle^2+\langle\hat{B}_1\hat{A}_2+\hat{A}_1\hat{B}_2+\hat{D}_1\hat{C}_2-\hat{C}_1\hat{D}_2+\hat{F}_1\hat{E}_2-\hat{E}_1\hat{F}_2-\hat{H}_1\hat{G}_2+\hat{G}_1\hat{H}_2\rangle^2\\
&\quad+\langle\hat{C}_1\hat{A}_2-\hat{D}_1\hat{B}_2+\hat{A}_1\hat{C}_2+\hat{B}_1\hat{D}_2+\hat{G}_1\hat{E}_2+\hat{H}_1\hat{F}_2-\hat{E}_1\hat{G}_2-\hat{F}_1\hat{H}_2\rangle^2+\langle\hat{D}_1\hat{A}_2+\hat{C}_1\hat{B}_2-\hat{B}_1\hat{C}_2+\hat{A}_1\hat{D}_2+\hat{H}_1\hat{E}_2-\hat{G}_1\hat{F}_2+\hat{F}_1\hat{G}_2\\
&\quad-\hat{E}_1\hat{H}_2\rangle^2+\langle\hat{E}_1\hat{A}_2-\hat{F}_1\hat{B}_2-\hat{G}_1\hat{C}_2-\hat{H}_1\hat{D}_2+\hat{A}_1\hat{E}_2+\hat{B}_1\hat{F}_2+\hat{C}_1\hat{G}_2+\hat{D}_1\hat{H}_2\rangle^2+\langle\hat{F}_1\hat{A}_2+\hat{E}_1\hat{B}_2-\hat{H}_1\hat{C}_2+\hat{G}_1\hat{D}_2-\hat{B}_1\hat{E}_2+\hat{A}_1\hat{F}_2\\
&\quad-\hat{D}_1\hat{G}_2+\hat{C}_1\hat{H}_2\rangle^2+\langle\hat{G}_1\hat{A}_2+\hat{H}_1\hat{B}_2+\hat{E}_1\hat{C}_2-\hat{F}_1\hat{D}_2-\hat{C}_1\hat{E}_2+\hat{D}_1\hat{F}_2+\hat{A}_1\hat{G}_2-\hat{B}_1\hat{H}_2\rangle^2\\
&\quad+\langle\hat{H}_1\hat{A}_2-\hat{G}_1\hat{B}_2+\hat{F}_1\hat{C}_2+\hat{E}_1\hat{D}_2-\hat{D}_1\hat{E}_2-\hat{C}_1\hat{F}_2+\hat{B}_1\hat{G}_2+\hat{A}_1\hat{H}_2\rangle^2\\
&\leq\langle(\hat{A}_1^2+\hat{B}_1^2+\hat{C}_1^2+\hat{D}_1^2+\hat{E}_1^2+\hat{F}_1^2+\hat{G}_1^2+\hat{H}_1^2)(\hat{A}_2^2+\hat{B}_2^2+\hat{C}_2^2+\hat{D}_2^2+\hat{E}_2^2+\hat{F}_2^2+\hat{G}_2^2+\hat{H}_2^2)\rangle.
\end{aligned} \tag{17}$$

It is a bipartite Bell-type inequality with eight observables on each site. To get a general multipartite inequality we need to use the algebra of octonions. Note that applications of octonions in quantum mechanics have been considered for the quantum Hall effect [13]. Moreover, based on the algebraic properties of octonions the secure continuous-variable quantum key distribution can be significantly improved [14]. The algebra of octonions is an eight-dimensional algebra over the reals, so each octonion o can be written as

$$o = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 i_5 + x_6 i_6 + x_7 i_7 \quad (18)$$

in a unique way, where x_l, $l=0,\ldots,7$ are reals and i_l, $l=1,\ldots,7$ are imaginary units, whose multiplication rules are given by Table I. The conjugation o^* of the octonion (18) is defined as

$$o^* = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3 - x_4 i_4 - x_5 i_5 - x_6 i_6 - x_7 i_7. \quad (19)$$

The norm $|o|$ is also defined in the standard way as

$$|o| = \sqrt{o^* o} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}. \quad (20)$$

The Degen eight square identity (11) expresses the fact that this norm is multiplicative, $|o'o''| = |o'||o''|$. This norm also satisfies the triangle inequality $|o_1 + o_2| \leqslant |o_1| + |o_2|$. We can define octonionic operators as

$$\hat{o} = \hat{A} + i_1\hat{B} + i_2\hat{C} + i_3\hat{D} + i_4\hat{E} + i_5\hat{F} + i_6\hat{G} + i_7\hat{H}, \quad (21)$$

where $\hat{A},\ldots,\hat{H}$ are Hermitian operators. Here we identify the first imaginary unit i_1 with the standard complex unit i. For the product of the other imaginary units with ordinary operators $\hat{f}$ we have the relation $i_l\hat{f} = \hat{f}^\dagger i_l$, $l=2,\ldots,7$. But now the product is not associative, so whenever we deal with a product of more than two terms we must explicitly group the terms.

Now we can generalize the inequality (15) as follows: If $\hat{o}_m$ are octonionic operators and $\hat{F}_m$ are Hermitian operators acting on different degrees of freedom such that $|\langle\hat{o}_m\rangle|^2 \leqslant \langle\hat{F}_m\rangle$, $m=1,\ldots,n$, then each completely separable state satisfies the inequality

$$|\langle\hat{o}_1 \cdots \hat{o}_n\rangle|^2 \leqslant \langle\hat{F}_1 \cdots \hat{F}_n\rangle, \quad (22)$$

for all C_n possible groupings of the terms on the left-hand side (we did not show the brackets explicitly), where C_n is the nth Catalan number, which is explicitly given as $C_n = (2n-2)!/[n!(n-1)!]$. Here we can take $\hat{F}_m = \hat{A}_m^2 + \hat{B}_m^2 + \hat{C}_m^2 + \hat{D}_m^2 + \hat{E}_m^2 + \hat{F}_m^2 + \hat{G}_m^2 + \hat{H}_m^2$. Upon taking the product of $\hat{o}_1,\ldots,\hat{o}_n$, the left-hand side of the inequality (22) will be a sum of eight squares of average values of some observables. Then the inequality (22) is the multipartite Bell-type inequality with eight observables on each site. In the case of $n=2$ it reduces to the inequality (17).

V. RELATION TO QUANTUM NONLOCALITY

The inequalities (1), (12), and (17) can be also obtained in another way. The integral form of the inequality (3) reads as

$$\left|\int p(\lambda)X(\lambda)d\lambda\right|^2 \leqslant \int p(\lambda)|X(\lambda)|^2 d\lambda, \quad (23)$$

where $p(\lambda)$ is a probability distribution on a measurable set Λ, and $X(\lambda)$ is a real- or complex-, quaternion- or octonion-valued function of $\lambda \in \Lambda$. The set Λ can be thought of as a set of hidden variables, which completely specify the state under study. The inequality (23) simply states that $|\langle X\rangle|^2 \leqslant \langle |X|^2\rangle$.

Let us take instead of X the operator $\hat{X}$, which is a product of ordinary non-Hermitian operators $\hat{f}_m$, quaternionic operators $\hat{q}_m$ or octonionic operators $\hat{o}_m$ defined in (21), acting on different degrees of freedom $m=1,\ldots,n$, and find the quantum mechanical analog of the quantity $|X|^2$. For example, in the case of ordinary non-Hermitian operators, we have

$$\hat{f}_m^\dagger \hat{f}_m = \hat{A}_m^2 + \hat{B}_m^2 + i[\hat{A}_m, \hat{B}_m]. \quad (24)$$

In a local hidden variable theory all commutators must be zero, so $|X|^2$ must be replaced by the product $\prod_{m=1}^n(\hat{A}_m^2 + \hat{B}_m^2)$. Analogously, in the case of quaternionic operators we make use of the replacement

$$|X|^2 \Rightarrow \prod_{m=1}^{n} (\hat{A}_m^2 + \hat{B}_m^2 + \hat{C}_m^2 + \hat{D}_m^2). \quad (25)$$

In the octonionic case, we use the replacement

$$|X|^2 \Rightarrow \prod_{m=1}^{n} (\hat{A}_m^2 + \cdots + \hat{H}_m^2). \quad (26)$$

Then we again arrive at the inequalities (1), (12), and (17), respectively. This approach may appear to be simpler than the one we started with, but it is ambiguous and does not strictly relate the obtained inequalities to separability.

It is noteworthy that the inequalities (1), (12), and (17) form a hierarchy—the inequality (1) is a special case of (12), which is in turn a special case of (17). As shown in [7], the inequality (1) can be violated. Thus it is clear that our inequalities (12) and (17) can be violated as well. Any violation of these inequalities is a clear signature of quantum nonlocality and hence of entanglement.

VI. SUMMARY AND CONCLUSIONS

To summarize, we have obtained Bell-type inequalities for observables with a general spectrum. They apply to measurements of up to eight observables per site for arbitrary systems. The derivation of the inequalities has been based on square identities. The multipartite forms of these inequalities are related to the algebras of quaternions and octonions.

Let us conclude with some remarks on the tightness of our inequalities. In the case of observables with discrete spectra, their pairwise, triplewise, etc., average values (with no two observables referring to the same degree of freedom), calculated for all separable states, form a convex polytope. It can be represented as an intersection of a finite number of half-spaces. Bell inequalities are exactly the linear inequalities which describe these half-spaces, so that they are tight in the sense that one cannot sharpen them. In the general case of observables with arbitrary spectra the corresponding set is no longer a polytope, i.e., it is not an intersection of a finite number of half-spaces. Our inequalities are not linear, so it is a more complicated problem to say whether they are tight or not.

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